Stress obtained by interpolation methods for a boundary value problem in linear viscoelasticity

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Abstract

We will solve the following boundary value problem in linear viscoelasticity: given the value of the stress on (a part of) the boundary of the domain find the stress in the whole body at all positive times. Using a state space setting we show that the stress field inside the body can be obtained from the boundary stress by a variation-of-parameters formula involving an analytic semigroup. The relation between the regularities of the boundary stress and the stress inside the body is therefore characterized by the well-known and rich regularity theory for analytic semigroups.

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1. Introduction

1.1. The problem

In this paper we consider a body made from a homogeneous isotropic linearly viscoelastic material. We assume that the material is synchronous, this means, that the

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relaxation of shear stress and normal stress is proportional to the same scalar kernel $\mu$. Our main assumption is that $\mu$ has a strong singularity at 0. For technical reasons we also require that $\mu$ is completely monotone. Given initial data for the strain history and the normal stress on the boundary, the stress field inside the body is to be determined. Our goal is to relate the space and time regularity of the stress inside the body to the space and time regularity of the stress on the boundary.

In Lagrangian coordinates, our body is given by a domain (bounded with Lipschitz continuous boundary) $\Omega \subset \mathbb{R}^n$. With given stress on the part of the boundary $\Gamma_1 \subset \partial \Omega$ we start with the model equations:

$$\begin{align*}
\frac{\partial}{\partial t} v(t,x) &= \text{div} \sigma(t,x), \quad x \in \Omega, \quad t > 0, \\
\sigma(t,x) &= \mu(\cdot) * P \frac{\nabla v(\cdot,x) + \nabla v^T(\cdot,x)}{2}(t), \quad x \in \Omega, \quad t > 0, \\
v(t,x) &= 0, \quad x \in \Gamma_0, \quad t > 0, \\
\sigma(t,x).n(x) &= g(t,x), \quad x \in \Gamma_1, \quad t > 0, \\
v(0,x) &= 0, \quad x \in \Omega, \quad t = 0.
\end{align*}$$

Here $t \geq 0$ is the time, $v(t,x) \in \mathbb{R}^n$ is the velocity at time $t$ and point $x$, $\sigma(t,x) \in \mathbb{R}^{n \times n}_{\text{sym}}$ is the stress, $\partial \Omega = \Gamma_0 \cup \Gamma_1$ (two relatively open disjoint sets). Further, $P(x)$ is a given symmetric positive definite tensor for any $x$ and $\mu(t)$ is a time-dependent kernel describing the material (e.g. a fractional derivative model). The velocity is fixed to be zero on $\Gamma_0$, and on $\Gamma_1$ the normal component of the stress is fixed to be a given function $g(t,x)$. Here $n(x)$ denotes the outer normal to $\partial \Omega$ at $x$. Since we are interested in the effect of the boundary conditions on the stress field, we have put the initial velocity to 0 for simplicity. The convolution is meant with respect to the time $t > 0$. For $t > 0$, the velocity field $v(t,x)$ and the stress field $\sigma(t,x)$ inside the material are the unknown functions.

This problem can be treated by a direct approach, using the theory of Volterra integral equations and their resolvents in Banach spaces [10], and we have done so in a previous paper [4]. However, in the sequel we will show that the problem can also be stated in the framework of analytic semigroups. In fact, instead of a single regularity theorem, this observation produces a whole family of regularity results and gives a key to quantify the difference of regularity between the boundary stress and the resulting interior stress field by a difference of orders of interpolation spaces. To be more precise, we summarize below some well-known facts on regularity of analytic semigroups, given in terms of interpolation spaces.

1.2. Some known regularity results for analytic semigroups

**Definition 1.1.** Let $\mathcal{A}$ be the generator of an analytic semigroup in some Banach space $\mathcal{X}$.

(i) $\text{dom} \, \mathcal{A}$ is the domain of $\mathcal{A}$, endowed with the graph norm of $\mathcal{A}$. 


(ii) For \( r \in [1, \infty) \) and \( \theta \in (0, 1) \), the interpolation space of order \( \theta \) (obtained by the \( K \)-method) between \( \mathcal{X} \) and \( \text{dom} \mathcal{A} \) is denoted by \( (\mathcal{X}, \text{dom} \mathcal{A})_{\theta,r} \).

With this notation, the following regularity results are well-known:

**Theorem 1.2.** Let \( \mathcal{A} \) be the generator of an analytic semigroup in some Banach space \( \mathcal{X} \). Let \( U, X \) be Banach spaces and let \( Q_1 : U \to (\mathcal{X}, \text{dom} \mathcal{A})_{\theta,r} \), \( Q_2 : U \to \mathcal{X} \), and \( J : (\mathcal{X}, \text{dom} \mathcal{A})_{\eta,r} \to X \) be bounded linear operators, where \( 0 < \eta < \theta < 1 \). For shorthand we denote \( \tau = \theta - \eta \). Let \( p \in [1, \infty) \).

Given a function \( g \in L^p([0, T], U) \) we denote by \( z(t) \) the mild solution to the abstract Cauchy problem

\[
\begin{align*}
  z'(t) &= \mathcal{A}(z(t) + Q_1 g(t)) + Q_2 g(t), \\
  z(0) &= 0,
\end{align*}
\]

obtained by the variation of parameters formula

\[
z(t) = \mathcal{A} \int_0^t e^{(t-s)\mathcal{A}} Q_1 g(s) \, ds + \int_0^t e^{(t-s)\mathcal{A}} Q_2 g(s) \, ds,
\]

and define

\[
y(t) = J z(t).
\]

Then the following regularity results are valid:

(i) If \( p \in [1, \frac{1}{\tau}) \), then \( y \in L^q([0, T], X) \) for all \( q \in [1, \frac{p}{1-\tau p}) \).

(ii) If \( p = \frac{1}{\tau} \), then \( y \in L^q([0, T], X) \) for all \( q \in [1, \infty) \).

(iii) If \( p \in (\frac{1}{\tau}, \infty) \), then \( y \in C_{0}^{\tau-1/p}([0, T], X) \).

Here, \( C_{0}^{\tau} \) denotes the space of \( \tau \)-Hölder continuous functions \( y \) with \( y(0) = 0 \). These results are scattered throughout the literature on analytic semigroups. A summary for the case \( Q_2 = 0 \) is given in [2, Proposition A.2]. It is easy to extend these results to \( Q_2 \neq 0 \). Further regularity results can be created with conditions like Hölder continuity of \( g \). In [3] one finds even a stochastic version, where \( g \) may be replaced by Hilbert space valued white noise \( dW \) if \( \theta - \eta > \frac{1}{2} \), but it will turn out that this does not pertain to our problem, since the gap \( \theta - \eta \) will always be less than \( \frac{1}{2} \).

1.3. Application of analytic semigroup theory to the problem of viscoelasticity

With regard to Theorem 1.2, we will obtain a fairly complete understanding of the regularity of the stress problem if we can put it into the following framework:

(i) We construct a state space \( \mathcal{X} \) and an operator \( \mathcal{A} \) so that the dynamics of the viscoelastic body with stress-free boundary conditions and nonzero initial conditions
is described by an abstract Cauchy problem $z'(t) = Az(t)$. The operator $A$ generates an analytic semigroup on $X$.

(ii) We construct an operator $J_0 : \text{dom}(A) \to L^2(\Omega, \mathbb{R}^{n \times n}_{\text{sym}})$ such that (for sufficiently smooth data) the stress field in the body can be computed from the state by $\sigma(t, \cdot) = J_0 z(t)$.

(iii) We put the inhomogeneous stress boundary condition $\sigma(t, x) \cdot n(x) = g(t, x)$ in the form $z'(t) = A(z(t) + Q_1 g(t)) + Q_2 g(t)$.

(iv) We show that the range of $Q_1$ is contained in $(X, \text{dom} A)_{\theta, 2}$ and that $J_0$ admits an extension as a continuous linear operator $J : (X, \text{dom} A)_{\eta, 2} \to L^2(\Omega, \mathbb{R}^{n \times n}_{\text{sym}})$, where $0 < \eta < \theta < 1$.

