Special functions as solutions to discrete Painlevé equations

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Received 17 October 2002

Abstract

We present results on special solutions of discrete Painlevé equations. These solutions exist only when one constraint among the parameters of the equation is satisfied and are obtained through the solutions of linear second-order (discrete) equations. These linear equations define the discrete analogues of special functions. © 2003 Elsevier B.V. All rights reserved.

Keywords: Special functions; Painlevé equations; Second-order discrete equations

1. Introduction

Painlevé equations \cite{7} have a double connection to the special functions of mathematical physics. First, their general solutions, the Painlevé transcendents, are the extensions to the nonlinear case of the functions of the Gauss hypergeometric family. Second, the Painlevé equations have special solutions (under appropriate constraints on their parameters) which are expressed in terms of these very same special functions. The parallel existing between the Painlevé transcendents and the functions of the Gauss hypergeometric family is best illustrated through their respective degeneration schemes. We have in fact for the Painlevé, the following cascade, obtained through singularity coalescence

\[
P_{VI} \rightarrow P_{V} \rightarrow \{P_{IV}, P_{III}\} \rightarrow P_{II} \rightarrow P_{I}.
\]

Similarly for the Gauss hypergeometric we find \cite{1}

\textit{hypergeometric} \rightarrow \textit{confluent hypergeometric} \rightarrow \{\textit{Hermite, Bessel}\} \rightarrow \textit{Airy}.

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doi:10.1016/S0377-0427(03)00635-6
From the above diagram it is easily inferred that, for instance, $P_V$ has solutions in terms of the confluent hypergeometric function when its parameters satisfy one appropriate condition.

The solutions of the Painlevé equations in terms of special functions are of double importance. First, they give a indication as to the qualitative behaviour of the solutions (something that would have necessitated a much more sophisticated approach). Second, they constitute the perfect testing ground for any numerical scheme which aims at simulating the behaviour of the Painlevé transcendents. Thus it is essential to have as complete a description of these solutions as possible.

Discrete Painlevé equations is the name given to nonautonomous integrable mappings, the continuous limits of which are (continuous) Painlevé equations [13]. Contrary to the continuous case where there exist just six canonical Painlevé equations (although the real situation is a little more complicated than this) the number of different discrete Painlevé equations is a priori infinite. Since in this paper we shall deal with discrete Painlevé equations, it is necessary to summarize what is known about them before examining their solutions.

• Discrete Painlevé equations appear in two varieties (and in fact in three). There exist difference equations where the independent variable appears in an additive way and equations where the independent variable enters multiplicatively. (The third variety corresponds to cases where the variables and the parameters appear through the arguments of elliptic functions [10]).

• Some difference Painlevé equations can be obtained from the contiguity relations of continuous Painlevé equations [3]. However, since discrete Painlevé equations can have more parameters than the continuous Painlevé equations (in fact up to seven [5]), there exist difference Painlevé equations which are not obtained from the contiguity of the second-order continuous Painlevé equations.

• A classification of discrete Painlevé equations can be obtained in terms of affine Weyl groups [14]. Starting from the exceptional $E_8^{(1)}$ group and coming down through its degeneration pattern, one can find discrete Painlevé equations associated to every one of these groups. (As a matter of fact, since one can define a discrete Painlevé equation for every nonclosed pattern periodically repeated in a given space there exist, in principle, infinitely many discrete Painlevé equations.) While a geometric description can be given for a discrete Painlevé equation only when it is considered in its full freedom, it is also possible, by artificially amputating some of these degrees of freedom, to obtain reduced version of these discrete Painlevé equations. Naturally the two discrete Painlevé equations with different number of parameters will not have the same continuous limit.

In this paper we shall consider the (discrete) special-function type solutions of discrete Painlevé equations. We shall start by introducing a general procedure which allows us to obtain them systematically and then apply this method to (a selection of) discrete Painlevé equations.

