PERGAMON

Applied Mathematics Letters 16 (2003) 791–796

Applied Mathematics Letters
www.elsevier.com/locate/aml

Partial Relaxed Monotonicity and General Auxiliary Problem Principle with Applications

R. U. VERMA
Department of Mathematics, University of Toledo
Toledo, OH 43606, U.S.A.
rverma@pop3.utoledo.edu

(Received October 2001; accepted March 2002)

Communicated by G. Fix

Abstract—First, a general framework for the auxiliary problem principle is introduced and then it is applied to the approximation-solvability of the following class of nonlinear variational inequality problems (NVIP) involving partially relaxed monotone mappings. Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geqq 0, \qquad \text{for all } x \in K,$$

where $T : K \to R^n$ is a mapping from a nonempty closed convex subset K of R^n into R^n, and $f : K \to R$ is a continuous convex functional on K. The general class of the auxiliary problem principles is described as follows: for a given iterate $x^k \in K$ and for a parameter $\rho > 0$, determine x^{k+1} such that

$$\left\langle \rho T\left(x^k\right) + h'\left(x^{k+1}\right) - h'\left(x^k\right), x - x^{k+1} \right\rangle + \rho\left[f(x) - f\left(x^{k+1}\right)\right] \geqq \left(-\sigma^k\right), \qquad \text{for all } x \in K,$$

where $h : K \to R$ is m-times continuously Frechet-differentiable on K and $\sigma^k > 0$ is a number.

Keywords—Auxiliary problem principle, Approximation-solvability, Approximate solutions, General auxiliary problem principle.

1. INTRODUCTION

In their ongoing research on generalized Newton's method, Argyros and Verma [1] used inexact Newton-like iterative procedures to approximate a solution of a class of nonlinear equations in a Banach space setting, since approximating a solution of a nonlinear equation using Newton-like iterates at each stage seems to be quite expensive in general.

On the top of that, it turns out that some of the auxiliary results from this work seem to have a great impact on auxiliary problem principle [2] and general auxiliary problem principle [3] and their applications to the approximation solvability in the general sense. For a better account on the auxiliary problem principle and nonlinear variational inequalities, we refer to [2–10].

PII: S0893-9659(03)00084-3

In this paper, we intend first to introduce a general version of the auxiliary problem principle and then apply it to the approximation-solvability of a class of nonlinear variational inequalities. The obtained results do complement the earlier works of Cohen [2] and Argyros and Verma [3] on the approximation-solvability of nonlinear variational inequalities in different space settings.

Let $T : K \to R^n$ be any mapping from K, a nonempty closed convex subset of R^n, into R^n. Let $f : K \to R$ be a continuous convex function on K. We consider a class of nonlinear variational inequality problems (abbreviated as NVIP) involving partially relaxed monotone mappings as follows. Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geqq 0, \qquad \text{for all } x \in K. \tag{1.1}$$

Let $\|x\|_B$ denote the norm induced by the positive definite matrix B, defined by

$$\|x\|_B = \langle Bx, x\rangle^{1/2}.$$

And let $\|x\|_2$ denote the standard Euclidean norm on R^n with respect to the dot product $\langle .,. \rangle$.

A mapping $T : K \to R^n$ is said to be γ-μ-partially relaxed monotone if for all $x, y, z \in K$, we have

$$\langle T(x) - T(y), z - y \rangle \geqq (-\gamma)\|z - x\|^2 + \mu\|x - y\|^2,$$

where $\gamma, \mu > 0$ are constants.

Clearly, it implies that

$$\langle T(x) - T(y), z - y \rangle \geqq (-\gamma)\|z - x\|^2.$$

The partial relaxed monotonicity is more general than the notions of strong monotonicity and cocoercivity. For more details on partial relaxed monotonicity and cocoercivity, we recommend [3,4,9].

2. GENERAL AUXILIARY PROBLEM PRINCIPLE

This section deals with a discussion of the approximation-solvability of the NVIP (1.1), based on a general version of the existing auxiliary problem principle (APP) introduced by Cohen [2] and later studied by others. This general version of auxiliary problem principle (GAPP) is described as follows.

