Discrete Mathematics 68 (1988) 245–255
North-Holland

POSITIVE SETS IN FINITE LINEAR FUNCTION SPACES

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Received 15 April 1985

This paper is mainly concerned with positive sets and positive functions in a finite linear function space. Our two main results characterize positive functions and minimal positive sets. We then show that certain morphisms preserve both the cone of positive functions and minimal positive sets. Finally we specialize these results to the case of measures on a hypergraph.

1. Introduction

In this paper we continue the work begun in [11] where we considered finite linear function spaces and measures on hypergraphs. A finite linear function space (FLFS) is a pair (X, V), where X is a nonempty finite set and V is a linear space of real-valued functions on X. In various applications it is important to know when $f \in V$ is in the cone V_+ of positive functions on X. For example, in the case of measures on a hypergraph H, the elements of V_+ correspond to unnormalized states (or stochastic functions) on H. The linear subspace J spanned by V_+ is called the space of Jordan functions. The elements of J are precisely those functions $f \in V$ which admit a Jordan decomposition $f = f_1 - f_2$, $f_1, f_2 \in V_+$. Our first main result gives a characterization of Jordan functions.

A subset $Y \subseteq X$ is called a positive set if it determines positive functions. That is, if $f \in V$ and $f(y) \geqslant 0$ for all $y \in Y$ imply $f \in V_+$. Such sets are useful since they reduce the labor required to find whether a function is positive or not. Of particular importance are the minimal positive sets, and our second main result characterizes such sets. We next use positive sets to characterize when V_+ is a simplicial cone.

In a subsequent section we consider morphisms between FLFS's. We show that certain types of morphisms preserve V_+, positive sets, and minimal positive sets. Finally, we specialize these results to the case of measures on a hypergraph. This case is not only important in hypergraph theory [2, 10, 13, 18], but also in operational statistics [3, 4, 5, 16], quantum logic [1, 6, 7, 12, 14, 15, 17, 20], and possibly elementary particle physics [8, 9].

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2. Notation and definitions

Let V be a finite-dimensional real linear space. We denote the linear hull of a subset S of V by $\operatorname{lin} S$. The *positive hull*, $\operatorname{pos} S$, is the set of all linear combinations of elements of S with nonnegative coefficients. By convention $\operatorname{pos}\emptyset=\{0\}$. A subset W of V is a *wedge* if $W+W\subseteq W$ and $\mathbb{R}_+W\subseteq W$, where $\mathbb{R}_+=\{\lambda\in\mathbb{R}:\lambda\geq 0\}$. Clearly, $\operatorname{pos} S$ is a wedge for any $S\subseteq V$. A wedge W is *generating* in V if $V=W-W$. A *cone* is a wedge W satisfying $W\cap(-W)=\{0\}$. Let $(V_1, V_2, (\cdot,\cdot))$ be a dual pair of finite-dimensional real linear spaces. For $A\subseteq V_1$, $B\subseteq V_2$, define

$$A^{\mathrm{w}}=\{g\in V_2: (f,g)\geq 0 \text{ for all } f\in A\},$$
$$B_{\mathrm{w}}=\{f\in V_1: (f,g)\geq 0 \text{ for all } g\in B\}.$$

It is clear that A^{w} is a wedge in V_2 and B_{w} is a wedge in V_1. The bipolar theorem [19] yields $A^{\mathrm{w}}_{\mathrm{w}}=\operatorname{cl}\operatorname{pos}A$, $B^{\mathrm{w}}_{\mathrm{w}}=\operatorname{cl}\operatorname{pos}B$, where the closure (cl) is taken in the Euclidean topology of the finite-dimensional vector spaces V_1, V_2, respectively.

Let X be a nonempty finite set and let V be a nonempty set of real-valued functions on X which is closed under scalar multiplication and pointwise addition. Then V is a real linear space with $\dim V\leq |X|$. We call (X, V) a *finite linear function space* (FLFS) [11]. We denote the dual of V by V^*. For $x\in X$, define $e_V(x)=x^*\in V^*$ by $x^*(f)=f(x)$ for all $f\in V$. If $Y\subseteq X$, the *V-closure* of Y is defined by

$$\bar{Y}=\{x\in X: f,g\in V, f\mid Y=g\mid Y\Rightarrow f(x)=g(x)\}.$$

We say that $Y\subseteq X$ is *V-dense* if $\bar{Y}=X$. One of the important properties of minimal V-dense sets is that they can be used to determine $\dim V$. In fact, it is shown in [11] that a V-dense set $Y\subseteq X$ is minimal V-dense if and only if $\dim V=|Y|$.