2. Constructing the solution of discrete Painlevé equations through linearisation

Before proceeding to the examination of specific cases, let us remind the general method for obtaining the special function solutions for the continuous Painlevé equations and then proceed to its analogue for the discrete ones. The general form of a continuous Painlevé equation is

$$w'' = f(w', w, z), \quad (2.1)$$
where \( f \) is polynomial in \( w' \), rational in \( w \) and analytic in \( z \). In order to find a solution of (2.1) in terms of special functions we assume that \( w \) is a solution of a Riccati

\[
w' = Aw^2 + Bw + C,
\]

(2.2)

where \( A, B, C \) are functions of \( z \) to be determined [6]. Substituting (2.2) into (2.1) yields an overdetermined system which allows the determination of \( A, B, C \) and fixes the parameters of (2.1). Eq. (2.2) is subsequently linearised through the transformation

\[
w = -\frac{u'}{Au}.
\]

(2.3)

The discrete equations we are going to consider can be given in the QRT [11] form. The symmetric QRT form is

\[
x_{n+1}x_nf_3(x_n) - x_{n+1}f_4(x_n) - x_nf_2(x_n) + f_1(x_n) = 0,
\]

(2.4)

where \( f_i \) are specific quartic polynomials of \( x_n \) with coefficients which may depend on the independent variable \( n \). Transposing the method presented in the previous paragraph to the discrete case we seek a solution of system (2.4) in the form [15]

\[
x_{n+1} = \frac{x_n + \beta}{\gamma x_n + \delta},
\]

(2.5)

i.e. in the form of a homographic mapping, since the latter is the discrete analogue of the Riccati equation. The coefficients \( \alpha, \beta, \gamma, \delta \) appearing in (2.5) depend in general on the independent variable \( n \). The existence of a solution in the form of (2.5) is possible only when some special relation exists between the parameters of the discrete Painlevé equation. We shall refer to this relation as “linearisability constraint”.

In some cases the discrete equations are given in the QRT asymmetric form

\[
x_{n+1}x_nf_3(y_n) - x_{n+1}f_4(y_n) - x_nf_2(y_n) + f_1(y_n) = 0,
\]

(2.6a)

\[
y_ny_{n-1}g_3(x_n) - y_{n-1}g_4(x_n) - y_ng_2(x_n) + g_1(x_n) = 0,
\]

(2.6b)

where again \( f_i, g_i \) are specific quartic polynomials of \( y_n \) and \( x_n \), respectively, with coefficients which may depend on the independent variable \( n \). Transposing the method above to the asymmetric case we seek a solution of system (2.6) in the form [16]

\[
x_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta},
\]

(2.7a)

\[
y_n = \frac{\varepsilon x_n + \zeta}{\eta x_n + \theta}.
\]

(2.7b)

Again the coefficients \( \alpha, \beta, \ldots, \theta \) appearing in (2.7) depend in general on the independent variable \( n \). The existence of a solution in the form of (2.7) is possible only if some linearisability constraint is satisfied. Eliminating \( y_n \) between (2.7a) and (2.7b) one can obtain a Riccati relation between \( x_{n+1} \)
and $x_n$ of the form

$$x_{n+1} = \frac{a_n x_n + b_n}{c_n x_n + d_n}.$$  \hspace{1cm} (2.8)

The linearisation we referred to above is the one obtained through a Cole–Hopf transformation $x_n = u_n/v_n$. Substituting into Eq. (2.8) we obtain a linear equation for $v_n$

$$c_n c_{n+2} v_{n+2} - (d_{n+1} c_n + a_n c_{n+1}) v_{n+1} + (a_n d_n - b_n c_n) v_n = 0.$$ \hspace{1cm} (2.9)

Through the use of an appropriate gauge in $v_n$, this equation can be arranged so as to redistribute the degrees of the independent variable among the three coefficients.

3. A selection of solutions of discrete Painlevé equations

Before proceeding to the presentation of solutions of discrete Painlevé equations let us give one explicit example of such a solution for a continuous one: the PV equation. We start from

$$w'' = w^2 \left( \frac{1}{2w} + \frac{1}{w-1} \right) - \frac{w'}{t} + \frac{(w-1)^2}{2t^2} \left( \alpha^2 w - \frac{\beta^2}{w} \right) + \frac{\gamma w}{t} - \frac{w(w+1)}{2(w-1)}.$$ \hspace{1cm} (3.1)

The linearisability condition, is just [15]

$$\varepsilon_1 \alpha + \varepsilon_2 \beta + \varepsilon_3 \gamma = 1,$$ \hspace{1cm} (3.2)

where $\varepsilon_i^2 = 1$, and the associated Riccati is

$$w' = \varepsilon_1 \frac{w(w-1)}{t} + \varepsilon_3 w + \varepsilon_2 \beta \frac{w-1}{t}.$$ \hspace{1cm} (3.3)