GAPP 2.1. For a given iterate x^k, determine an x^{k+1} such that (for $k \geqq 0$)

$$\left\langle \rho T\left(x^k\right) + h'\left(x^{k+1}\right) - h'\left(x^k\right), x - x^{k+1} \right\rangle + \rho\left[f(x) - f\left(x^{k+1}\right)\right] \geqq \left(-\sigma^k\right), \quad \text{for all } x \in K, \tag{2.1}$$

where $h : R^n \to R$ is m-times continuously Frechet-differentiable ($m \geqq 2$, an integer) on $R^n, \rho > 0$ a parameter and the sequence $\{\sigma^k\}$ satisfies

$$\sigma^k \geqq 0, \qquad \sum_{k=1}^{\infty} \sigma^k < \infty.$$

When $m = 2$, GAPP (2.1) reduces to the following.

GAPP 2.2. For a given iterate x^k, determine an x^{k+1} such that

$$\left\langle \rho T\left(x^k\right) + h'\left(x^{k+1}\right) - h'\left(x^k\right), x - x^{k+1} \right\rangle + \rho\left(f(x) - f\left(x^{k+1}\right)\right) \geqq \left(-\sigma^k\right), \quad \text{for all } x \in K, \tag{2.2}$$

where $h : R^n \to R$ is two-times continuously Frechet-differentiable on $R^n, \rho > 0$ is a parameter and the sequence $\{\sigma^k\}$ satisfies

$$\sigma^k \geqq 0, \qquad \sum_{k=1}^{\infty} \sigma^k < \infty.$$

Next, we recall some auxiliary results crucial to the approximation-solvability of the NVIP (1.1).

LEMMA 2.1. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \to E_2$ be a nonlinear mapping such that h is m-times continuously Frechet-differentiable ($m \geqq 2$ an integer) on E_1. Suppose that there exist an $x^* \in E_1$ and nonnegative numbers α, α_i ($i = 2, 3, \dots, m$) such that*

$$\left\langle h^{(m)}(x) - h^{(m)}(x^*), (x - x^*)^m \right\rangle \geqq \alpha \|x - x^*\|^{m+1}$$

and

$$\left\| h^{(i)}(x^*) \right\| \geqq \alpha_i, \qquad \text{for } i = 2, 3, \dots, m.$$

Then we have

$$\begin{aligned} &h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \\ &\geqq \left\{ \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha_3}{3!} \right) \|x - x^*\| + \cdots + \left(\frac{\alpha_m}{m!} \right) \|x - x^*\|^{m-2} \right] + \left[\frac{\alpha}{(m+1)!} \right] \|x - x^*\|^{m-1} \right\} \\ &\qquad \|x - x^*\|^2. \end{aligned}$$

PROOF. The proof follows from the following identity [1] and hypotheses of the Lemma 2.1:

$$\begin{aligned} h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle &= \int_0^1 \langle [h'(x^* + \theta_1 (x - x^*)) - h'(x^*)], x - x^* \rangle \, d\theta_1 \\ &= \int_0^1 \int_0^1 \left\langle [h''(x^* + \theta_2 \theta_1 (x - x^*)) - h''(x^*)], \theta_1 (x - x^*)^2 \right\rangle d\theta_1 \, d\theta_2 \\ &\quad + \int_0^1 \int_0^1 \left\langle h''(x^*), \theta_1 (x - x^*)^2 \right\rangle d\theta_1 \, d\theta_2 \\ &= \cdots = \int_0^1 \cdots \int_0^1 \left\langle \left\{ h^{(m)} [x^* + \theta_m \theta_{m-1} \dots \theta_1 (x - x^*)] - h^{(m)}(x^*) \right\}, \right. \\ &\qquad \left. \theta_{m-1} \theta_{m-2}^2 \dots \theta_2^{m-2} \theta_1^{m-1} (x - x^*)^m \right\rangle d\theta_m \, d\theta_{m-1} \dots d\theta_2 \, d\theta_1 \\ &\quad + \cdots + \int_0^1 \int_0^1 \left\langle h''(x^*), \theta_1 (x - x^*)^2 \right\rangle d\theta_1 \, d\theta_2. \end{aligned}$$

For $\alpha = 0$, in Lemma 2.1, we have the following.