For a FLFS (X, V) we define

$$V_+=\{f\in V: f(x)\geq 0 \text{ for all } x\in V\}.$$

Then V_+ is a cone in V. We define the subspace J of V by $J=\operatorname{lin}V_+=V_+-V_+$. We say that (X, V) is *Jordan* if V_+ is a generating cone; that is, if $J=V$. We also define

$$K=\{x\in X: f(x)=0 \text{ for all } f\in V_+\},$$
$$N=\{x\in X: f(x)=0 \text{ for all } f\in V\}.$$

Clearly, $N\subseteq K$, $K=X$ if and only if $V_+=\{0\}$, $N=X$ if and only if $V=\{0\}$. It is shown in [11] that the following statements are equivalent: (a) $J=V$, (b) $N=K$, (c) $|K\setminus N|\leq 1$, (d) V^{w}_+ is a cone. We say that V is *nucleonic* if for every $x\in X$ there exists $f\in V_+$ such that $f(x)=0$, $f(y)\neq 0$ for every $y\in X\setminus\{x\}$. If V is nucleonic, then of course V is separating. Nucleonic spaces of measures on a hypergraph have been studied in [13]. For $Y \subseteq X$ write

$$Y^+ = \{f \in V : f(y) \geqslant 0 \text{ for all } y \in Y\}.$$

If $Y, Z \subseteq X$ we call Y a *Z-positive set* if $Y^+ \subseteq Z^+$. An X-positive set is simply called a *positive set.*

It is not hard to show that the set $\mathscr{K} = \{\ker f : f \in V_+\}$ is a lattice under the ordering $\subseteq$. Indeed, if $A = \ker f$, $B = \ker g$, $f, g \in V_+$, then

$$A \wedge B = \ker(f+g) = A \cap B.$$

Moreover,

$$A \vee B = \bigwedge \{\ker f : A \cup B \subseteq \ker f\}$$

In this lattice, the infimum coincides with the set-intersection, X is the largest element and K is the least element. Clearly, the following are equivalent: $|\mathscr{K}| = 1$, $X = K$, $V_+ = \{0\}$. Denote by $\mathscr{A}$ the collection of atoms in the lattice $(\mathscr{K}, \subseteq)$.

Let (X_1, V_1) and (X_2, V_2) be FLFS's. A map $T : X_1 \to X_2$ is a *morphism* if for every $f \in V_2$ we have $f \circ T \in V_1$. If $T : X_1 \to X_2$ is a morphism, we define the linear map $\hat{T} : V_2 \to V_1$ by $\hat{T}f(x) = f(Tx)$ for all $x \in X_1$, $f \in V_2$. We also define the linear map $\hat{T}^* : V_1^* \to V_2^*$ by $\hat{T}^*F(f) = F(\hat{T}f)$ for all $F \in V_1^*$, $f \in V_2$. It is not hard to show that $\hat{T}^* e_{V_1} = e_{V_2} T$. If $T : X_1 \to X_2$ is bijective and both T and T^{-1} are morphisms, we call T an *isomorphism* and say that (X_1, V_1), (X_2, V_2) are *isomorphic*. It is easy to check that if T is an isomorphism, then $\hat{T}$ and $\hat{T}^*$ are linear isomorphisms and that $(\hat{T}^{-1})^* = (\hat{T}^*)^{-1}$.

3. Positive sets

In the sequel, (X, V) will always denote a FLFS. The complement of a subset A of X will be denoted by A'.

Lemma 1. (a) $\operatorname{pos} e_V(K')$ *is a cone in* V^*.
(b) $e_V(K')_w$ *is a generating wedge in* V.