(v) According to Theorem 1.2, the regularity of the stress field depends on the gap $\theta - \eta$.

The first technical task required by this approach is to define a suitable state space. Since the original system is written in terms of convolution equations, the state must contain some information on the past history. A solution for this problem has been given in [6,5], where one also finds the observation that—unlike many other state space settings for problems with memory—a strong singularity of the convolution kernel at 0 gives rise to an analytic semigroup. The state consists of the velocity field at time $t$ and a function containing information on the history. The stress field is not part of the state, but it can be computed from the state by an unbounded operator.

Interpolation techniques in context with this type of state space setting were first used to get regularity for a (different) scalar convolution equation by [7,2]. The ultimate intention of these papers was to handle stochastic convolution equations by the semigroup methods of [3]. Since the original problem was a scalar equation, and the infinite dimensional Hilbert space structure was only a consequence of the bookkeeping of the history, all regularity results concerned time-regularity.

The present paper is the first attempt to expand the interpolation techniques to the equations of viscoelasticity, which are operator equations in Hilbert spaces. Our main effort concerns the tradeoff between space and time regularity. While this paper is finished, the experiences gained here are used in [1] to treat stochastic forcing of a Hilbert space valued version of the equation considered in [7]. So we understand this paper as a step in the development of a general method to treat regularity of convolution equations with singular completely monotone kernels in Hilbert spaces.

1.4. Structure of the paper

In Section 2 we state our assumptions and some notation for the abstract setting. We will need a result on the space regularity of the stress field in an elliptic problem of elastostatics. This problem will be treated in Section 3. Section 4 introduces the semigroup setting. In Section 5 we state our main theorem and give a comparison to the regularity result proved by the direct method in [4]. The subsequent Sections 6 and 7 are devoted to the proof of the main theorem.
2. Preliminaries

2.1. Assumptions

We make the following hypotheses:

(H1) For each \( x \in \Omega \), the tensor \( P(x) \), considered as an operator mapping \( \mathbb{R}^{n \times n} \) to \( \mathbb{R}^{n \times n} \), is symmetric and positive definite. Moreover, \( P(x) \) is a measurable bounded function of \( x \), and the inverse operator \( P^{-1}(x) \) is uniformly bounded.

(H2) The kernel \( \mu \) is completely monotonic, i.e. there exists a nondecreasing function \( \mu : [0, \infty) \rightarrow [0, \infty) \) such that
\[
\mu(t) = \int_0^\infty e^{-t\zeta} \, d\nu(\zeta), \quad t > 0.
\]

(H3) \( \int_0^1 |\mu(s)| \, ds < \infty \).

(H4) There exists a constant \( \beta < \pi/2 \) such that \( \hat{\mu} : \{ \Re \lambda > 0 \} \rightarrow \{ |\text{Arg} \lambda| < \beta \} \setminus \{0\} \) (i.e. \( \mu \) is sectorial with angle less then \( \pi/2 \), [10, Definition 3.2]).

(H5) There exists a constant \( c > 0 \) such that \( |\lambda \hat{\mu}'(\lambda)| \leq c |\hat{\mu}(\lambda)| \) for all \( \Re \lambda > 0 \) (i.e. \( \mu \) is 1-regular [10]).

(H6) There exists a \( \delta \in (0, 1) \) such that for all \( \lambda > \lambda_0 \) one has \( \hat{\mu}(\lambda) \geq c \lambda^{-\delta} \), for some \( \lambda_0 > 0, c > 0 \).

(H7) There exists a constant \( \gamma \in (0, 1) \) such that for all \( \lambda > \lambda_0 \), one has
\[
\left| \frac{\hat{\mu}(\lambda)^2}{t \hat{\mu}(\lambda)} \right| \leq c \lambda^{-\gamma},
\]
for some \( \lambda_0 > 0, c > 0 \).

Here and subsequently, \( c \) will denote different positive constants varying possibly from line to line. As usual, \( \hat{\mu} \) denotes the Laplace transform of \( \mu \) and its analytic extension to the slotted plane \( \mathbb{C} \setminus \mathbb{R}_- \).

Note that (H5) follows from (H2) by [10, Proposition 3.3]. Assumption (H7) is in a sense opposite to the inequalities in (H5)–(H6).

(H2) has the following simple consequence:

**Lemma 2.1** (Desch and Grimmer [5, Lemma 2.3]). For \( \lambda > 0 \), \( k \in \mathbb{N} \) one has
\[
\hat{\mu}^{(k)}(\lambda) = (-1)^k k! \int_0^\infty \frac{1}{(\lambda + \zeta)^{k+1}} \, d\nu(\zeta).
\]

2.2. Mathematical setting

We introduce the following spaces and operators:

\[
Y := L^2(\Omega, \mathbb{R}^n);
\]
\[
X := L^2(\Omega, \mathbb{R}^{n \times n}_\text{sym});
\]
\(\tilde{D} : \begin{cases} \text{dom } \tilde{D} \subset X & \rightarrow Y, \\ \tilde{D}\sigma & = -\text{div } \sigma, \end{cases}\)

with \(\text{dom } \tilde{D} = \{ \sigma \in X, \text{ div } \sigma \in L^2(\Omega, \mathbb{R}^n) \};\)

\(D : \begin{cases} \text{dom } D \subset X & \rightarrow Y, \\ D\sigma & = -\text{div } \sigma, \end{cases}\)

with \(\text{dom } D = \{ \sigma \in X, \text{ div } \sigma \in L^2(\Omega, \mathbb{R}^n), \sigma.n = 0 \text{ on } \Gamma_1 \};\)

\(D^* : \begin{cases} \text{dom } D^* \subset Y & \rightarrow X \\ (D^*v)_{ij} & = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right), \end{cases}\)

with \(\text{dom } D^* := \{ v \in Y, v \in H^1(\Omega, \mathbb{R}^n), v = 0 \text{ on } \Gamma_0 \};\)

\(P : \begin{cases} X & \rightarrow X, \\ (P\sigma)(x) & = P(x)\sigma(x). \end{cases}\)

The derivatives are meant in the sense of distributions, the boundary conditions are meant in the sense of traces.

**Lemma 2.2** (Desch and Grimmer [5, Lemma 4.1]). \(D^*\) is the adjoint operator to \(D\). \(P\) is a self-adjoint, positive definite bounded linear operator on \(X\).

With this notation (1.1) can be rewritten as follows: find \(v : [0, \infty) \rightarrow \text{dom } D^* \subset Y\) and \(\sigma : [0, \infty) \rightarrow \text{dom } \tilde{D} \subset X\) such that

\[
\frac{\partial}{\partial t} v = -\tilde{D}\sigma, \quad t > 0, \\
\sigma = \mu \ast P D^* v, \quad t > 0, \\
\sigma.n = g, \quad x \in \Gamma_1, \quad t > 0, \\
v = 0, \quad t = 0,
\]

when a classical solution is considered. However, we obtain a solution in a mild sense.

2.3. Example

A linearly viscoelastic, homogeneous, isotropic, synchronous medium in \(\mathbb{R}^3\) or \(\mathbb{R}^2\), with the fractional derivative model fits in our setting. In this case (see also [5,10,4]) \(P\) is given by \(p_{ijkl} = (\beta - \frac{2}{3})\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{ij}\delta_{jk}, \beta > 0,\) and \(\mu(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} e^{-\varepsilon t}, \alpha \in (0, 1), \varepsilon \geq 0.\) In this case \(\hat{\mu}(\lambda) = (\lambda + \varepsilon)^{\alpha-1}, \nu(t) = \frac{1}{2\Gamma(1-\alpha)} (t - \varepsilon)^\alpha\) for \(t > \varepsilon\) and \(\nu(t) = 0\) otherwise. In (H6), (H7) we take \(\delta = \gamma = 1 - \alpha,\) and we can choose the
function $z(t)$ (see (4.9) later) to be an appropriate constant on $[0, 2\varepsilon]$, and to be 0 otherwise.

