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A general iterative method for nonexpansive mappings in Hilbert spaces

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Abstract

Let H be a real Hilbert space. Consider on H a nonexpansive mapping T with a fixed point, a contraction f with coefficient $0 < \alpha < 1$, and a strongly positive linear bounded operator A with coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \bar{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated by the iterative method $x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n)$ converges strongly to a fixed point $\tilde{x} \in \mathrm{Fix}(T)$ which solves the variational inequality $\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leqslant 0$ for $x \in \mathrm{Fix}(T)$.

Keywords: Nonexpansive mapping; Iterative method; Variational inequality; Fixed point; Projection; Viscosity approximation

1. Introduction

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [1,4,5,7,8] and the references therein. A typical

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problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geqslant \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{2}$$

Recall that $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is the set $\operatorname{Fix}(T) := \{x \in H\colon Tx = x\}$. We assume that $\operatorname{Fix}(T) \neq \emptyset$ and $C = \operatorname{Fix}(T)$. It is well known that $\operatorname{Fix}(T)$ is closed convex (cf. [2]). In [5] (see also [7]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geqslant 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem (1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions that will be made precise in Section 3.

On the other hand, Moudafi [3] introduced the viscosity approximation method for nonexpansive mappings (see [6] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geqslant 0, \tag{4}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [3,6] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* in C of the variational inequality

$$\big\langle (I - f)x^*, x - x^*\big\rangle \geqslant 0, \quad x \in C. \tag{5}$$

In this paper we will combine the iterative method (3) with the viscosity approximation method (4) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geqslant 0. \tag{6}$$

We will prove in Section 3 that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of the variational inequality

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geqslant 0, \quad x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

2. Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 2.1. [4] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [2] *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

The following lemma is not hard to prove.

Lemma 2.3. *Let H be a Hilbert space, K a closed convex subset of H, $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\alpha$,*

$$\big\langle x-y, (A-\gamma f)x - (A-\gamma f)y\big\rangle \geqslant (\bar{\gamma}-\gamma\alpha)\|x-y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Recall the metric (nearest point) projection P_K from a real Hilbert space H to a closed convex subset K of H is defined as follows: given $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf\big\{\|x-y\|:\ y \in K\big\}.$$

P_K is characterized as follows.

Lemma 2.4. *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geqslant 0, \quad \forall z \in K.$$

Lemma 2.5. *Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leqslant \|A\|^{-1}$. Then $\|I - \rho A\| \leqslant 1 - \rho\bar{\gamma}$.*

Proof. Recall that a standard result in functional analysis is that if V is linear bounded self-adjoint operator on H, then

$$\|V\| = \sup\big\{\big|\langle Vx, x\rangle\big|:\ x \in H,\ \|x\| = 1\big\}.$$

Now for $x \in H$ with $\|x\| = 1$, we see that

$$\big\langle (I - \rho A)x, x\big\rangle = 1 - \rho\langle Ax, x\rangle \geqslant 1 - \rho\|A\| \geqslant 0$$

(i.e., $I - \rho A$ is positive). It follows that

$$\begin{aligned}\|I - \rho A\| &= \sup\big\{\big\langle (I - \rho A)x, x\big\rangle\colon\ x \in H,\ \|x\| = 1\big\}\\ &= \sup\big\{1 - \rho\langle Ax, x\rangle\colon\ x \in H,\ \|x\| = 1\big\}\\ &\leqslant 1 - \rho\bar{\gamma} \quad \text{by (2).} \qquad \Box\end{aligned}$$

Notation. We use $\to$ for strong convergence and $\rightharpoonup$ for weak convergence.

3. A general iterative method

Let H be a real Hilbert space, let A be a bounded linear operator on H, and let T be a nonexpansive mapping on H (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$ for $x, y \in H$). Assume the set $\mathrm{Fix}(T)$ of fixed points of H is nonempty; that is, $\mathrm{Fix}(T) = \{x \in H\colon\ Tx = x\} \neq \emptyset$. Since $\mathrm{Fix}(T)$ is closed convex, the nearest point projection from H onto $\mathrm{Fix}(T)$ is well defined.

Throughout the rest of this paper, we always assume that A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geqslant \bar{\gamma}\|x\|^2, \quad x \in H. \tag{8}$$

(Note: $\bar{\gamma} > 0$ is throughout reserved to be the constant such that (8) holds.)