LEMMA 2.2. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \to E_2$ be a nonlinear mapping such that h is m-times continuously Frechet-differentiable ($m \geqq 2$ an integer) on E_1. Suppose that there exist an $x^* \in E_1$ and nonnegative numbers α_i ($i = 2, 3, \dots, m$) such that*

$$\left\langle h^{(m)}(x) - h^{(m)}(x^*), (x - x^*)^m \right\rangle \geqq 0$$

and

$$\left\| h^{(i)}(x^*) \right\| \geqq \alpha_i.$$

Then we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \geqq \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha_3}{3!} \right) \|x - x^*\| + \cdots + \left(\frac{\alpha_m}{m!} \right) \|x - x^*\|^{m-2} \right] \|x - x^*\|^2.$$

LEMMA 2.3. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \to E_2$ be a nonlinear mapping such that h is m-times continuously Frechet-differentiable ($m \geqq 2$ an integer) on E_1. Suppose that there exist an $x^* \in E_1$ and nonnegative numbers β, β_i ($i = 2, 3, \dots, m$) such that*

$$\left\langle h^{(m)}(x) - h^{(m)}(x^*), (x - x^*)^m \right\rangle \leqq \beta \|x - x^*\|^{m+1}$$

and

$$\left\| h^{(i)}\left(x^*\right)\right\| \leqq \beta_i, \qquad \text{for } i = 2,3,\dots,m.$$

Then we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle$$
$$\leqq \left\{ \left[\frac{\beta_2}{2!} + \left(\frac{\beta_3}{3!}\right)\|x - x^*\| + \dots + \left(\frac{\beta_m}{m!}\right)\|x - x^*\|^{m-2}\right] + \left[\frac{\beta}{(m+1)!}\right]\|x - x^*\|^{m-1}\right\}$$
$$\cdot \|x - x^*\|^2.$$

We are just about ready to present, based on the GAPP (2.1), the approximation-solvability of the NVIP (1.1).

THEOREM 2.1. *Let $T : K \to R^n$ be γ-μ-partially relaxed monotone from a nonempty closed convex subset K of R^n into R^n. Let $f : K \to R$ be proper, convex and lower semicontinuous on K and $h : K \to R$ be m-times continuously Frechet-differentiable ($m \geqq 2$ an integer) on K. Suppose that there exist an $x' \in K$ and nonnegative numbers α and α_i ($i = 2,3,\dots,m$) such that*

$$\left\langle h^{(m)}(x) - h^{(m)}(x'), (x - x')^m\right\rangle \geqq \alpha\|x - x'\|^{m+1}, \tag{2.3}$$

$$\left\| h^{(i)}(x')\right\| \geqq \alpha_i, \tag{2.4}$$

$$\left\langle h^{(m)}(x) - h^{(m)}(x'), (x - x')^m\right\rangle \leqq \beta\|x - x'\|^{m+1}, \tag{2.5}$$

$$\left\| h^{(i)}(x')\right\| \leqq \beta_i, \qquad \text{for } i = 2,3,\dots,m, \tag{2.6}$$

and

$$\sigma^k \geqq 0, \qquad \sum_{k=1}^{\infty} \sigma^k < \infty. \tag{2.7}$$

If in addition, $x^* \in K$ is any fixed solution of the NVIP (1.1) and

$$0 < \rho < \left(\frac{\alpha_2}{2\gamma}\right),$$

then the sequence $\{x^k\}$ converges strongly to x^*.

PROOF. To show the sequences $\{x^k\}$ converges to x^*, a solution of the NVIP (1.1), we need to compute the estimates. Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x\rangle .$$

Then, by Lemma 2.1, we have

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x\rangle$$
$$\geqq \left\{ \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha_3}{3!}\right)\|x^* - x\| + \dots + \left(\frac{\alpha_m}{m!}\right)\|x^* - x\|^{m-2}\right] + \left(\frac{\alpha}{(m+1)!}\right)\|x^* - x\|^{m-1}\right\} \tag{2.8}$$
$$\cdot \|x^* - x\|^2$$

for $x \in K$, where x^* is any fixed solution of the NVIP (1.1). It follows that:

$$\Lambda^*\left(x^{k+1}\right) = h\left(x^*\right) - h\left(x^{k+1}\right) - \left\langle h'\left(x^{k+1}\right), x^* - x^{k+1}\right\rangle .$$