Proof. (a) Clearly, $\operatorname{pos} e_V(K')$ is a wedge. Let

$$F \in \operatorname{pos}_V(K') \cap [-\operatorname{pos} e_V(K')].$$

Then for suitable scalars $s_x, t_x \geqslant 0$ we have

$$F = \sum_{x \in K'} s_x e_V(x) = -\sum_{x \in K'} t_x e_V(x).$$

This shows that for every $f \in V_+$, $F(f) \geqslant 0$ and $F(f) \leqslant 0$, hence, $F(f) = 0$. Thus, $\sum_{x \in K'} s_x f(x) = 0$ for all $f \in V_+$ which implies $s_x = 0$ for all $x \in K'$. Therefore, $F = 0$.

(b) Suppose that $e_V(K')_w$ is not generating. Then there exists $0 \neq F \in V^*$ such that

$$e_V(K')_w - e_V(K')_w \subseteq \ker F.$$

By the bipolar theorem we have

$$\pm F \in e_V(K')_w^w = \operatorname{pos} e_V(K').$$

By Part (a), $\operatorname{pos} e_V(K')$ is a cone, so $F = 0$. This is a contradiction. □

Theorem 2. *Let (X, V) be a FLFS and let $f \in V$. Then the following statements are equivalent.* (a) $f \in J$, (b) $K \setminus N \subseteq \ker f$, (c) $K \subseteq \ker f$.

Proof. (a)⇒(b)⇒(c) are obvious. We now show that (c)⇒(a). If $K = X$, then $K \subseteq \ker f$ implies $f = 0$. Hence, $f \in J$. We now assume that $K' \neq \emptyset$. Select for each $x \in K'$ an $f_x \in V_+$ such that $f_x(x) > 0$ and define $f_0 = \sum_{x \in K'} f_x$. Then $f_0 \in V_+$ and $f_0(x) > 0$ for all $x \in K'$. Suppose $K \subseteq \ker f$. Then $f \in \bigcap_{y \in K} \ker e_V(y)$. Since, by Lemma 1(b), $e_V(K')_w$ is a generating wedge, there exist $g_1, g_2 \in e_V(K')_w$ such that $f = g_1 - g_2$. Notice that $f_0 \in e_V(K')_w$ as well. Define

$$t_i = \max_{x \in K'} g_i(x)/f_0(x), \quad i = 1, 2.$$

Then $t_1, t_2 \geq 0$ and for all $x \in K'$ we have

$$0 \leq g_i(x) \leq t_i f_0(x), \quad i = 1, 2.$$

Now for all $x \in K'$ we have

$$-t_1 f_0(x) \leq g_1(x) \leq t_1 f_0(x), \qquad -t_2 f_0(x) \leq -g_2(x) \leq t_2 f_0(x),$$

so it follows that

$$-(t_1 + t_2) f_0(x) \leq f(x) \leq (t_1 + t_2) f_0(x).$$

Letting $s = t_1 + t_2$, we conclude that

$$s f_0 - f,\ f + s f_0 \in e_V(K')_w \cap \bigcap_{y \in K} \ker e_V(y),$$

since $f_0 \in J \subseteq \bigcap_{y \in K} \ker e_V(y)$. But

$$f = \tfrac{1}{2}(f + s f_0) - \tfrac{1}{2}(s f_0 - f),$$

and

$$e_V(K')_w \cap \bigcap_{y \in K} \ker e_V(y) \subseteq V_+.$$

Thus, $f \in V_+ - V_+ = J$. □

Lemma 3. *A subset Y is a Z-positive set if and only if* $\operatorname{pos} e_V(Z) \subseteq \operatorname{pos} e_V(Y)$.

Proof. If Y is a Z-positive set, then $e_V(Y)_w \subseteq e_V(Z)_w$. Applying the bipolar theorem, we conclude that

$$\operatorname{pos} e_V(Z) = e_V(Z)_w^w \subseteq e_V(Y)_w^w = \operatorname{pos} e_V(Y).$$

The converse is easily shown. □

Corollary 4. *A positive set is V-dense.*

Proof. If Y is a positive set, then applying Lemma 3 we have

$$V_+^w = \operatorname{pos} e_V(X) \subseteq \operatorname{pos} e_V(Y).$$

But $V_+ = V_{+w}^w$ is a cone and hence V_+^w is a generating wedge of V^*. Thus $V^* = \operatorname{lin} e_V(Y)$. It follows from [11, Lemma 1] that Y is V-dense. □

The next theorem, which is our main result, characterizes minimal positive sets.