3. The elliptic problem

We are going to construct an operator $S : H^{-1/2}(\Gamma_1, \mathbb{R}^n) \to \text{dom}(D^*\tilde{D}) \subset X$ such that the boundary condition $(Sg)(x).n(x) = g(x)$ is satisfied on $\Gamma_1$ in the sense of traces. This operator will be needed to rephrase the inhomogeneous boundary condition $\sigma(x).n(x) = g(t, x)$ in the semigroup formulation. A particularly convenient choice of $S$ is the solution operator of the following elliptic problem:

$$D^*\tilde{D}r + r = 0 \text{ in } \Omega,$$
$$r.n = g \text{ on } \Gamma_1,$$
$$\tilde{D}r = 0 \text{ on } \Gamma_0.$$  \hspace{1cm} (3.1)

This problem can be solved by Lax–Milgram theorem, see [4], to get the following result. (In fact, one solves the equation for $w := -\tilde{D}r$ in $\{u \in L^2, \ u \in H^1, \ u = 0 \text{ on } \Gamma_0\}$ by Lax–Milgram, making use of Korn’s inequality.)

**Lemma 3.1** (Desch and Fašanga [4, Section 3]). For any $g \in H^{-1/2}(\Gamma_1, \mathbb{R}^n)$ there exists a (unique) $r \in \text{dom} D^*\tilde{D}$ such that (3.1) holds. Moreover, $\tilde{D}r \in Y$ and $r = -D^*\tilde{D}r \in X$ depend continuously on $g$.

Therefore we can define:

**Definition 3.2.** For $g \in H^{-1/2}(\Gamma_1, \mathbb{R}^n)$ let $Sg$ be the unique solution $r$ of (3.1).

For sharper estimates on the regularity of $S$, we need the following lemma:

**Lemma 3.3.** For $s > 0$ we consider the solution $r(s)$ of

$$D^*\tilde{D}r(s) + sr(s) = 0 \text{ in } \Omega,$$
$$r(s).n = g \text{ on } \Gamma_1,$$
$$\tilde{D}r(s) = 0 \text{ on } \Gamma_0.$$  \hspace{1cm} (3.2)

Then $Sg \in (X, \text{dom } D^*D)_{q,p}$ if and only if

$$\int_2^\infty s^{pq-1}\|r(s)\|_X^p ds < \infty.$$  \hspace{1cm} (3.3)
Proof. From [8, Proposition 2.2.6] we know that

\[ r(1) \in (X, \text{dom } D^* D)_{q, p} \iff s^{q - \frac{1}{p}} \| D^* D(s + D^* D)^{-1} r(1) \|_X \in L^p((2, \infty), \mathbb{R}). \]

A straightforward computation shows that

\[ r(s) = r(1) - (s - 1)(s + D^* D)^{-1} r(1) \]

\[ = D^* D(s + D^* D)^{-1} r(1) + (s + D^* D)^{-1} r(1). \]

Since

\[ \int_2^{\infty} s^{q - \frac{1}{p}} \| (s + D^* D)^{-1} r(1) \|_X^p ds \leq \int_2^{\infty} s^{q - \frac{1}{p}} \| r(1) \|_X ds < \infty, \]

we infer that

\[ s^{q - \frac{1}{p}} \| D^* D(s + D^* D)^{-1} r(1) \|_X \in L^p((2, \infty), \mathbb{R}) \]

iff \( s^{q - \frac{1}{p}} \| r(s) \|_X \in L^p((2, \infty), \mathbb{R}). \)

Proposition 3.4. Let \( \kappa \in (0, \frac{1}{2}) \) and \( \varrho \in (0, \frac{1}{2} - \frac{2\kappa}{4}). \) Let \( p \in [1, \infty). \) Then \( S \) is a continuous linear operator from \( H^{-\kappa}(\Gamma_1, \mathbb{R}^n) \) into \( (X, \text{dom } D^* D)^{\varrho, p} \).

Proof. The proof of this proposition is a standard application of the variational formulation of an elliptic problem. However, it is hard to find a reference for tensor valued problems. Therefore we give a sketch of the proof below: Using Lemma 3.3 we have to estimate the solution \( r(s) \) of (3.2). With \( w(s) = \tilde{D}r(s) \) we obtain

\[ \tilde{D} D^* w(s) + s w = 0 \quad \text{in } \Omega, \]

\[ D^* w(s) n = -s g \quad \text{on } \Gamma_1, \]

\[ w = 0 \quad \text{on } \Gamma_0. \]

(3.4)

Using the divergence theorem we can set up the variational formulation of (3.4)

\[ a(u, w) + (s - 1) \int_{\Omega} \langle u, w \rangle \, dx = -s \int_{\Gamma_1} \langle u, g \rangle \, dS \]

(3.5)

with

\[ a(u, w) = \int_{\Omega} \langle D^* u, D^* w \rangle \, dx + \int_{\Omega} \langle u, w \rangle \, dx. \]

Throughout this proof, \( M \) will denote a generic constant independent of \( u \) and \( w(s) \). By Korn’s inequality, we know that \( \| u \|^2_{H^1(\Omega, \mathbb{R}^n)} \leq M a(u, u) \). Now we utilize that
$g \in H^{-\kappa}(\Gamma, \mathbb{R}^n)$ and the trace theorem for $u \in H^{\kappa+\frac{1}{2}}(\Omega, \mathbb{R}^n)$:

$$\left| \int_{\Gamma} \langle u, g \rangle dS \right| \leq M \| u \|_{H^{\kappa+\frac{1}{2}}(\Omega, \mathbb{R}^n)} \leq M \| u \|_{H^{1}(\Omega, \mathbb{R}^n)} \| u \|_{\frac{1}{2} - \kappa}.$$  

We use the inequality

$$\xi^{1+2\kappa} \eta^{-2\kappa} \leq 2(\xi^2 + \eta^2)$$

with $\xi = \| w(s) \|_{H^{1}(\Omega, \mathbb{R}^n)}$ and $\eta = s^{\frac{1}{2}} \| w(s) \|_Y$. Thus for $s \geq 2$,

$$\xi^2 + \eta^2 = \| w(s) \|^2_{H^{1}(\Omega, \mathbb{R}^n)} + s \| w(s) \|^2_Y \leq M (\alpha(w(s), w(s)) + (s - 1)\langle w(s), w(s) \rangle) = Ms \left| \int_{\Gamma} \langle w(s), g \rangle dS \right| \leq Ms^{\frac{3}{2} + \xi} \| w(s) \|^2_{H^{1}(\Omega, \mathbb{R}^n)} \| s^{\frac{1}{2}} w(s) \|_{\frac{1}{2} - \kappa} = Ms^{\frac{3}{2} + \xi} \xi^2 + \eta^2 - \kappa$$

Thus

$$\| w(s) \|^2_{H^{1}(\Omega, \mathbb{R}^n)} + s \| w(s) \|^2_Y \leq \xi^2 + \eta^2 \leq Ms^{\kappa + \frac{3}{2}}$$

and

$$\| r(s) \|^2_X = s^{-2} \| D^s w(s) \|^2_X \leq Ms^{\kappa - \frac{1}{2}}.$$  

Finally, we can check Eq. (3.3):

$$\int_2^{\infty} s^{pq-1} \| r(s) \|^2_X ds \leq M \int_2^{\infty} s^{pq-1 + \frac{p}{2} - \frac{p}{4}} ds < \infty$$

if $q + \frac{p}{2} < \frac{1}{4}$. □

4. The semigroup method

This section will settle tasks (i)–(iii) outlined in Section 1.3.

