Computational homogenization in magneto-mechanics

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This work presents a geometrically nonlinear homogenization framework for composites with magneto-mechanical behavior whereby the composite can be subject to large deformation processes. The magneto-mechanical governing equations in the material description for both the overall body and its microstructure are presented, and the connections between micro- and macro-scale field variables are identified. Considering periodic boundary conditions for the microscopic unit cell, a finite element framework for computing the macroscopic field variables and the effective tangent moduli is developed. The proposed methodology is utilized to study a variety of two- and three-dimensional numerical examples. In particular, the behavior of fiber and particle reinforced composites with magneto-mechanical constitutive laws are illustrated. Finally, a specific physically motivated problem of a magnetorheological elastomer, consisting of a polymer matrix and iron particles, under finite deformation and applied magnetic field is analyzed and the results are given for several combinations of deformation modes and applied magnetic fields.

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1. Introduction

The effective macroscopic properties of a heterogeneous material can be estimated from the response of the underlying microstructure using homogenization procedures. These mature procedures need to be extended in certain situations (e.g. when the microstructure contains magnetic inclusions) to account for the role of the magnetic field. The objective of this contribution is to present a novel micro-to-macro transition (computational homogenization) framework that accounts for the magneto-mechanical coupling at the micro-scale.

The two main ingredients of the work presented here are (i) continuum formulations of magneto-mechanical problems, in particular, with applications to magnetorheological elastomers and (ii) homogenization as pioneered by Hill (1963). A brief review of these topics is now given in Sections 1.1 and 1.2.

1.1. State-of-the-art review of magneto-mechanics with applications to magnetorheological elastomers

Magnetorheological elastomers are magneto-sensitive composite materials whose mechanical behavior depends on the applied magnetic field. The increased research attention on elastomers filled with magnetic particles is due to the variety of applications that they can be incorporated into, including adaptive engine mounts, vibration absorbers, suspensions and automotive bushing (Kordonsky, 1993; Jolly et al., 1996; Carlson and Jolly, 2000; Ginder et al., 2002; Bellan and Bossis, 2002; Varga et al., 2006).

Thorough studies have been carried out in the last decades for identifying the electro-mechanical and magneto-mechanical responses of solids. Combining the laws of elasticity, electricity and magnetism, the general equations have been derived and several boundary-value problems have been solved (e.g. Pao, 1978; Eringen and Maugin, 1989; Maugin, 1993; Kovetz, 2000; Vu and Steinmann, 2007; Bustamante et al., 2008). Especially for magnetorheological elastomers, modeling efforts have been made recently (Brigadnov and Dorfmann, 2003; Dorfmann and Ogden, 2004; Steigmann, 2004; Kankanala and Triantafyllidis, 2004; Bustamante, 2010; Danas et al., 2012, among others) that focus on phenomenological macroscopic laws which describe the material behavior.

1.2. State-of-the-art review of homogenization

Extensive studies on composite materials show that their overall behavior depends strongly on the properties of the material constituents and the microscopic geometry (volume fraction, shape and orientation of constituents). The determination of the macroscopic response considering the material microstructure is generally a difficult task and involves numerical techniques that have significant computational cost, if possible at all. Homogenization methods, as pioneered by Hill (Hill, 1963; Hill and Rice, 1972), allow to study the effective mechanical behavior of composites...
with general and periodic microstructures (Hashin and Shtrikman, 1963; Hill, 1963; Bensoussan et al., 1978; Sanchez-Palencia, 1978; Mura, 1987; Murat and Tartar, 1995; Kouznetsova et al., 2001; Miehe and Koch, 2002; Su et al., 2011) by considering scales separation for the microstructure and the macrostructure.