Theorem 5. *Let (X, V) be a FLFS with $V_+ \neq \{0\}$. Then $Y \subseteq X$ is a minimal positive set if and only if*

(a) $|Y \cap (A \setminus K)| = 1$ *for every* $A \in \mathscr{A}$;
(b) $Y \cap (K \setminus N)$ *is a minimal* $(K \setminus N)$*-positive set*;
(c) $Y \cap N = \emptyset$.

Proof. We first show that if Y is a positive set, then $Y \cap (A \setminus K) \neq \emptyset$ for every $A \in \mathscr{A}$ and $Y \cap (K \setminus N)$ is a $(K \setminus N)$-positive set. If $K \setminus N = \emptyset$, then clearly $Y \cap (K \setminus N)$ is a $(K \setminus N)$-positive set so suppose $K \setminus N \neq \emptyset$. If $y \in K \setminus N$, then $e_V(y) \neq 0$. Since Y is positive, we have

$$\begin{aligned} \operatorname{pos} e_V(X) = \operatorname{pos} e_V(Y) &= \operatorname{pos} e_V(Y \cap K') + \operatorname{pos} e_V[Y \cap (K \setminus N)] \\ &= \operatorname{pos} e_V(K') + \operatorname{pos} e_V[Y \cap (K \setminus N)]. \end{aligned}$$

If $Y \cap (K \setminus N) = \emptyset$, then $e_V(y) = \sum_{x \in K'} s_x e_V(x)$ for suitable scalars $s_x \geq 0$. Then for every $f \in V_+$ we have

$$0 = e_V(y)(f) = \sum_{x \in K'} s_x f(x).$$

Hence, $s_x = 0$, $x \in K'$, which implies $e_V(y) = 0$, a contradiction. Therefore, $Y \cap (K \setminus N) \neq \emptyset$. We then have

$$e_V(y) = \sum_{x \in K'} r_x e_V(x) + \sum_{z \in Y \cap (K \setminus N)} t_z e_V(z),$$

for suitable scalars r_x, $t_x \geq 0$. Again, for every $f \in V_+$ we have

$$0 = e_V(y)(f) = \sum_{x \in K'} r_x f(x) + \sum_{z \in Y \cap (K \setminus N)} t_z f(z) = \sum_{x \in K'} r_x f(x),$$

and it follows that $r_x = 0$ for all $x \in K'$. Hence, $e_V(y) \in \text{pos}\, e_V[Y \cap (K \setminus N)]$. Thus,

$$\text{pos}\, e_V(K \setminus N) \subseteq \text{pos}\, e_V[Y \cap (K \setminus N)],$$

and by Lemma 3 we conclude that $Y \cap (K \setminus N)$ is a $(K \setminus N)$-positive set.

Notice that $Y \cap K' \neq \emptyset$ since otherwise $Y \subseteq K$ making K a positive set. Now if $f \in J$, then $f(x) = 0$ for all $x \in K$, hence $f \in V_+$. It follows that $J = V_+$. But V_+ is a cone and J is a linear subspace of V so $V_+ = \{0\}$ which is a contradiction.

Moreover, since Y is a positive set, we have

$$V_+ = \{f \in J : f(y) \geq 0 \text{ for all } y \in Y \cap K'\}.$$

Thus, in the duality (J, J^*) we have $V_+ = [e_J(Y \cap K')]_{w'}$ and hence, $V_+^{w'} = \text{pos}\, e_J(Y \cap K')$. Now let $f \in V_+$ satisfy $\ker f \in \mathscr{A}$ and select $x \in \ker f \setminus K$. Then

$$e_J(x) = \sum_{y \in Y \cap K'} t_y e_J(y),$$

for suitable scalars $t_y \geq 0$. Since $e_J(x) \neq 0$, there exists $y' \in Y \cap K'$ with $t_{y'} > 0$. Then

$$0 = f(x) = \sum_{y \in Y \cap K'} t_y f(y)$$

implies that $f(y') = 0$. Thus,

$$y' \in \ker f \cap (Y \cap K') = Y \cap (\ker f \setminus K).$$

We conclude that $Y \cap (A \setminus K) \neq \emptyset$ for any $A \in \mathscr{A}$.