Our state space will be $\mathcal{X} := Y \times L^2((0, \infty), d\mu; X)$, where $\mu$ is the Bernstein measure corresponding to the convolution kernel $\mu$. For shorthand we will denote $L^2_\mu := L^2((0, \infty), d\mu; X)$. Formally, defining

$$\Phi(t, \zeta) := P^{\frac{1}{2}} \int_0^t e^{-\zeta(t-s)} D^s v(s) \, ds,$$  

(4.1)
we obtain
\[\sigma(t) = P \mu \ast D^s v(t) = P \int_0^t \int_0^\infty e^{-(t-s)\zeta} d\nu(\zeta) D^s v(s) \, ds = P^{1/2} \int_0^\infty \Phi(t, \zeta) \, d\nu(\zeta),\]
(4.2)
and
\[\frac{\partial}{\partial t} \Phi(t, \zeta) = -\zeta \Phi(t, \zeta) + P^{1/2} D^s v(t).\]
(4.3)
Therefore, in terms of \( \begin{pmatrix} v(t) \\ \Phi(t) \end{pmatrix} \in \mathcal{X} \) our problem (2.1) may be rewritten as
\[\frac{\partial}{\partial t} \begin{pmatrix} v(t) \\ \Phi(t) \end{pmatrix} = \begin{pmatrix} -D P^{1/2} \int_0^\infty \Phi(t, \zeta) \, d\nu(\zeta) \\ -\zeta \Phi(t, \zeta) + P^{1/2} D^s v(t) \end{pmatrix}, \quad t > 0\]
(4.4)
\[P^{1/2} \int_0^\infty \Phi(t, \zeta) \, d\nu(\zeta) \cdot n = g(t), \quad t > 0, \quad x \in \Gamma_1.\]
(4.5)
While the velocity field \( v(t) \) is part of the state, the stress field is obtained by
\[\sigma(t) = \mathcal{J}_0 \begin{pmatrix} v(t) \\ \Phi(t, \zeta) \end{pmatrix} := P^{1/2} \int_0^\infty \Phi(t, \zeta) \, d\nu(\zeta).\]
(4.6)
The operator \( \mathcal{J}_0 \) is not defined everywhere on \( \mathcal{X} \), but it is well-defined whenever \( -\zeta \Phi(\zeta) + u \in L^2_v \) for some constant \( u \) (see [5]).
Consider the operator
\[\mathcal{A} : \text{dom} \mathcal{A} \subset \mathcal{X} \rightarrow \mathcal{X}, \quad \begin{pmatrix} v \\ \Phi \end{pmatrix} \mapsto \begin{pmatrix} -D P^{1/2} \int_0^\infty \Phi(\zeta) \, d\nu(\zeta) \\ -\zeta \Phi(\zeta) + P^{1/2} D^s v \end{pmatrix}\]
(4.7)
with domain
\[\text{dom} \mathcal{A} = \left\{ \begin{pmatrix} v \\ \Phi \end{pmatrix} : v \in \text{dom} D^s, -\zeta \Phi(\zeta) + P^{1/2} D^s v \in L^2_v, \quad P^{1/2} \int_0^\infty \Phi(\zeta) \, d\nu(\zeta) \in \text{dom} D \right\}.\]
(4.8)
The last condition implies that for \( \begin{pmatrix} v \\ \Phi \end{pmatrix} \in \text{dom} \mathcal{A} \) the stress-free boundary condition holds on \( \Gamma_1 \):
\[P^{1/2} \int_0^\infty \Phi(\zeta) \, d\nu(\zeta) \cdot n = 0.\]
It is known that under conditions (H1)–(H4) the operator \( \mathcal{A} \) generates an analytic \( C_0 \)-semigroup of contractions on \( \mathcal{X} \) (see [5, Theorem 2.1, Corollary 4.4]).
We will write our problem in terms of $A$. To handle the inhomogeneous boundary condition, we let $z$ be an auxiliary function such that

$$z(\zeta) \geq 0, \quad \int_0^\infty z(\zeta) \, dv(\zeta) = 1, \quad \int_0^\infty (1 + \zeta^2) z^2(\zeta) \, dv(\zeta) < \infty \tag{4.9}$$

(note that then $z(\zeta) \zeta \in L^2_\zeta$). Moreover, let $S$ be the operator defined in Definition 3.2. Then

$$Sg = P^{\frac{1}{2}} \int_0^\infty z(\zeta) P^{-\frac{1}{2}} Sg \, dv(\zeta) \tag{4.10}$$

so that on $\Gamma_1$

$$\left[ P^{\frac{1}{2}} \int_0^\infty z(\zeta) P^{-\frac{1}{2}} Sg \, dv(\zeta) \right] n = g.$$ 

Hence the boundary condition (4.5) can be rewritten as

$$\left[ P^{\frac{1}{2}} \int_0^\infty (\Phi(t, \zeta) - z(\zeta) P^{-\frac{1}{2}} Sg(t)) \, dv(\zeta) \right] n = 0.$$

Moreover,

$$\begin{pmatrix} -\tilde{D} \int_0^\infty z(\zeta) P^{-\frac{1}{2}} Sg \, dv(\zeta) \\ -\zeta z(\zeta) P^{-\frac{1}{2}} Sg \end{pmatrix} = z(\zeta) P^{-\frac{1}{2}} Sg.$$

Thus Eq. (4.4) is equivalent to

$$\frac{\partial}{\partial t} \begin{pmatrix} v(t) \\ \Phi(t, \zeta) \end{pmatrix} = A \begin{pmatrix} v(t) \\ \Phi(t, \zeta) \end{pmatrix} - \begin{pmatrix} 0 \\ z(\zeta) P^{-\frac{1}{2}} Sg(t) \end{pmatrix} - \begin{pmatrix} \tilde{D} Sg(t) \\ \zeta z(\zeta) P^{-\frac{1}{2}} Sg(t) \end{pmatrix},$$

where

$$Q_1 g = -\begin{pmatrix} 0 \\ z(\zeta) P^{-\frac{1}{2}} Sg \end{pmatrix}, \quad Q_2 g = -\begin{pmatrix} \tilde{D} Sg(t) \\ \zeta z(\zeta) P^{-\frac{1}{2}} Sg(t) \end{pmatrix}.$$ 

The choice of $z$ and Lemma 3.1 imply that the operators $Q_1$ and $Q_2$ are continuous operators from $H^{-1/2}(\Gamma_1, \mathbb{R}^n)$ into $\mathcal{X}$.

Using the analyticity of $e^{tA}$ and the variation-of-parameters formula, the mild solution of (4.11) has the following form:

$$\begin{pmatrix} v(t) \\ \Phi(t) \end{pmatrix} = A \int_0^t e^{(t-s)A} Q_1 g(s) \, ds + \int_0^t e^{(t-s)A} Q_2 g(s) \, ds. \tag{4.12}$$
5. The main result

5.1. The regularity theorem by semigroups

The following lemmas settle tasks (iii) and (iv) of the outline given in Section 1.3.

**Lemma 5.1.** Assume (H1), (H2), (H3), (H7). Let \( \eta \in \left( \frac{1-\gamma}{2}, 1 \right) \) (\( \gamma \) as in (H7)), then \( J_0 \) can be extended to a continuous linear operator \( J : (X, \text{dom } A)_{\eta,2} \rightarrow X \).

**Lemma 5.2.** Assume (H1)—(H6). Let \( \kappa \) be given by (H6), let \( \theta \in \left( 0, \frac{1-\delta}{2} + \frac{1}{4} \frac{1+\delta}{2} \right) \). Then the operator \( Q_1 \) maps \( H^{-\kappa}(\Gamma_1, \mathbb{R}^n) \) continuously into \( (X, \text{dom } A)_{\theta,2} \).

The proof of these two lemmas will be deferred to the next two sections. In this section we proceed to combine all results and characterize the regularity properties of our problem:

**Theorem 5.3.** Assume (H1)–(H7). Let \( \tau > 0, \kappa \in (0, \frac{1}{2}) \), \( \gamma \) as in (H7), \( \delta \) as in (H6), such that

\[
0 < \frac{1}{\tau} < \frac{1 - \delta}{2} + \frac{1 - 2\kappa}{4} \frac{1 + \delta}{2} - \frac{1 - \gamma}{2}.
\]  

For some \( p \in [1, \infty) \) let \( g \in L^p_{\text{loc}}([0, \infty), H^{-\kappa}(\Gamma_1, \mathbb{R}^n)) \). Let \( \begin{pmatrix} v(t) \\ \Phi(t) \end{pmatrix} \) be the mild solution to problem (1.1), i.e. (4.12) holds in \( X \). Then for \( t > 0 \) the stress \( \sigma(t) = J \begin{pmatrix} v(t) \\ \Phi(t) \end{pmatrix} \in X \) exists and has the following regularity properties:

(i) If \( p \in [1, \frac{1}{\tau}) \), then \( \sigma \in L^q([0, T], X) \) for all \( q \in [1, \frac{p}{1-\tau p}] \).
(ii) If \( p = \frac{1}{\tau} \), then \( \sigma \in L^q([0, T], X) \) for all \( q \in [1, \infty) \).
(iii) If \( p \in (\frac{1}{\tau}, \infty) \), then \( \sigma \in C_0^{T-1/p}([0, T], X) \).