Reviews of different multi-scales approaches are available in the literature (Kanouté et al., 2009; Pindera et al., 2009; Charalambakis, 2010; Geers et al., 2010). Homogenization of composites with linear as well as nonlinear constituents has been studied extensively (Suquet, 1987; Guedes and Kikuchi, 1990; Terada and Kikuchi, 1995; Ponte Castañeda, 1996; Smit et al., 1998; Michel et al., 1999; Feyel and Chaboche, 2000; Terada and Kikuchi, 2001; Miehe, 2002; Aboudi et al., 2003; Asada and Ohno, 2007; Yvonnet et al., 2009, among many others). Several authors have examined the mechanical response of composites considering finite strains (Hill, 1972; Nemat-Nasser, 1999; Miehe et al., 1999; Kouznetsova et al., 2002; Miehe, 2003; Costanzo et al., 2005; Hirschberger et al., 2008; Temizer and Wriggers, 2008; Ricker et al., 2010; McBride et al., 2012; Javili et al., 2013).

Within the framework of the present manuscript, the response of the macroscopic problem is governed by a model of finite elasticity. The constitutive response of a macroscopic material point is obtained from the (numerical) solution of a representative problem at the microscopic scale. The macro- and micro-scale problems satisfy the assumption of scales separation.

In the current study we analyze the overall response of composites with magneto-mechanical constituents and we present a homogenization framework and its numerical implementation, that accounts for large deformations and is applicable to any microstructural geometry and linear or nonlinear magneto-mechanical constitutive law. Very recently, homogenization approaches have been considered for the magneto-mechanical response of magnetorheological elastomers (Borcea and Bruno, 2001; Yin et al., 2002; Wang et al., 2003; Yin et al., 2006; Ponte Castañeda and Galipeau, 2011; Galipeau and Ponte Castañeda, 2012). In these efforts the effective response is studied under certain assumptions on the constitutive behavior, the boundary conditions and the micro-geometry. In our own contribution (Chatzigeorgiou et al., 2013) the general principles and conditions for the consistent homogenization of such composites are presented. In that theoretical study we identified appropriate boundary conditions in the material formulation and checked if the homogenized response was compatible with the spatial formulation in terms of volume averaging and macroscopic quantities. We have proven that the use of kinematic and magnetic field potentials instead of kinetic field and magnetic induction potentials provides a more appropriate homogenization process, in which averaging over the RVE in the material and spatial description renders equivalent counterparts. This logical, and rather natural, choice is employed in this work which serves additionally as an advantage in the numerical implementation procedure. Chatzigeorgiou et al. (2013) describe a general theoretical homogenization framework for finite deformation magneto-elasticity with regard to material and spatial descriptions. The current contribution proposes a computational framework for periodic homogenization of magneto-elastic materials with arbitrary geometry in the microstructure.

1.3. Structure of the present work

The structure of the paper is the following: In Section 2 we describe the theoretical framework by presenting the field variables, the balance equations and the constitutive laws governing both the micro- and the macro-problem. Section 3 introduces the numerical scheme for solving the two-scale problems and provides the tangents for an FE2 procedure. A magneto-mechanical constitutive model that accounts for large deformation processes and is used for our numerical examples is presented in Section 4. The theory is elucidated next via a series of numerical examples in Section 5 performed using the finite element method. The numerical examples include two-dimensional fiber reinforced composites as well as three-dimensional particle reinforced composites. Next, in Section 6, the developed and numerically tested framework is employed for an application to a magnetorheological elastomer with iron particles. Finally, Section 7 concludes the paper.

2. Problem definition

The objective of this section is to study composites with periodic microstructure and magneto-mechanical behavior. In homogenization theory it is typical to consider two separate scales, the macro-scale, which describes the continuum body, and the micro-scale, which describes the representative volume element (RVE) of the microstructure. As it is depicted in Fig. 1, both macro-scale and micro-scale can be expressed in material or in spatial configuration. Detailed expositions on nonlinear continuum mechanics can be found in Truesdell and Noll (2004), Gurtin et al. (2009), Marsden and Hughes (1994), Ogden (1997), Maugin (2010), among others. With regard to electromagnetism, extensive discussion can be found in Pao (1978), Kovetz (2000) and Griffiths (2013). In the current work we focus only on the material configuration. The theoretical aspects of the homogenization procedure have been developed in Chatzigeorgiou et al. (2013) and in this section.
we present the necessary parts of the theory that are essential for the current work.