To complete the proof we show that a subset $Y \subseteq X$ satisfying conditions (a), (b), and (c) is a positive set. Let $h \in V$ and suppose that $h(y) \geq 0$ for all $y \in Y$. Then, in particular,

$$h \in [\text{pos}\, e_V(Y \cap (K \setminus N)]_w.$$

Now $Y \cap (K \setminus N)$ is a $(K \setminus N)$-positive set, so by [11, Lemma 22] we have

$$\text{pos}\, e_V[Y \cap (K \setminus N)] = \text{pos}\, e_V(K \setminus N) = \text{lin}\, e_V(K \setminus N).$$

Then by Theorem 2 we have

$$\begin{aligned}[\text{lin}\, e_V(K \setminus N)]_w &= \{f \in V : F(f) \geq 0 \text{ for all } F \in \text{lin}\, e_V(K \setminus N)\} \\ &= \{f \in V : F(f) = 0 \text{ for all } F \in \text{lin}\, e_V(K \setminus N)\} \\ &= \{f \in V : f(z) = 0 \text{ for all } z \in K \setminus N\} = J.\end{aligned}$$

Therefore, $h \in J$.

Since $V_+^{w'}$ is a polyhedral cone, each of its faces is exposed [19]. Let F be a generator of an extreme ray (one-dimensional face) of the cone $V_+^{w'}$. Then $F \neq 0$ and there exists an $f \in V_+$ such that

$$\{G \in V_+^{w'} : G(f) = 0\} = \mathbb{R}_+ F.$$

We now show that $\ker f \in \mathscr{A}$. If $\ker f = K$, then $f(x) > 0$ for all $x \in K'$. Thus,

$G(f)>0$ for all $0\neq G\in \operatorname{pos} e_J(K')=V_+^{w'}$. Hence, $F=0$ which is a contradiction. Suppose there exists $g\in V_+$ with $K\subseteq \ker g\subseteq \ker f$, $K\neq \ker g$. Let $x\in \ker g\setminus K$. Then $e_J(x)\neq 0$ and $x\in \ker f$. Hence, $e_J(x)=t_xF$ for some $t_x>0$. If $y\in \ker f\setminus K$, then $e_J(y)=t_yF$ for some $t_y>0$. Then $e_J(x)=se_J(y)$ for some $s>0$. We then have $0=g(x)=sg(y)$ so $g(y)=0$. Hence, $y\in \ker g$ which proves that $\ker g=\ker f$.

Now let $y\in Y\cap(\ker f\setminus K)$. Again, we have $e_J(y)=tF$ for some scalar $t>0$. This proves that $\operatorname{pos} e_J(Y\cap K')$ contains the generators of the extreme rays of the cone $V_+^{w'}$ and hence $V_+^{w'}=\operatorname{pos} e_J(Y\cap K')$. We have shown above that if $h\in V$ satisfies $h(y)\geqslant 0$ for all $y\in Y$, then $h\in J$. But then $e_J(y)(h)\geqslant 0$ and hence

$$h\in[\operatorname{pos} e_J(Y\cap K')]_{w'}=V_{+w'}^{w'}=V_+. \quad \square$$

If $A_1,\ldots,A_n$ are disjoint sets, we use the notation $\dot{\bigcup} A_i$ for $\bigcup A_i$.

Corollary 6. *Let (X, V) be a FLFS and let $Y\subseteq X$ be a minimal positive set. Then*

(a) $Y\cap K'=\dot{\bigcup}_{A\in\mathscr{A}}[Y\cap(A\setminus K)]$;

(b) $|\mathscr{A}|=|Y\cap K'|$.

Proof. We may assume that $X\neq K$. If A_1, $A_2\in\mathscr{A}$ and $A_1\neq A_2$, then $K=A_1\wedge A_2=A_1\cap A_2$ and hence, $(A_1\setminus K)\cap(A_2\setminus K)=\emptyset$. By Theorem 5 is now follows that

$$\dot{\bigcup}_{A\in\mathscr{A}}[Y\cap(A\setminus K)]\dot{\bigcup}[Y\cap(K\setminus N)]$$

is a minimal positive set and thus equals Y. Condition (a) follows immediately. Condition (b) is a straightforward consequence of (a). $\square$