Eq. (3.3) is linearised through a Cole–Hopf transformation $w = -\varepsilon_1 t u'/(\varepsilon u)$ to a confluent hypergeometric equation

$$u'' - (\varepsilon_3 - \frac{1}{t} - \varepsilon_2 \beta + \varepsilon_1 \alpha) u' - \frac{\varepsilon_1 \varepsilon_2 \beta}{t^2} u = 0,$$ \hspace{1cm} (3.4)

the solution of which can be given in terms of a Kummer (or equivalently, a Whittaker) function.

We now turn to discrete equations and start with the $q$-discrete PII [12]:

$$(x_n x_{n+1} - 1)(x_{n-1} x_n - 1) = \frac{a q_n^2 x_n}{x_n - q_n},$$ \hspace{1cm} (3.5)

where $q_n = \lambda^n$. We seek a special solution in the form (2.5). It turns out that if we take $\alpha = q_{n+1}$, $\beta = 1$, $\gamma = -1$, $\delta = 0$ such a solution does exist provided $a = \lambda$. In this case the solution is given by the discrete Riccati

$$x_{n+1} = q_{n+1} + \frac{1}{x_n},$$ \hspace{1cm} (3.6)
which can be linearised through the Cole–Hopf transformation $x_n = u_n/v_n$. We thus obtain the equation

$$v_{n+1} - q_n v_n - v_{n-1} = 0$$  \hfill (3.7)

which is a $q$-discrete analogue of the Airy equation.

Next, we turn to a $q$-discrete equation which was shown in [8] to be a discrete analogue of PIII

$$x_n x_{n+1} = \frac{a q_n y_n + b q_n^2}{y_n(y_n - 1)}, \quad (3.8a)$$

$$y_n y_{n-1} = \frac{c q_n x_n + b q_n q_{n-1}^2}{x_n(x_n - 1)}. \quad (3.8b)$$

In this case the linearisation proceeds in two steps: First, one finds a homographic transformation between $x_n$ and $y_n$ and then obtain a discrete Riccati of the form (2.5) for one of the variables, say $x_n$. We find

$$x_{n+1} = \frac{c q_{n+1} + 1}{y_n}$$  \hfill (3.9a)

and

$$y_n = \frac{a q_n + 1}{x_n} \quad (3.9b)$$

the linearisation condition being $b = \lambda a c$. Eliminating $y_n$ between the two homographic relations results to

$$x_{n+1} = \frac{(c q_{n+1} + 1)x_n + a q_n}{a q_n + x_n}. \quad (3.10)$$

The standard Cole–Hopf transformation $x_n = u_n/v_n$ results to the linear equation

$$v_{n+1} - ((c + a)q_n + 1)v_n + a c q_n v_{n-1} = 0$$  \hfill (3.11)

which is expected to be a $q$-discrete analogue of the Bessel equation (up to some simple gauge). By introducing a gauge $v_n = \phi_n w_n$ with $\phi_{n+1} = q_n \phi_n$ we can rewrite Eq. (3.11) as

$$w_{n+1} - ((c + a) + q_n^{-1})w_n + c w_{n-1} = 0. \quad (3.12)$$

The next equation we shall examine is a $q$-discrete form of Painlevé IV derived in [12]

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = q_n^2 a (x_n^2 + 1) + b x_n.$$  \hfill (3.13)

In this case the linearisability condition reads: $b = a^2 \lambda + \frac{1}{\lambda}$. The discrete Riccati in this case is

$$x_{n+1} = \frac{1 + a q_{n+1}}{x_n - q_n} \quad (3.14)$$

and the linearisation through the standard Cole–Hopf leads to the linear equation

$$v_{n+1} + q_n v_n - (1 + a q_n) v_{n-1} = 0 \quad (3.15)$$

which is a $q$-analogue of the Weber–Hermite equation.
As an example of a discrete analogue of discrete PV we choose an equation which was obtained from the contiguity relations of PV1 [2,9]. It is again given here in “asymmetric”, two component difference equation:

\[
\begin{align*}
\frac{2z_n}{y_n - x_n} + z_n + z_{n+1} &= \frac{z_n + a}{y_n} + \frac{z_n + b}{y_n + s} + \frac{z_n + c}{y_n + 1/s}, \\
\frac{2z_n}{x_n - y_n} + z_n + z_{n-1} &= \frac{z_n - a}{x_n} + \frac{z_n - b}{x_n + s} + \frac{z_n - c}{x_n + 1/s},
\end{align*}
\] (3.16a)

where \(z_n = \alpha(n - n_0)\). The linearisability condition is given by \(a + c = \alpha\) and it results to the system

\[
\begin{align*}
y_n &= \frac{(b + z_n)x_n + 2sz_n}{b - z_n}, \\
x_{n+1} &= \frac{(a - z_{n+1})y_n}{sy_n(z_n + z_{n+1}) + a + z_n}.
\end{align*}
\] (3.17a)

Eliminating \(y_n\) between the two equations we obtain the discrete Riccati for \(x_n\) in the form

\[
x_{n+1} = \frac{x_n(a - z_{n+1})(b + z_n) + 2sz_n(a - z_{n+1})}{sx_n(z_n + z_{n+1})(b + z_n) + (a + z_n)(b - z_n) + 2s^2z_n(z_n + z_{n+1})}. \\
\] (3.18)

For its linearisation we introduce \(x_n = u_n/v_n\) and find

\[
v_{n+1}(z_n + z_{n-1})(a + z_n)(b - z_n) - 2v_nz_n(s^2(4z_n^2 - x^2) + 2ab - 2z^2 + \alpha(a - b)) + v_{n-1}(z_n + z_{n+1})(a - z_n)(b + z_n) = 0
\] (3.19)

with the appropriate gauge transformation. The \(q\)-discrete analogue of PV1 was introduced in [4]. Its form is

\[
\frac{(x_nx_{n+1} - q_{n+1})(x_nx_{n-1} - q_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} = \frac{(x_n - aq_n)(x_n - q_{n+1})(x_n - bq_n)(x_n - q_{n-1})}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)},
\] (3.20)

where \(a, b, c, d\) and \(q_n = q_0\lambda^n\). Its linearisable solutions were also presented in [4]. It turns out that when the condition

\[
ab = cd\lambda
\] (3.21)

holds, the mapping

\[
\frac{x_nx_{n+1} - q_{n+1}x_{n+1}}{x_nx_{n+1} - 1} = \frac{(x_n - aq_n)(x_n - bq_n)}{(x_n - c)(x_n - d)}
\] (3.22)

becomes homographic. Its linearisation can be obtained through a Cole–Hopf transformation \(x_n = u_n/v_n\), resulting to the linear equation

\[
\begin{align*}
u_{n+1}(aq_n - d)(aq_n - c)((a + b)q_{n-1} - c - d) \\
+ au_n((a + b)q_{n-1}((cd\lambda - 1)q_n^2 + \lambda - cd) - (c + d)((ab - 1/\lambda)q_n^2 + 1 - ab/\lambda)) \\
- u_{n-1}(a - dq_n)(a - cq_n)((a + b)q_n - c - d) = 0.
\end{align*}
\] (3.23)

As was shown in [4] Eq. (3.23) has the hypergeometric equation as continuous limit.
4. Conclusion

In this paper we have presented solutions of discrete Painlevé equations which are given through the solution of nonautonomous linear second-order equations. These linear mappings define special functions of the discrete variable. Several open problems remains at this stage. First, the study of the discrete special functions has to be undertaken systematically, so as to construct the special functions related to the discrete Painlevé equations and also complete the gaps in our knowledge of discrete special functions. Second, the higher solutions of discrete Painlevé equations must be constructed from the existing results. We can summarize that the appropriate use of the auto-Bäcklund/Schlesinger transformations together with the bilinear formalism will make possible the expression of these solutions in the form of Casorati determinants for the $\tau$-functions. Finally, the other elementary solutions of discrete Painlevé equations which do not involve special functions but which can be expressed as rational functions of the independent variable are also of interest and their study should be the object of some future work.

Acknowledgements

K.M. Tamizhmani acknowledges invitations from both Centre de Physique Théorique and Centre des Mathématiques de Ecole Polytechnique. T. Tamizhmani acknowledges an invitation from the Groupe de Modélisation Physique et Interfaces Biologie of Paris VII University.

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