**Proof.** Choose \( \eta > \frac{1-\gamma}{2} \) and \( \theta < \frac{1-\delta}{2} + \frac{1-2\kappa}{4} \frac{1+\delta}{2} \) such that \( \tau = \theta - \eta \). The preceding two lemmas state that the assumptions of Theorem 1.2 are satisfied. Therefore, its regularity results carry over to our problem verbatim. \( \Box \)

**Remark 5.4.** For the kernel \( \mu(t) = t^{-\alpha} \) the assumptions are automatically satisfied and the theorem may be applied. With \( \gamma = \delta = 1 - \alpha \) we obtain the condition \( \tau < \frac{2-\alpha}{2} \frac{1-2\kappa}{4} \).

5.2. Comparison to a result obtained by direct methods

We cite, for comparison, the main result from [4], where the same problem is solved by the method of Volterra equations.
Theorem 5.5 (Desch and Fašanga [4, Theorem 2.2]). Suppose that the hypotheses (H1), (H3), (H4), (H5) hold. Then for any \( g \in W^{1,1}_{\text{loc}}([0, \infty); H^{-\frac{1}{2}}(\Gamma_1, \mathbb{R}^n)) \) there exists a function \( \sigma \in C([0, \infty), X) \) solving (1.1) in the following sense:

\[
1 \ast \mu \ast \sigma(t) \in \text{dom } D^\ast \hat{D}, \quad t \geq 0,
\]

\[
\sigma(t) = -PD^\ast \hat{D}(1 \ast \mu \ast \sigma(t)), \quad t \geq 0,
\]

\[
(1 \ast \mu \ast \sigma)(t, \cdot), n = (1 \ast \mu \ast g(t, \cdot))(t) \text{ in } H^{-\frac{1}{2}}(\Gamma_1), \quad t \geq 0.
\]

Assume in addition that \(|\lambda^2 m''(\lambda)| \leq M|\hat{m}(\lambda)|\) whenever \( \Re \lambda > 0 \), and let \( \alpha \in (0,1) \). Then for each \( g \in C^0_{\text{loc}}([0, \infty); H^{-\frac{1}{2}}(\Gamma_1, \mathbb{R}^n)) \) there exists \( \sigma \in C^\alpha_0([0, \infty); X) \) solving (1.1) in the sense above.

We see that the result obtained by direct methods requires less space regularity of the boundary stress: While the semigroup result needs that \( \sigma(t) \in H^{-\kappa}(\Gamma_1, \mathbb{R}^n) \) a.e., we need only \( \sigma(t) \in H^{-1/2} \) in the direct approach. On the other hand, the direct approach requires much more time regularity. While the semigroup needs only that \( \sigma \) is locally \( L^p \) with respect to time, the direct result requires \( W^{1,p} \). The assumptions on the kernel are weaker and more natural in the direct approach. The semigroup approach relies on complete monotonicity, which is not a consequence of physical considerations, although most kernels used in practice will have this property. Energy considerations lead only to the fact that the relaxation modulus must be of positive type. A generalization of the semigroup method to more general kernels has been obtained by [11]. The analyticity of the semigroup and the characterization of the interpolation spaces, however, has not yet been investigated in this general situation.

There is, of course, no doubt that the direct approach could be modified to obtain different regularity results, and we expect that any result obtained by semigroups can be reproduced and even improved by the direct approach. It is the reference to a very general abstract result; however, that makes the semigroup approach appealing to us.

6. Proof of Lemma 5.1

Proof. We define a subspace \( E \subset L^2_v \) by

\[
E := \{ \Phi \in L^2_v; \text{there exists a } w \in X \text{ such that } \zeta \Phi(\zeta) - w \in L^2_v \},
\]

\[
\| \Phi \|_E^2 := \| \Phi \|_{L^2_v}^2 + \| (\zeta + 1) \Phi(\zeta) - w \|_{L^2_v}^2.
\]

(6.1)

Evidently, if \( \begin{pmatrix} v \\ \Phi \end{pmatrix} \in \text{dom } A \), then \( \Phi \in E \) (with \( w = P^{1/2} D^\ast v \)) and

\[
\| \Phi \|_E^2 \leq \left( \| 2 \Phi \|_{L^2_v}^2 + \| \zeta \Phi(\zeta) - P^{1/2} D^\ast v \|_{L^2_v}^2 \right)^{1/2} \leq 2 \left\| \begin{pmatrix} v \\ \Phi \end{pmatrix} \right\|_{\text{dom } A}.
\]
Therefore we have the embedding

$$\text{dom } A \hookrightarrow (Y, E) \hookrightarrow \mathcal{X} = \left( \frac{Y}{L^2_v} \right).$$

(6.2)

Hence for all \( \eta \in (0, 1), p > 1, \)

$$\mathcal{X}, \text{dom } A \hookrightarrow \left[ \left( \frac{Y}{L^2_v} \right), \left( \frac{Y}{E} \right) \right] \hookrightarrow \left( \frac{Y}{(L^2_v, E)} \right).$$

(6.3)

Let \( \left( \frac{v}{0} \right) \in (\mathcal{X}, \text{dom } A)_{\eta, p} \) be fixed. Then \( \Phi \in (L^2_v, E)_{\eta, p} \hookrightarrow L^2_v \).

We shall show that for any \( w \in X \) the constant function \( \zeta \mapsto w \) lies in the dual space \( (L^2_v, E)^*_{\eta, p} \) with the pairing \( \langle w, \Phi \rangle := \int_0^\infty \langle \Phi (\zeta), w \rangle_X d\zeta \), and that \( \| w \|_{(L^2_v, E)^*_{\eta, p}} \leq c\| w \|_X \). If this will be done then for our fixed \( \Phi \)

$$\left| \int_0^\infty \langle \Phi (\zeta), w \rangle d\zeta \right| \leq c\| \Phi \|_{(L^2_v, E)_{\eta, p}} \| w \|_X,$$

(6.4)

therefore \( w \mapsto \int \langle \Phi, w \rangle \) is an element of \( X^* \equiv X \) and we shall call it \( \mathcal{J} (\Phi (\zeta)) \in X \).

With this notation, (6.4) implies that

$$\left\| \mathcal{J} \left( \frac{v}{0} \right) \right\|_X \leq c\| \Phi \|_{(L^2_v, E)_{\eta, p}} \leq c\left\| \left( \frac{v}{0} \right) \right\|_{(\mathcal{X}, \text{dom } A)_{\eta, p}}.$$

Observe that \( L^2_v \hookrightarrow E^* \) and that \( (L^2_v, E)^*_{\eta, p} = (L^2_v, E^*)_{\eta, p'} \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \), (cf. [9]). Using the characterization of real interpolation spaces by the \( K \)-method [12], and a straightforward equivalent modification of \( k \) we may proceed by three steps:

(i) compute \( \| f \|_{E^*}; \)

(ii) compute and estimate \( k^2(t, w) := \inf_{w = f + g, f \in L^2_v, g \in E^*} \| f \|_{L^2_v}^2 + t^2 \| g \|_{E^*}^2; \)

(iii) check that \( \| w \|_{(L^2_v, E^*)_{\eta, p'}} := \left( \int_0^\infty \left( t^{-\eta} k(t, w) \right)^p d\frac{t}{T} \right)^{\frac{1}{p'}} \leq c\| w \|_X. \)

Proving (i)–(iii) and taking \( p = 2 \) will finish the proof of Lemma 5.1.