Corollary 7. *Let (X, V) be a Jordan FLFS with $V_+\neq\{0\}$. Then V is nucleonic if and only if X is a minimal positive set.*

Proof. Let V be nucleonic. It follows that $K=\emptyset$. By definition, each singleton set $\{x\}$, $x\in X$, is an atom in the lattice $\mathscr{K}$. Hence, by Theorem 5, X is a minimal positive set. Conversely, suppose that X is a minimal positive set. Since (X, V) is Jordan, it follows that $K=N$. But $X\cap N=\emptyset$, hence, $K=N=\emptyset$. Let $x\in X$. By Corollary 6, there exists an $A\in\mathscr{A}$ such that $x\in A$. If there exists a $y\in A$ with $y\neq x$, then $|X\cap A|>1$, contradicting the assumption that X is a minimal positive set. Thus, $\{x\}=A$ so $\{x\}\in\mathscr{K}$. Hence, there exists an $f\in V_+$ such that $\{x\}=\ker f$. Then clearly V is nucleonic. $\square$

Theorem 8. *If (X, V) is a FLFS, then V_+ is a simplicial cone if and only if there exists a positive set Y such that $Y\cap K'$ is a minimal J-dense set.*

Proof. If $X=K$, then $V_+=\{0\}$ and the assertions hold (for $Y=X$). We now assume that $X\neq K$. Suppose $Y\subseteq X$ has the required properties. Then, as in the proof of Theorem 5, we get $V_+^{w'}=\operatorname{pos} e_J(Y\cap K')$. Also, $e_J(Y\cap K')$ is a linearly independent set in J^*. Since $e_J(Y\cap K')$ generates (positive hull) the polyhedral cone $V_+^{w'}$, we conclude that for each extreme ray, there exists $y\in Y\cap K'$ such that it coincides with $\mathbb{R}_+e_J(y)$. Notice that if $x_1, x_2\in Y\cap K'$ and $x_1\neq x_2$, then $e_J(x_1)\neq e_J(x_2)$. Define the set

$$Z=\{x\in Y\cap K' : \mathbb{R}_+e_J(x) \text{ is an extreme ray of } V_+^{w'}\}.$$

Then $\operatorname{pos} e_J(Z)=V_+^{w'}$. Suppose there exists a $y\in Y\cap K'\setminus Z$. Then $e_J(y)=\sum_{x\in Z} t_x e_J(x)$ for suitable scalars $t_x\geq 0$. Since $e_J(y)\neq 0$, this contradicts the linear independence of $e_J(Y\cap K')$. Thus $Y\cap K'=Z$ and the number of extreme rays equals $\dim J^*$ which equals $\dim V_+^{w'}$ (since $V_+^{w'}$ is a generating cone). This proves that $V_+^{w'}$ and, by duality, V_+ is a simplicial cone.

Conversely, suppose that V_+ and hence $V_+^{w'}$ are simplicial cones. Since $V_+^{w'}=\operatorname{pos} e_J(K')$, to each extreme ray there corresponds a $z\in K'$ such that it coincides with $\mathbb{R}_+e_J(z)$. Select for each extreme ray exactly one such z and denote the set of z's by Z. Since $V_+^{w'}$ is a simplicial cone, it follows that $e_J(Z)$ is linearly independent. Then $Y=Z\cup K$ is a positive set. Indeed, if $f(x)\geq 0$ for all $x\in Z\cup K$ then, in particular, $f(x)\geq 0$ for all $x\in K$. Thus, by [11, Lemma 22] we have

$$f\in[\operatorname{pos} e_V(K)]_w=[\operatorname{lin} e_V(K)]_w.$$

Hence, $f(x)=0$ for all $x\in K$. By Theorem 2, we conclude that $f\in J$. But $f(x)\geq 0$ for all $x\in Z$, so

$$f\in[\operatorname{pos} e_J(Z)]_{w'}=V_{+w'}^{w'}.$$

Thus, $f\in V_+$. Then $Z=Y\cap K'$ and since $\operatorname{pos} e_J(Z)=V_+$, we conclude that Z is a J-dense set. Hence, by the observation above, Z is minimal J-dense. □

4. Positive sets and morphisms

When considering two FLFS's (X_1, V_1) and (X_2, V_2), subscripted sets such as V_{i+}, N_i, $i=1,2$, refer to subsets of X_i, $i=1, 2$, and are defined as usual.