Recall also that a contraction on H is a self-mapping f of H such that

$$\big\|f(x) - f(y)\big\| \leqslant \alpha\|x - y\|, \quad x, y \in H,$$

where $\alpha \in [0, 1)$ is a constant.

Denote by Π the collection of all contractions on H; namely,

$$\Pi = \{f\colon\ f \text{ a contraction on } H\}.$$

Now given $f \in \Pi$ with contraction coefficient $0 < \alpha < 1$, $t \in (0, 1)$ such that $t < \|A\|^{-1}$ and $0 < \gamma < \bar{\gamma}/\alpha$. Consider a mapping S_t on H defined by

$$S_t x = t\gamma f(x) + (I - tA)Tx, \quad x \in H. \tag{9}$$

It is easy to see that S_t is a contraction. Indeed, by Lemma 2.5, we have:

$$\begin{aligned}\|S_t x - S_t y\| &\leqslant t\gamma\big\|f(x) - f(y)\big\| + \big\|(I - tA)(Tx - Ty)\big\|\\ &\leqslant \big(1 - t(\bar{\gamma} - \gamma\alpha)\big)\|x - y\|.\end{aligned}$$

Hence S_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t. \tag{10}$$

Note that x_t indeed depends on f as well, but we will suppress this dependence of x_t on f for simplicity of notation throughout the rest of this paper. We will also always use γ to mean a number in $(0, \bar{\gamma}/\alpha)$.

The next proposition summarizes the basic properties of $\{x_t\}$.

Proposition 3.1. *Let x_t be defined via* (10).

(i) $\{x_t\}$ *is bounded for* $t \in (0, \|A\|^{-1})$.
(ii) $\lim_{t\to 0} \|x_t - Tx_t\| = 0$.
(iii) x_t *defines a continuous curve from* $(0, \|A\|^{-1})$ *into* H.

Proof. First observe that for $t \in (0, \|A\|^{-1})$, we have $\|I - tA\| \leqslant 1 - t\bar{\gamma}$ by Lemma 2.5.

To show (i) pick a $p \in \mathrm{Fix}(T)$. We then have

$$\begin{aligned}
\|x_t - p\| &= \big\|(I - tA)(Tx_t - p) + t\big(\gamma f(x_t) - Ap\big)\big\| \\
&\leqslant (1 - \bar{\gamma}t)\|x_t - p\| + t\big\|\gamma f(x_t) - Ap\big\| \\
&= (1 - \bar{\gamma}t)\|x_t - p\| + t\big\|\gamma\big(f(x_t) - f(p)\big) + \big(\gamma f(p) - Ap\big)\big\| \\
&\leqslant (1 - \bar{\gamma}t)\|x_t - p\| + t\big[\gamma\alpha\|x_t - p\| + \big\|\gamma f(p) - Ap\big\|\big] \\
&= \big(1 - t(\bar{\gamma} - \gamma\alpha)\big)\|x_t - p\| + t\big\|\gamma f(p) - Ap\big\|.
\end{aligned}$$

It follows that

$$\|x_t - p\| \leqslant \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}.$$

Hence $\{x_t\}$ is bounded.

(ii) We have $\|x_t - Tx_t\| = t\|\gamma f(x_t) - ATx_t\| \to 0$ since the boundedness of $\{x_t\}$ implies that of $\{f(x_t)\}$ and of $\{ATx_t\}$.

To prove (iii) take $t,\ t_0 \in (0, \|A\|^{-1})$ and calculate

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \big\|(t - t_0)\gamma f(x_t) + t_0\gamma\big(f(x_t) - f(x_{t_0})\big) - (t - t_0)ATx_t \\
&\qquad + (I - t_0A)(Tx_t - Tx_{t_0})\big\| \\
&\leqslant \big(\gamma\big\|f(x_t)\big\| + \|ATx_t\|\big)|t - t_0| + \big(1 - t_0(\bar{\gamma} - \gamma\alpha)\big)\|x_t - x_{t_0}\|.
\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \leqslant \frac{\gamma\|f(x_{t_0})\| + \|ATx_t\|}{t_0(\bar{\gamma} - \gamma\alpha)}|t - t_0|.$$

This shows that x_t is locally Lipschitzian and hence continuous. □

Our first main result below shows that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T which solves some variational inequality.

Theorem 3.2. *We have that* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a fixed point* $\tilde{x}$ *of* T *which solves the variational inequality*:

$$\big\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\big\rangle \leqslant 0, \quad z \in \mathrm{Fix}(T). \tag{11}$$

Equivalently, we have $P_{\mathrm{Fix}(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$.