(i) We consider the following reparametrization of \( E \): For \( \Phi \in E \) let \( \Psi (\zeta) := (\zeta + 1)\Phi (\zeta) - w \). It is easily checked that this is a homeomorphism from \( E \) to \( L^2_v \). We
compute:

\[ | \langle f, \Phi \rangle | = \left| \int_0^\infty \langle f(\zeta), \Phi(\zeta) \rangle_X \, d\nu(\zeta) \right| = \left| \int_0^\infty \left\langle f, \frac{\psi + w}{\zeta + 1} \right\rangle \, d\nu(\zeta) \right| \]
\[ \leq \sqrt{\int \left\| \frac{f}{\zeta + 1} \right\|^2 \sqrt{\int \| \psi \|^2} + \int \left\| \frac{f}{\zeta + 1} \right\| \| w \|_X} \]
\[ = \left\| \frac{f}{\zeta + 1} \right\|_{L^2_X} \| \psi \|_{L^2} + \int \frac{f}{\zeta + 1} \left\| \frac{1}{|\bar{\mu}'(1)|} \int \| w - (\zeta + 1)\Phi \| + \| (\zeta + 1)\Phi \|_{(\zeta + 1)^2} \right\| \]
\[ \leq \left\| \frac{f}{\zeta + 1} \right\|_{L^2_X} \| \psi \|_{L^2} + \int \frac{f}{\zeta + 1} \left\| \frac{1}{|\bar{\mu}'(1)|} \right\| \]
\[ \times \left( \| w - (\zeta + 1)\Phi \|_{L^2} \sqrt{\int \frac{1}{(\zeta + 1)^4} + \| \Phi \| \sqrt{\int \frac{1}{(\zeta + 1)^2}} \right) \]
\[ = \left\| \frac{f}{\zeta + 1} \right\|_{L^2_X} \| \psi \|_{L^2} + \int \frac{f}{\zeta + 1} \left( \frac{1}{|\bar{\mu}'(1)|} \left( \| w - (\zeta + 1)\Phi \|_{L^2} \frac{\sqrt{\bar{\mu}''(1)}}{6} + \| \Phi \| \sqrt{\bar{\mu}'(1)} \right) \right) \]
\[ \leq c \left( \left\| \frac{f}{\zeta + 1} \right\|_{L^2_X} + \int \frac{f}{\zeta + 1} \right) \| \Phi \|_E, \]
so

\[ \| f \|_{E^*}^2 := \sup_{\Phi \in E, \| \Phi \|_E \leq 1} | \langle f, \Phi \rangle |^2 \leq c \left( \left\| \frac{f(\zeta)}{\zeta + 1} \right\|_{L^2_X}^2 + \int_0^\infty \frac{f(\zeta)}{\zeta + 1} \, d\nu(\zeta) \right)^2. \] (6.5)

(In fact, a more careful computation would show that the right-hand side is an equivalent norm on \( E^* \), but this will not be needed in the sequel.)
(ii) Let \( w \in X \) be fixed. We consider for \( t > 0 \) (an equivalent “\( k \)-norm”)

\[
k^2(t, w) := \inf_{f \in L^2_\nu} \| f \|_{L^2_\nu}^2 + t^2 \| w - f \|_{E^*}^2
\]

\[
\leq c \inf \int_0^\infty \| f \|^2 dv + t^2 \int_0^\infty \left\| f - \frac{w}{1 + \zeta} \right\|^2 dv + t^2 \int_0^\infty \frac{f - w}{1 + \zeta} dv \|^2 \cdot \tag{6.6}
\]

The infimum is attained at an \( f \) where the variation is zero, and from this equation we will compute \( f \). Denote

\[
\int_0^\infty \left\| f - \frac{w}{1 + \zeta} \right\|^2 dv = \gamma.
\]

If \( f \) is a minimizer then for all \( u \in L^2_\nu \) one must have

\[
\int_0^\infty \langle f, u \rangle dv + t^2 \int_0^\infty \left\langle \frac{f - w}{1 + \zeta} , u \right\rangle (1 + \zeta)^2 dv + \| \gamma \|^2 = 0. \tag{6.7}
\]

We provide the following computations step by step:

\[
0 = f + t^2 \frac{f - w}{(1 + \zeta)^2} + t^2 \frac{\gamma}{1 + \zeta} \quad \text{in } L^2_\nu, \tag{6.8}
\]

\[
f(\zeta) = t^2 \frac{w - (1 + \zeta) \gamma}{(1 + \zeta)^2 + t^2}, \tag{6.9}
\]

\[
\int_0^\infty \| f \|^2 dv = t^4 \int_0^\infty \| \frac{f - (1 + \zeta) \gamma}{(1 + \zeta)^2 + t^2} \|^2 dv(\zeta), \tag{6.10}
\]

\[
\frac{f(\zeta) - w}{1 + \zeta} = -t^2 \gamma + w(1 + \zeta) \frac{(1 + \zeta)^2 + t^2}{(1 + \zeta)^2 + t^2}, \tag{6.11}
\]

\[
\gamma = -t^2 \int_0^\infty \frac{1 + \zeta}{(1 + \zeta)^2 + t^2} dv \quad \text{in } L^2_\nu, \tag{6.12}
\]

\[
\| \gamma \|^2 = \left( \frac{t^2}{(1 + \zeta)^2 + t^2} \right)^2 \int_0^\infty \| f \|^2 dv, \tag{6.13}
\]

\[
\int_0^\infty \left\| f - \frac{w}{1 + \zeta} \right\|^2 dv = \int \frac{t^4 \| \gamma \|^2 + 2t^2 \langle \gamma, w \rangle (1 + \zeta) + \| w \|^2 (1 + \zeta)^2}{(1 + \zeta)^2 + t^2} dv. \tag{6.14}
\]
We are now in the position to estimate $k$:

\[
\frac{1}{c} k^2(t, w) \leq \int \| f \|^2 \mathrm{d}v + t^2 \int \left\| \frac{f - w}{1 + \zeta} \right\|^2 \mathrm{d}v + t^2 \| \gamma \|^2 \\
= \int t^4 \| w \|^2 + t^4 \left( \frac{1 + \zeta}{\| \gamma \|^2 + t^2} \right)^2 + t^2 \| w \|^2 (1 + \zeta)^2 + t^2 \| \gamma \|^2 \\
= \| w \|^2 \int \frac{t^2}{(1 + \zeta)^2 + t^2} \mathrm{d}v + \| \gamma \|^2 \left( \int \frac{t^4}{(1 + \zeta)^2 + t^2} \mathrm{d}v + t^2 \right) \\
= t^2 \| w \|^2 \left[ \int \frac{1}{(1 + \zeta)^2 + t^2} \mathrm{d}v + \frac{\left( \int \frac{1 + \zeta}{(1 + \zeta)^2 + t^2} \mathrm{d}v \right)^2}{1 + \int \frac{t^2}{(1 + \zeta)^2 + t^2} \mathrm{d}v} \right] \\
= t^2 \| w \|^2 \left[ -\frac{1}{t} \Im \mu(1 + it) + \frac{(\Re \mu(1 + it))^2}{1 - t \Im \mu(1 + it)} \right].
\]

(iii) We have to check the convergence of the integral

\[
\int_0^1 (t^{-\eta} k(t, w)) p' \frac{dt}{t} \leq c \| w \|^p \int_0^\infty t^{p'(1-\eta)-1} \left[ -\frac{1}{t} \Im \mu(1 + it) + \frac{(\Re \mu(1 + it))^2}{1 - t \Im \mu(1 + it)} \right]^{p'/2}. 
\]

Behavior at $t \to 0$. The term in the above bracket converges to $-\Im \mu(1 + \mu(1) < \infty$ as $t \to 0$ (use Lemma 2.1 and the Lebesgue dominated convergence theorem). Therefore

\[
\int_0^1 (t^{-\eta} k(t, w)) p' \frac{dt}{t} \leq c \| w \|^p \int_0^1 t^{(1-\eta)p'-1} \mathrm{d}t \leq c \| w \|^p. 
\]

Behavior at $t \to \infty$. Note that (using Lemma 2.1)

\[
|\mu(1 + it)| \leq |\Re \mu(1 + it)| + |\Im \mu(1 + it)| = \int_0^\infty \frac{1 + \zeta + t}{(1 + \zeta)^2 + t^2} \mathrm{d}v(\zeta) \\
\leq 2 \int \frac{1}{1 + \zeta + t} \mathrm{d}v(\zeta) = 2 \mu(1 + t),
\]

\[
\]
\[-\Im \hat{\mu}(1 + it) = \int_0^\infty \frac{t}{(1 + \zeta)^2 + t^2} d\nu(\zeta) \geq t \int_0^\infty \frac{1}{(1 + \zeta + t)^2} d\nu(\zeta),\]
\[= -t \hat{\mu}'(1 + t).\]