Lemma 9. *Let (X_1, V_1), (X_2, V_2) be FLFS's and let $T: X_1\to X_2$ be a morphism*: (a) $\hat{T}(V_{2+})\subseteq V_{1+}$; (b) $\hat{T}^*(V_{1+}^{w})\subseteq V_{2+}^{w}$; (c) $t(N_1)\subseteq N_2$, *and* (d) $T(K_1)\subseteq K_2$.

Proof. (a) If $f\in V_{2+}$, then for all $x\in X_1$ we have $\hat{T}f(x)=f(Tx)\geq 0$. Hence, $\hat{T}f\in V_{1+}$ and $\hat{T}(V_{2+})\subseteq V_{1+}$.

(b) Let $F\in V_{1+}^{w}$ and $f\in V_{2+}$. Then $\hat{T}f\in V_{1+}$ by Part (a). Hence, $\hat{T}^*F(f)=F(\hat{T}f)\geq 0$ so $\hat{T}^*F\in V_{2+}^{w}$. Thus, $\hat{T}^*(V_{1+}^{w})\subseteq V_{2+}^{w}$.

(c) If $x \in N_1$ and $f \in V_2$, then $f(Tx) = \hat{T}f(x) = 0$. Hence, $Tx \in N_2$ so $T(N_1) \subseteq N_2$.

(d) If $x \in K_1$ and $f \in V_{2+}$, then by Part (a) $\hat{T}f \in V_{1+}$ so $f(Tx) = \hat{T}f(x) = 0$. Thus $Tx \in K_2$ so $T(K_1) \subseteq K_2$. □

Corollary 10. *Let $T : X_1 \to X_2$ be an isomorphism*: (a) $\hat{T}(V_{2+}) = V_{1+}$; (b) $\hat{T}^*(V_{1+}^{w}) = V_{2+}^{w}$; (c) $T(N_1) = N_2$, *and* (d) $T(K_1) = K_2$.

For $x, y \in X$, if $f(x) = f(y)$ for all $f \in V$ we write $x \sim y$. Then $\sim$ is an equivalence relation. A map $T : X_1 \to X_2$ is *$\sim$surjective* if for every $y \in X_2$ there exists an $x \in X_1$ such that $Tx \sim y$. It can be shown [11, Theorem 18] that if T is $\sim$surjective, then $\hat{T}$ is injective.

Lemma 11. *Let $T : X_1 \to X_2$ be a $\sim$surjective morphism. If $Y \subseteq X_1$ is a positive set, then $T(Y) \subseteq X_2$ is a positive set.*

Proof. Suppose $Y \subseteq X_1$ is a positive set and suppose $f \in V_2$ satisfies $f(Ty) \geq 0$ for all $y \in Y$. Then $\hat{T}f(y) \geq 0$ for all $y \in Y$ so $\hat{T}f \in V_{1+}$. If $x_2 \in X_2$ there exists an $x_1 \in X$ such that $Tx_1 \sim x_2$. Hence,

$$f(x_2) = f(Tx_1) = \hat{T}f(x_1) \geq 0.$$

Thus, $f \in V_{2+}$ so $T(Y)$ is a positive set. □

Theorem 12. *Let $T : X \to X$ be a $\sim$surjective morphism.* (a) $\hat{T}(V_+) = V_+$; (b) $\hat{T}^*(V_+^{w}) = V_+^{w}$, *and* (c) *if $Y \subseteq X$ is a minimal positive set, then $T(Y)$ is a minimal positive set.*

Proof. It follows from our previous remark that $\hat{T} : V \to V$ is injective. Hence, $\hat{T}$ is bijective. Moreover, $\hat{T}^*$ is bijective. (a) Applying Lemma 9(a), we have $\hat{T}(V_+) \subseteq V_+$. Now let $g \in V_+$. Then there exists an $f \in V$ such that $\hat{T}f = g$. If $y \in X$, then there exists an $x \in X$ with $Tx \sim y$. Hence,

$$f(y) = f(Tx) = \hat{T}f(x) = g(x) \geq 0.$$

Therefore, $f \in V_+$ and $\hat{T}(V_+) = V_+$.

The proof of (b) is similar.