Proof. We first show the uniqueness of a solution of the variational inequality (11), which is indeed a consequence of the strong monotonicity of $A - \gamma f$. Suppose $\tilde{x} \in \mathrm{Fix}(T)$ and $\hat{x} \in \mathrm{Fix}(T)$ both are solutions to (11); then

$$\big\langle (A-\gamma f)\tilde{x}, \tilde{x}-\hat{x}\big\rangle \leqslant 0 \tag{12}$$

and

$$\big\langle (A-\gamma f)\hat{x}, \hat{x}-\tilde{x}\big\rangle \leqslant 0. \tag{13}$$

Adding up (12) and (13) gets

$$\big\langle (A-\gamma f)\tilde{x} - (A-\gamma f)\hat{x}, \tilde{x}-\hat{x}\big\rangle \leqslant 0.$$

The strong monotonicity of $A - \gamma f$ (Lemma 2.3) implies that $\tilde{x} = \hat{x}$ and the uniqueness is proved. Below we use $\tilde{x} \in \mathrm{Fix}(T)$ to denote the unique solution of (11).

To prove that $x_t \to \tilde{x}$ ($t \to 0$), we write, for a given $z \in \mathrm{Fix}(T)$,

$$x_t - z = t\big(\gamma f(x_t) - Az\big) + (I - tA)(Tx_t - z)$$

to derive that

$$\begin{aligned}\|x_t - z\|^2 &= t\big\langle \gamma f(x_t) - Az, x_t - z\big\rangle + \big\langle (I-tA)(Tx_t - z), x_t - z\big\rangle \\ &\leqslant (1 - t\bar{\gamma})\|x_t - z\|^2 + t\big\langle \gamma f(x_t) - Az, x_t - z\big\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_t - z\|^2 &\leqslant \frac{1}{\bar{\gamma}}\big\langle \gamma f(x_t) - Az, x_t - z\big\rangle \\ &= \frac{1}{\bar{\gamma}}\big\{\gamma\big\langle f(x_t) - f(z), x_t - z\big\rangle + \big\langle \gamma f(z) - Az, x_t - z\big\rangle\big\} \\ &\leqslant \frac{1}{\bar{\gamma}}\big\{\gamma\alpha\|x_t - z\|^2 + \big\langle \gamma f(z) - Az, x_t - z\big\rangle\big\}.\end{aligned}$$

Therefore,

$$\|x_t - z\|^2 \leqslant \frac{1}{\bar{\gamma} - \gamma\alpha}\big\langle \gamma f(z) - Az, x_t - z\big\rangle. \tag{14}$$

Since $\{x_t\}$ is bounded as $t \to 0$, we see that if $\{t_n\}$ is a sequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$, then by (14), we see $x_{t_n} \to x^*$. Moreover, by Proposition (3.1)(ii), we have $x^* \in \mathrm{Fix}(T)$. We next prove that x^* solves the variational inequality (11). Since

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \tag{15}$$

we derive, that

$$(A - \gamma f)x_t = -\frac{1}{t}(I - tA)(I - T)x_t.$$

It follows that, for $z \in \mathrm{Fix}(T)$,

$$\begin{aligned}\big\langle (A-\gamma f)x_t, x_t - z\big\rangle &= -\frac{1}{t}\big\langle (I-tA)(I-T)x_t, x_t - z\big\rangle \\ &= -\frac{1}{t}\big\langle (I-T)x_t - (I-T)z, x_t - z\big\rangle + \big\langle A(I-T)x_t, x_t - z\big\rangle \\ &\leqslant \big\langle A(I-T)x_t, x_t - z\big\rangle \end{aligned} \tag{16}$$

since $I - T$ is monotone (i.e., $\langle x - y, (I - T)x - (I - T)y\rangle \geqslant 0$ for $x, y \in H$. This is due to the nonexpansivity of T). Now replacing t in (16) with t_n and letting $n \to \infty$, we, noticing that $(I - T)x_{t_n} \to (I - T)x^* = 0$ for $x^* \in \mathrm{Fix}(T)$, obtain

$$\big\langle (A-\gamma f)x^*, x^* - z\big\rangle \leqslant 0.$$

This is, $x^* \in \mathrm{Fix}(T)$ is a solution of (11); hence $x^* = \tilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$.