We continue to estimate \(k(t, w)\) using hypothesis (H7):
\[k(t, w)^2 \leq ct^2 \|w\|^2 \left[\frac{-\frac{1}{t} \Im \hat{\mu}(1 + it) + \Im^2 \hat{\mu}(1 + it) + \Re^2 \hat{\mu}(1 + it)}{1 - t \Im \hat{\mu}(1 + it)}\right].\]
\[\leq c \|w\|^2 t^2 \left[\frac{1}{t^2} + \frac{|\hat{\mu}(1 + it)|^2}{t \Im \hat{\mu}(1 + it)}\right] \leq c \|w\|^2 \left(1 + \frac{|\hat{\mu}(1 + t)|^2}{|\hat{\mu}'(1 + t)|}\right)\]
\[\leq c \|w\|^2 t^{1-\gamma}.\]

Therefore,
\[\int_1^\infty t^{-\eta p'} k(t, w)^{p'} \frac{dt}{t} \leq c \int_1^\infty t^{-1 + p'\left(-\eta + \frac{1-\gamma}{2}\right)} dt\|w\|^{p'} \leq c\|w\|^{p'}.\quad \square\]

7. Proof of Lemma 5.2

We rely on the following characterization of the interpolation space:

**Proposition 7.1** (Lunardi [8, Proposition 2.2.6]). Let \(A\) generate an analytic semigroup. Then the following is an equivalent norm on the interpolation space \((X, \dom A)_\theta, p^\prime\):
\[\|x\|_{X'} + \left(\int_2^\infty \|\lambda^\theta A(\lambda - A)^{-1} x\|_X^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}.\quad (7.1)\]

Remember also that
\[Q_1 g = \left(\begin{array}{c} 0 \\ -\gamma(\zeta) P^{-1/2} S g \end{array}\right),\]
where \(S\) is the solution operator defined in Definition 3.2 and \(\gamma\) is as in (4.9). We choose some \(\theta > 0\) such that
\[\theta < \frac{1 - 2\kappa}{4} \quad \text{and} \quad \theta < \frac{1 - \delta}{2} - \theta + \frac{1 + \delta}{2}.\quad (7.2)\]
Notice that by Lemma 3.4 the operator \(S\) is continuous from \(H^{-\kappa}(\Gamma_1, \mathbb{R}^n)\) into \((X, \dom D^\theta D)_{\theta, 2}\). For shorthand we will define
\[h(\lambda) = \int_0^\infty \frac{\gamma(\zeta)}{\lambda + \zeta} d\nu(\zeta).\quad (7.3)\]
We will split the proof into three parts:

(i) For \( \lambda > 2 \), compute \( \mathcal{A}(\lambda - \mathcal{A})^{-1}Q_1g \) (Lemma 7.2).
(ii) Estimate the norm of \( \|\mathcal{A}(\lambda - \mathcal{A})^{-1}Q_1g\| \) in \( \mathcal{X} \) (Lemma 7.3).
(iii) Use this estimate in Proposition 7.1.

**Lemma 7.2.**
\[
\mathcal{A}(\lambda - \mathcal{A})^{-1}Q_1g = \left( h(\lambda)D \left( I + \frac{\mu}{\lambda} PD^*D \right)^{-1} Sg \right) - \left( \frac{\zeta(\lambda)}{\lambda + \zeta} P^{-\frac{1}{2}} Sg + \frac{1}{\lambda + \zeta} h(\lambda) P^\frac{1}{2} D^*D \left( I + \frac{\mu}{\lambda} PD^*D \right)^{-1} Sg \right).
\]

**Proof.** First we compute
\[
\begin{pmatrix} v \\ \Phi \end{pmatrix} = (\lambda - \mathcal{A})^{-1} \begin{pmatrix} 0 \\ \Psi \end{pmatrix}.
\]
This is equivalent to
\[
\begin{align*}
\lambda v + DP^\frac{1}{2} \int_0^\infty \Phi(\zeta) dv(\zeta) &= 0, \quad (7.4) \\
\lambda \Phi(\zeta) + \zeta \Phi(\zeta) - P^\frac{1}{2} D^*v &= \Psi(\zeta). \quad (7.5)
\end{align*}
\]
We compute \( \Phi \) from (7.5), integrate and use Lemma 2.1:
\[
P^\frac{1}{2} \int_0^\infty \Phi(\zeta) dv(\zeta) = P^\frac{1}{2} \int_0^\infty \frac{1}{\lambda + \zeta} \Psi(\zeta) dv(\zeta) + \hat{\mu}(\lambda) PD^*v.
\]
Now we insert \( v \) from (7.4):
\[
P^\frac{1}{2} \int \Phi dv = \left( I + \frac{\hat{\mu}(\lambda)}{\lambda} PD^*D \right)^{-1} P^\frac{1}{2} \int_0^\infty \frac{1}{\lambda + \zeta} \Psi(\zeta) dv(\zeta),
\]
and hence
\[
v = -\frac{1}{\lambda} DP^\frac{1}{2} \int \Phi dv = -\frac{1}{\lambda} D \left( I + \frac{\hat{\mu}}{\lambda} PD^*D \right)^{-1} P^\frac{1}{2} \int_0^\infty \frac{1}{\lambda + \zeta} \Psi(\zeta) dv(\zeta),
\]
\[
\Phi(\zeta) = \frac{\Psi(\zeta)}{\lambda + \zeta} \frac{1}{\lambda + \zeta} P^\frac{1}{2} D^*v
\]
\[
= \frac{\Psi(\zeta)}{\lambda + \zeta} - \frac{1}{\lambda + \zeta} \frac{1}{\lambda} P^\frac{1}{2} D^*D \left( I + \frac{\hat{\mu}}{\lambda} PD^*D \right)^{-1} P^\frac{1}{2} \int_0^\infty \frac{\Psi(\zeta)}{\lambda + \zeta'} dv(\zeta').
\]
Thus (with $\Psi = -\lambda P^{-1/2} S g$)

$$
\begin{align*}
A(\lambda - A)^{-1} & \begin{pmatrix} 0 \\ -\lambda P^{-1/2} S g \end{pmatrix} \\
= & (-I + \lambda(\lambda - A)^{-1}) \begin{pmatrix} 0 \\ -\lambda P^{-1/2} S g \end{pmatrix} \\
= & \begin{pmatrix} h(\lambda) D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} S g \\
& \left( \frac{\xi^2 \alpha(\zeta)}{\lambda + \zeta} P^{-1/2} S g + \frac{1}{\lambda + \zeta} h(\lambda) P \frac{1}{\lambda} D^* D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} S g \right) \end{pmatrix}.
\end{align*}
\tag{7.6}
$$

\textbf{Lemma 7.3.} Suppose (H4), (H5) and (4.9) hold. There exists a constant $c$ such that for all $g \in H^{-k}(\Gamma_1, \mathbb{R}^n)$ and all $\lambda > 2$ we have

$$
\|A(\lambda - A)^{-1} Q_1 g\| \leq c \|S g\|_{(X, \text{dom } D^* D)_{q, p}} \lambda^{\frac{q}{2} - \frac{1}{2} - q} \lambda^{\frac{q}{2} - \frac{1}{2} - q}. \tag{7.7}
$$

\textbf{Proof.} Using Lemma 7.2, we estimate

\begin{align*}
\left\| A(\lambda - A)^{-1} \begin{pmatrix} 0 \\ -\lambda P^{-1/2} S g \end{pmatrix} \right\| \\
\leq & |h(\lambda)| \left\| D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} S g \right\| + \left\| \frac{\xi \alpha(\zeta)}{\lambda + \zeta} \right\|_{L^2} \left\| P^{-1/2} S g \right\| \\
+ & |h(\lambda)| \left\| \frac{1}{\lambda + \zeta} \right\|_{L^2} \left\| P \frac{1}{\lambda} \right\| \\
\times & \left\| D^* D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} \right\|_{L((X, \text{dom } D^* D)_{q, p}, X)} \left\| S g \right\|_{q, p}. \tag{7.9}
\end{align*}