(c) Suppose $Y \subseteq X$ is a minimal positive set. By Lemma 11, $T(Y)$ is a positive set. Let $z \in T(Y)$ and let $Y_1 = Y \setminus T^{-1}(\{z\})$. Then $Y_1 \subseteq Y$ and $Y_1 \neq Y$, Since Y_1 is not a positive set, there exists an $f \in V \setminus V_+$ such that $f(y) \geq 0$ for all $y \in Y_1$. It follows from Part (a) and the fact that $\hat{T}$ is bijective that $\hat{T}^{-1}f \notin V_+$. Let $x \in T(Y) \setminus \{z\}$. Then there exists a $y \in Y_1$ such that $Ty = x$. Then

$$\hat{T}^{-1}f(x) = \hat{T}^{-1}f(Ty) = f(y) \geq 0.$$

Hence, $T(Y) \setminus \{z\}$ is not a positive set so $T(Y)$ is minimal positive. □

5. Hypergraphs

In this section we shall consider an example of an FLFS, namely a hypergraph together with its set of measures. We shall then specialize our previous results to this example.

Let $H=(X, 0)$ be a hypergraph [2]. A function $\mu:X\to\mathbb{R}$ is a *measure* if there exists a constant $\hat{\mu}\in\mathbb{R}$ such that $\sum_{x\in E}\mu(x)=\hat{\mu}$, for all $E\in O$. We denote the set of measures on H by $M=M(H)$. It is clear that (X, M) is an FLFS. The set M_+ consists of the *positive measures* on X. If $\mu\in M_+$ and $\hat{\mu}=1$, we call μ a *state* (or *stochastic function*). The set of states is denoted by $\Omega=\Omega(H)$, and Ω is a convex set which forms a base for M_+ (i.e., $M_+=\mathbb{R}_+\Omega$). In this context, Ω is of more importance and is more basic than M_+. For this reason, some of the definitions are given in terms of Ω instead of in terms of M_+. However, the definitions are equivalent to those given earlier. For example, we say that Ω is *nucleonic* if for any $x\in X$ there exists a $\mu\in\Omega$ such that $\mu(x)=0$ and $\mu(y)\neq 0$ for all $y\in X\setminus\{x\}$. We can now apply our previous results to hypergraphs. For example, we have the following results.

Theorem 2′. *Let H be a hypergraph and let* $\mu\in M$. *Then the following statements are equivalent*: (a) $\mu\in J$;
(b) $K\setminus N\subseteq\ker\mu$, *and* (c) $K\subseteq\ker\mu$.

Theorem 5′. *Let H be a hypergraph with* $\Omega\neq\emptyset$. *Then* $Y\subseteq X$ *is a minimal positive set if and only if*

(a) $|Y\cap(A\setminus K)|=1$, *for every* $A\in\mathcal{A}$;
(b) $Y\cap(K\setminus N)$ *is a minimal* $(K\setminus N)$*-positive set*;
(c) $Y\cap N=\emptyset$.

Corollary 7′. *Let H be a Jordan hypergraph with* $\Omega\neq\emptyset$. *Then* Ω *is nucleonic if and only if X is a minimal positive set.*

Let $H_1=(X_1, 0_1)$ and $H_2=(X_2, 0_2)$ be hypergraphs. A map $T:X_1\to X_2$ is a *hypergraph isomorphism* if T is a bijection satisfying $T(E)\in 0_2$ if and only if $E\in 0_1$. It is clear that if $T:X_1\to X_2$ is a hypergraph isomorphism, then T is an FLFS isomorphism for the FLFS's (X_1, M_1), (X_2, M_2). Using our results in Section 4, the following theorem is easily proved.

Theorem 13. *Let* $T:X_1\to X_2$ *be a hypergraph isomorphism for the hypergraphs* $(X_1, 0_1)$ *and* $(X_2, 0_2)$: (a) $\hat{T}(M_{2+})=M_{1+}$; (b) $\hat{T}^*(M_{1+}^w)=M_{2+}^w$; (c) $T(N_1)=N_2$; (d) $T(K_1)=K_2$, *and* (e) *if* $Y\subseteq X$ *is a* (*minimal*) *positive set, then* $T(Y)$ *is a* (minimal) *positive set.*

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