The variational inequality (11) can be rewritten as

$$\big\langle \big[(I - A + \gamma f)\tilde{x}\big] - \tilde{x}, \tilde{x} - z\big\rangle \geqslant 0, \quad z \in \mathrm{Fix}(T).$$

This, by Lemma 2.4, is equivalent to the fixed point equation

$$P_{\mathrm{Fix}(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}. \qquad \square$$

Taking $A = I$ and $\gamma = 1$ in Theorem 3.2, we get

Corollary 3.3. [6] *Let $z_t \in H$ be the unique fixed point of the contraction $z \mapsto (1-t)Tz + tf(z)$. Then $\{z_t\}$ converges strongly as $t \to 0$ to the unique solution $\tilde{z} \in \mathrm{Fix}(T)$ of the variational inequality*

$$\big\langle (I - f)\tilde{z}, z - \tilde{z}\big\rangle \geqslant 0, \quad z \in \mathrm{Fix}(T).$$

Next we study a general iterative method as follows. The initial guess x_0 is selected in H arbitrarily, and the $(n+1)$th iterate x_{n+1} is recursively defined by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geqslant 0, \tag{17}$$

where $\{\alpha_n\}$ is a sequence in (0, 1) satisfying the following conditions:

(C1) $\alpha_n \to 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

Below is the second main result of this paper.

Theorem 3.4. *Let $\{x_n\}$ be generated by algorithm* (17) *with the sequence $\{\alpha_n\}$ of parameters satisfying conditions* (C1)–(C3). *Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Theorem* 3.2.

Proof. Since $\alpha_n \to 0$ by condition (C1), we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$ for all n.

We now observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in \mathrm{Fix}(T)$ to obtain

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|(I-\alpha_n A)(Tx_n-p)+\alpha_n\big(\gamma f(x_n)-Ap\big)\big\| \\ &\leqslant \|I-\alpha_n A\|\|Tx_n-p\|+\alpha_n\big\|\gamma f(x_n)-Ap\big\| \\ &\leqslant (1-\alpha_n\bar{\gamma})\|x_n-p\|+\alpha_n\big[\gamma\big\|f(x_n)-f(p)\big\|+\big\|\gamma f(p)-Ap\big\|\big] \\ &\leqslant \big(1-(\bar{\gamma}-\gamma\alpha)\alpha_n\big)\|x_n-p\|+\alpha_n\big\|\gamma f(p)-Ap\big\| \\ &= \big(1-(\bar{\gamma}-\gamma\alpha)\alpha_n\big)\|x_n-p\|+(\bar{\gamma}-\gamma\alpha)\alpha_n\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}.\end{aligned}$$

It follows from induction that

$$\|x_n-p\|\leqslant \max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\},\quad n\geqslant 0. \tag{18}$$

As a result, noticing $x_{n+1}-Tx_n=\alpha_n(\gamma f(x_n)-ATx_n)$ and $\alpha_n\to 0$, we obtain

$$x_{n+1}-Tx_n\to 0. \tag{19}$$

But the key is to prove that

$$x_{n+1}-x_n\to 0. \tag{20}$$

To see this, we calculate

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|(I-\alpha_n A)(Tx_n-Tx_{n-1})-(\alpha_n-\alpha_{n-1})ATx_{n-1} \\ &\qquad +\gamma\big[\alpha_n\big(f(x_n)-f(x_{n-1})\big)+(\alpha_n-\alpha_{n-1})f(x_{n-1})\big]\big\| \\ &\leqslant (1-\alpha_n\bar{\gamma})\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|ATx_{n-1}\| \\ &\qquad +\gamma\big[\alpha_n\alpha\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\big\|f(x_{n-1})\big\|\big] \\ &\leqslant \big(1-(\bar{\gamma}-\gamma\alpha)\alpha_n\big)\|x_n-x_{n-1}\|+M|\alpha_n-\alpha_{n-1}|,\end{aligned} \tag{21}$$

where $M:=\sup\{\max\{\|ATx_n\|,\|f(x_n)\|\}\colon n\geqslant 0\}<\infty$.

An application of Lemma 2.1 to (21) implies (20) which, combined with (19), in turns, implies

$$x_n-Tx_n\to 0. \tag{22}$$

Next we show that

$$\limsup_{n\to\infty}\big\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle\leqslant 0, \tag{23}$$

where $\tilde{x}$ is obtained in Theorem 3.2.