Here the behavior (estimates from above) of the particular terms as $\lambda \to \infty$ is the following:

- $|h(\lambda)| = \int_0^{\infty} \frac{\xi \alpha(\zeta)}{\lambda + \zeta} d v(\zeta) \leq \frac{1}{\lambda} \int_0^{\infty} \alpha(\zeta) d v(\zeta) = \frac{1}{\lambda}$;
- $\left\| \frac{\xi \alpha(\zeta)}{\lambda + \zeta} \right\|_{L^2} = \frac{1}{\xi} \left( \int \frac{\xi^2 \alpha^2(\zeta)}{\lambda + \zeta} d v(\zeta) \right)^{1/2} \leq \frac{1}{\lambda} \int \xi^2 \alpha^2(\zeta) d v = \frac{\xi}{\lambda}$;
- $\left\| \frac{1}{\lambda + \zeta} \right\|_{L^2} = \left\| \frac{\xi \alpha(\zeta)}{\lambda + \zeta} \right\|_{L^2} \leq c |\xi|^1_{L^2} \lambda^{-1/2}$;
- $\| D^* D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} \|_{L((X, \text{dom } D^* D)_{q, p}, X)} \leq c \left\| \frac{1}{\lambda} \alpha(\zeta) \right\|_{L^2} \left\| S g \right\|_{q, p}$;
- $\| D \left( I + \frac{\xi}{\lambda} P D^* D \right)^{-1} S g \| \leq c \left\| \frac{1}{\lambda} \alpha(\zeta) \right\|_{L^2} \left\| S g \right\|_{q, p}$.
To prove the last two estimates, let first $f = \left( I + \frac{\widehat{\mu}}{\lambda} PD^*D \right)^{-1} u$, and take the scalar product of $u$ with $P^{-1} f$:

$$\langle u, P^{-1} f \rangle = \langle f, P^{-1} f \rangle + \left( \frac{\widehat{\mu}}{\lambda} PD^*Df, P^{-1} f \right).$$

Hence using the self-adjointness of $P$ we obtain

$$\left| \left| \frac{\lambda}{\mu} u \right| \right| P^{-1} \| f \| \geq \left( \frac{\lambda}{\mu} \langle u, P^{-1} f \rangle \right),$$

and get

$$\| f \| \leq c \| P^{-1} \| \| P^{\frac{1}{2}} \| \| u \| \leq c \| u \|.$$ 

This means that $\left\| \left( I + \frac{\widehat{\mu}}{\lambda} PD^*D \right)^{-1} \right\|_{(X, \text{dom } D^*, X)} \leq c$. Further,

$$\left\| D^* D \left( I + \frac{\widehat{\mu}}{\lambda} PD^*D \right)^{-1} \right\|_{(X, \text{dom } D^*, X)} = \left\| \frac{\lambda}{\mu} P^{-1} - \frac{\lambda}{\mu} \left( I + \frac{\widehat{\mu}}{\lambda} PD^*D \right)^{-1} \right\|_{(X, \text{dom } D^*, X)} \leq c \left| \frac{\lambda}{\mu} \right|,$$

and for $u \in (X, \text{dom } D^* D)_{q, p}$ one has

$$\left\| D^* D \left( I + PD^*D \frac{\widehat{\mu}}{\lambda} \right)^{-1} u \right\|_{X} = \left\| \left( I + PD^*D \frac{\widehat{\mu}}{\lambda} \right)^{-1} D^* Du \right\|_{X} \leq c \| D^* Du \|_{X} \leq c \| u \|_{\text{dom } D^* D},$$

i.e.

$$\left\| D^* D \left( I + PD^*D \frac{\widehat{\mu}}{\lambda} \right)^{-1} \right\|_{(\text{dom } D^* D, X)} \leq c.$$

Hence by interpolation (cf. [8, Proposition 1.2.6]) from (7.10) and (7.11) we obtain

$$\left\| D^* D \left( I + \frac{\widehat{\mu}}{\lambda} PD^*D \right)^{-1} \right\|_{(X, \text{dom } D^* D)_{q, p, X}} \leq c \left| \frac{\lambda}{\mu(\lambda)} \right|^{1-q}.$$
Finally,

\[
\left\| D\left( I + \frac{\hat{\mu}}{\hat{\lambda}} P D^* D\right)^{-1} S_g \right\|_{\mathcal{X}}^2 \\
= \left( I + \frac{\hat{\mu}}{\hat{\lambda}} P D^* D\right)^{-1} S_g, D^* D \left( I + \frac{\hat{\mu}}{\hat{\lambda}} P D^* D\right)^{-1} S_g \right) \\
\leq c \left| \frac{\hat{\lambda}}{\hat{\mu}} \right|^{1-q} \| S_g \|_{\mathcal{X}} \| S_g \|_{q,p} \leq c \left| \frac{\hat{\lambda}}{\hat{\mu}} \right|^{1-q} \| S_g \|_{q,p}^2.
\]

Inserting these estimates into (7.8) and using (H6) we get for \( \hat{\lambda} > 2 \)

\[
\left\| A(\hat{\lambda} - A)^{-1} \begin{pmatrix} 0 \\ -\beta P^{-\frac{1}{2}} S_g \end{pmatrix} \right\| \leq c \| S_g \|_{q,p} \left( \left| \frac{\hat{\lambda}}{\hat{\mu}} \right|^{1-q} \left| \frac{1}{\hat{\lambda}} \right| + \left| \frac{\hat{\mu}}{\hat{\lambda}} \right|^{\frac{1}{2}} \left| \frac{\hat{\lambda}}{\hat{\mu}} \right|^{\frac{1}{2}} \right) \\
\leq c \| S_g \|_{q,p} \left( \frac{1}{\hat{\lambda}^{\frac{1}{2}-\frac{\alpha}{2}+\frac{1+\delta}{2}}} + \frac{1}{\hat{\lambda}^{\frac{1}{2}-\frac{\alpha}{2}+\delta(1+\delta)}} \right) \\
\leq c \| S_g \|_{q,p} \hat{\lambda}^{\frac{\delta-1}{2} - \frac{\alpha}{2} - \frac{1+\delta}{2}},
\]

so the claim (7.7) is proved. □

Now we can finish the proof of Lemma 5.2, using Proposition 7.1:

\[
\| Q_1 g \|_{(\mathcal{X}, \text{dom} A)_{\delta,p}} \leq c \left( 2P^{-\frac{1}{2}} S_g \right)_{\mathcal{X}} + c \left( \int_2^\infty \hat{\lambda}^{\theta_p} \left\| A(\hat{\lambda} - A)^{-1} \begin{pmatrix} 0 \\ -\beta P^{-\frac{1}{2}} S_g \end{pmatrix} \right\|_{\mathcal{X}} d\frac{\lambda}{\hat{\lambda}} \right)^{\frac{1}{p}} \\
\leq c \| \beta P^{-\frac{1}{2}} S_g \| + c \left( \int_2^\infty \hat{\lambda}^{\theta_p} \left( \| S_g \|_{(\mathcal{X}, \text{dom} D^* D)_{q,p}} \hat{\lambda}^{\frac{\delta-1}{2} - \frac{\alpha}{2} - \frac{1+\delta}{2}} \right) d\frac{\lambda}{\hat{\lambda}} \right)^{\frac{1}{p}} \\
\leq c \int_0^\infty 2^2(\zeta) d\nu(\zeta) \| \beta P^{-\frac{1}{2}} \| \| S_g \|_{q,p} \\
+ c \int_2^\infty \hat{\lambda}^{\theta_p(\frac{\delta-1}{2} - \frac{\alpha}{2} - \frac{1+\delta}{2})} d\frac{\lambda}{\hat{\lambda}} \| S_g \|_{q,p} \\
\leq c \| S_g \|_{q,p}.
\]

Choosing \( p = 2 \) and using Proposition 3.4 we obtain the assertion of Lemma 5.2.
References