Now we can write

$$
\begin{aligned}
&\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right)\\
&=h\left(x^{k+1}\right)-h\left(x^k\right)-\left\langle h'\left(x^k\right),x^{k+1}-x^k\right\rangle+\left\langle h'\left(x^{k+1}\right)-h'\left(x^k\right),x^*-x^{k+1}\right\rangle\\
&\geqq\left\{\left[\frac{\alpha_2}{2!}+\left(\frac{\alpha_3}{3!}\right)\left\|x^{k+1}-x^k\right\|+\cdots+\left(\frac{\alpha_m}{m!}\right)\left\|x^{k+1}-x^k\right\|^{m-2}\right]\right.\\
&\qquad\left.+\left(\frac{\alpha}{(m+1)!}\right)\left\|x^{k+1}-x^k\right\|^{m-1}\right\}\\
&\qquad\cdot\left\|x^{k+1}-x^k\right\|^2+\left\langle h'\left(x^{k+1}\right)-h'\left(x^k\right),x^*-x^{k+1}\right\rangle\\
&\geqq\left\{\left[\frac{\alpha_2}{2!}+\left(\frac{\alpha_3}{3!}\right)\left\|x^{k+1}-x^k\right\|+\cdots+\left(\frac{\alpha_m}{m!}\right)\left\|x^{k+1}-x^k\right\|^{m-2}\right]\right.\\
&\qquad\left.+\left(\frac{\alpha}{(m+1)!}\right)\left\|x^{k+1}-x^k\right\|^{m-1}\right\}\left\|x^{k+1}-x^k\right\|^2\\
&\qquad+\rho\left\langle T\left(x^k\right),x^{k+1}-x^*\right\rangle+\rho\left(f\left(x^{k+1}\right)-f\left(x^*\right)\right)-\sigma^k
\end{aligned}
\tag{2.9}
$$

for $x=x^*$ in (2.1).

If we replace x by x^{k+1} in (1.1) and combine with (2.9), we obtain

$$
\begin{aligned}
&\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right)\\
&\geqq\left\{\left[\frac{\alpha_2}{2!}+\left(\frac{\alpha_3}{3!}\right)\left\|x^{k+1}-x^k\right\|+\cdots+\left(\frac{\alpha_m}{m!}\right)\left\|x^{k+1}-x^k\right\|^{m-2}\right]+\left(\frac{\alpha}{(m+1)!}\right)\left\|x^{k+1}-x^k\right\|^{m-1}\right\}\\
&\qquad\cdot\left\|x^{k+1}-x^k\right\|^2+\rho\left\langle T\left(x^k\right),x^{k+1}-x^*\right\rangle-\rho\left\langle T\left(x^*\right),x^{k+1}-x^*\right\rangle-\sigma^k\\
&=\left\{\left[\frac{\alpha_2}{2!}+\left(\frac{\alpha_3}{3!}\right)\left\|x^{k+1}-x^k\right\|+\cdots+\left(\frac{\alpha_m}{m!}\right)\left\|x^{k+1}-x^k\right\|^{m-2}\right]+\left(\frac{\alpha}{(m+1)!}\right)\left\|x^{k+1}-x^k\right\|^{m-1}\right\}\\
&\qquad\cdot\left\|x^{k+1}-x^k\right\|^2+\rho\left\langle T\left(x^k\right)-T\left(x^*\right),x^{k+1}-x^*\right\rangle-\sigma^k.
\end{aligned}
$$

Since T is λ-μ-partially relaxed monotone, it implies that

$$
\begin{aligned}
&\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right)\\
&\geqq\left\{\left[\frac{\alpha_2}{2!}+\left(\frac{\alpha_3}{3!}\right)\left\|x^{k+1}-x^k\right\|+\cdots+\left(\frac{\alpha_m}{m!}\right)\left\|x^{k+1}-x^k\right\|^{m-2}\right]+\left(\frac{\alpha}{(m+1)!}\right)\left\|x^{k+1}-x^k\right\|^{m-1}\right\}\\
&\qquad\cdot\left\|x^{k+1}-x^k\right\|^2-\rho\gamma\left\|x^{k+1}-x^k\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2-\sigma^k\\
&\geqq\left[\frac{\alpha_2}{2}\right]\left\|x^{k+1}-x^k\right\|^2-\rho\gamma\left\|x^{k+1}-x^k\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2-\sigma^k\\
&=\left(\frac{1}{2}\right)\left[\alpha_2-2\rho\gamma\right]\left\|x^{k+1}-x^k\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2-\sigma^k\\
&\geqq\left(-\sigma^k\right),\qquad\text{for }\alpha_2-2\rho\gamma>0,
\end{aligned}
$$

that is,

$$\Lambda^*\left(x^k\right)-\Lambda*\left(x^{k+1}\right)\geqq\left(-\sigma^k\right). \tag{2.10}$$