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\big\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle=\lim_{k\to\infty}\big\langle x_{n_k}-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle.$$

We may also assume that $x_{n_k}\rightharpoonup z$. Note that $z\in\operatorname{Fix}(T)$ in virtue of Lemma 2.2 and (22). It follows from the variational inequality (11) that

$$\limsup_{n\to\infty}\big\langle x_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle=\big\langle z-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle\leqslant 0.$$

So (23) holds, thanks to (22).

Finally, we prove $x_n\to\tilde{x}$. To this end, we calculate

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \big\|(I-\alpha_n A)(Tx_n-\tilde{x})+\alpha_n\big(\gamma f(x_n)-A\tilde{x}\big)\big\|^2 \\
&= \big\|(I-\alpha_n A)(Tx_n-\tilde{x})\big\|^2+\alpha_n^2\big\|\gamma f(x_n)-A\tilde{x}\big\|^2 \\
&\quad +2\alpha_n\big\langle(I-\alpha_n A)(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\big\rangle \\
&\leqslant (1-\alpha_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2+\alpha_n^2\big\|\gamma f(x_n)-A\tilde{x}\big\|^2 \\
&\quad +2\alpha_n\big\langle Tx_n-\tilde{x},\gamma f(x_n)-A\tilde{x}\big\rangle-2\alpha_n^2\big\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\big\rangle \\
&\leqslant (1-\alpha_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2+\alpha_n^2\big\|\gamma f(x_n)-A\tilde{x}\big\|^2 \\
&\quad +2\alpha_n\gamma\big\langle Tx_n-\tilde{x},f(x_n)-f(\tilde{x})\big\rangle+2\alpha_n\big\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle \\
&\quad -2\alpha_n^2\big\langle A(Tx_n-\tilde{x}),\gamma f(x_n)-A\tilde{x}\big\rangle \\
&\leqslant \big[(1-\alpha_n\bar{\gamma})^2+2\alpha_n\gamma\alpha\big]\|x_n-\tilde{x}\|^2+\alpha_n\big[2\big\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle \\
&\quad +\alpha_n\big(\big\|\gamma f(x_n)-A\tilde{x}\big\|^2+2\alpha_n\big\|A(Tx_n-\tilde{x})\big\|\cdot\big\|\gamma f(x_n)-A\tilde{x}\big\|\big)\big] \\
&= \big(1-2(\bar{\gamma}-\gamma\alpha)\alpha_n\big)\|x_n-\tilde{x}\|^2+\alpha_n\big\{2\big\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle \\
&\quad +\alpha_n\big(\big\|\gamma f(x_n)-A\tilde{x}\big\|^2+2\alpha_n\big\|A(Tx_n-\tilde{x})\big\|\cdot\big\|\gamma f(x_n)-A\tilde{x}\big\| \\
&\quad +\alpha_n\bar{\gamma}^2\|x_n-\tilde{x}\|^2\big)\big\}.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we can take a constant $L>0$ such that

$$L\geqslant\big\|\gamma f(x_n)-A\tilde{x}\big\|^2+2\alpha_n\big\|A(Tx_n-\tilde{x})\big\|\cdot\big\|\gamma f(x_n)-A\tilde{x}\big\|+\alpha_n\bar{\gamma}^2\|x_n-\tilde{x}\|^2$$

for all $n\geqslant 0$. It then follows that

$$\|x_{n+1}-\tilde{x}\|^2\leqslant\big(1-2(\bar{\gamma}-\gamma\alpha)\alpha_n\big)\|x_n-\tilde{x}\|^2+\alpha_n\beta_n, \tag{24}$$

where

$$\beta_n=2\big\langle Tx_n-\tilde{x},\gamma f(\tilde{x})-A\tilde{x}\big\rangle+L\alpha_n.$$

By (23), we get $\limsup_{n\to\infty}\beta_n\leqslant 0$. Now applying Lemma 2.1 to (24) concludes that $x_n\to\tilde{x}$. □

Corollary 3.5. [6] *Let $\{x_n\}$ be generated by the following algorithm*:

$$x_{n+1}=(1-\alpha_n)Tx_n+\alpha_n f(x_n),\quad n\geqslant 0.$$

Assume the sequence $\{\alpha_n\}$ satisfies conditions (C1)–(C3). *Then $\{x_n\}$ converges strongly to $\tilde{z}$ obtained in Corollary* 3.3.

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