It implies that

$$\Lambda^*\left(x^{k+1}\right)-\Lambda^*\left(x^k\right)\leq\sigma^k. \tag{2.11}$$

If we sum from $k=1,2,\ldots,N$, we arrive at

$$\sum^{N}\left[\Lambda^*\left(x^{k+1}\right)-\Lambda^*\left(x^k\right)\right]\leqq\sum^{\infty}\sigma^k.$$

As a result of this, we can get

$$\Lambda^*\left(x^{N+1}\right) - \Lambda^*\left(x^1\right) \leqq \sum_{k=1}^{\infty} \sigma^k. \tag{2.12}$$

It follows using (2.8) from (2.12) that

$$\left[\frac{\alpha_2}{2}\right] \left\|x^{N+1} - x^*\right\|^2 \leqq \Lambda^*\left(x^1\right) + \sum_{k=1}^{\infty} \sigma^k. \tag{2.13}$$

Under the hypotheses of the theorem, it follows from (2.13) that the sequence $\{x^k\}$ is bounded. Let $x\hat{}$ be a cluster point of the sequence $\{x^k\}$. Then taking the limit of (2.1) results $x\hat{}$ a solution of the NVIP (1.1).

If we replace x^* by $x\hat{}$ in the above proof, the proof holds for $x\hat{}$ and the corresponding sequence $\{\Lambda\hat{}(x^k)\}$ still turns out to be strictly decreasing. As a result, using Lemma 2.3, we have

$$\Lambda\hat{}(x^k) \leq \left\{\left[\frac{\beta_2}{2!} + \left(\frac{\beta_3}{3!}\right)\left\|x^k - x\hat{}\right\| + \cdots + \left(\frac{\beta m}{m!}\right)\left\|x^k - x\hat{}\right\|^{m-2}\right] + \left[\frac{\beta}{(m+1)!}\right]\left\|x^k - x\hat{}\right\|^{m-1}\right\} \cdot \left\|x^k - x\hat{}\right\|^2.$$

This clearly implies that

$$\Lambda\hat{}\left(x^k\right) \to 0, \qquad \text{as } k \to \infty.$$

Similarly, by applying Lemma 2.1, we can have

$$\Lambda\hat{}\left(x^k\right) \geqq \left\{\left[\frac{\alpha_2}{2!} + \left(\frac{\alpha_3}{3!}\right)\left\|x^k - x\hat{}\right\| + \cdots + \left(\frac{\alpha_m}{m!}\right)\left\|x^k - x\hat{}\right\|^{m-2}\right] + \left[\frac{\alpha}{(m+1)!}\right]\left\|x^k - x\hat{}\right\|^{m-1}\right\} \cdot \left\|x^k - x\hat{}\right\|^2.$$

Based on the above inequality arguments, we conclude that the entire sequence $\{x^k\}$ converges to $x\hat{}$. This concludes the proof.

REFERENCES

1. A.K. Argyros and R.U. Verma, Forcing sequences and inexact iterates involving m-Frechet differentiable operators, *Math. Sci. Res. Hot-Line* **4** (9), 35–46, (2000).
2. G. Cohen, Auxiliary problem principle extended to variational inequalities, *J. Optim. Theo. Appl.* **59** (2), 325-333, (1988).
3. A.K. Argyros and R.U. Verma, On general auxiliary problem principle and nonlinear mixed variational inequalities, *Nonlinear Funct. Anal. Appl.* (to appear).
4. R.U. Verma, Approximation-solvability of nonlinear variational inequalities involving partially relaxed monotone (prm) mappings, *Adv. Nonlinear Var. Inequal.* **2** (2), 137–148, (1999).
5. Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York, (1995).
6. R.U. Verma, Nonlinear variational and constrained hemivariational inequalities involving relaxed operators, *ZAMM* **77** (5), 387-391, (1997).
7. R.U. Verma, A class of projection-contraction methods applied to monotone variational inequalities, *Appl. Math. Lett.* **13** (8), 55–62, (2000).
8. R.U. Verma, A new class of iterative algorithms for approximation-solvability of nonlinear variational inequalities, *Computers Math. Applic.* **41** (3/4), 505-512, (2001).
9. D.L. Zhu and P. Marcotte, Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities, *SIAM J. Optim.* **6** (3), 714–726, (1996).
10. R. Glowinski, J.-L. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, (1981).