2.1. Micro-scale

In the micro-scale we consider that the RVE in the undeformed (material) configuration occupies the space \( B_0 \) with volume \( V_0 \), bounded by the surface \( \partial B_0 \) (Fig. 1). The RVE consists of at least two materials, which are separated by coherent interphases. The normal vector to the boundary surfaces \( \partial B_0 \) is denoted \( \mathbf{N} \).

Following standard continuum mechanics arguments, we consider a one to one correspondence between the position vector \( \mathbf{x} \) of a point in the spatial control region \( B_l \) and the position vector \( \mathbf{X} \) of the point in the material control region \( B_0 \), which are related through the nonlinear deformation maps \( \mathbf{x} = \mathbf{Y}(\mathbf{X}) \) and \( \mathbf{X} = \mathbf{y}(\mathbf{x}) \). The deformation gradient \( \mathbf{F} \) is connected with the deformation map \( \mathbf{Y} \) through the relation

\[
\mathbf{F} = \text{Grad} \mathbf{Y}.
\]

With regard to magneto-statics, we ignore any free current density in the micro-scale and we identify the scalar magnetic potential \( \psi(\mathbf{X}) \) in the undeformed configuration. The Lagrangian magnetic field \( \mathbf{B}_l \) is connected with the scalar magnetic potential through the relations

\[
\mathbf{H} = \text{Grad} \psi, \quad \mathbf{B}_l = \mathbf{B}_0 + \mathbf{B}_l = \mathbf{B}_0 + \mathbf{B}_l = \mathbf{B}_0 + \mathbf{B}_l.
\]

In this work we prefer to use the potential \( \psi \) instead of the traditional vector magnetic potential, connected with the magnetic induction, in order to guarantee that the magneto-mechanical process has meaningful space averages in both undeformed and deformed configurations (for further details see Chatzigeorgiou et al., 2013). Note that, in contrast to Eq. (2), the magnetic field is frequently defined in the literature as the negative of the scalar magnetic potential gradient (e.g. Steigmann, 2004).

The total Piola stress tensor \( \mathbf{P} \) is considered as the sum of the mechanical stress, the Maxwell stress and the stress due to magnetic flux. The conservation of magnetic flux in terms of the magnetic field is written

\[
\text{Div} \mathbf{B} = 0 \quad \text{in} \quad B_0.
\]

Moreover, the conservation of magnetic flux in terms of the magnetic induction \( \mathbf{B}_l \) is written

\[
\text{Div} \mathbf{B}_l = 0 \quad \text{in} \quad B_0.
\]

Energetic considerations (Dorfmann and Ogden, 2004; Vu and Steinmann, 2007) allow us to identify the constitutive relations for the Piola stress \( \mathbf{P} \) and the magnetic induction \( \mathbf{B}_l \) in terms of the deformation gradient and the magnetic field through an energy functional \( \Psi(\mathbf{F}, \mathbf{H}) \),

\[
\mathbf{P} = \mathbf{P}(\mathbf{F}, \mathbf{H}) = \frac{\partial \Psi(\mathbf{F}, \mathbf{H})}{\partial \mathbf{F}}, \quad \mathbf{B}_l = \mathbf{B}_0(\mathbf{F}, \mathbf{H}) = -\frac{\partial \Psi(\mathbf{F}, \mathbf{H})}{\partial \mathbf{H}}.
\]

In the absence of mechanical body forces, the macroscopic equilibrium equation is written in terms of the macroscopic Piola stress \( \mathbf{P} \) as

\[
\text{Div} \mathbf{P} = 0 \quad \text{in} \quad B_0, \quad \text{Div} \mathbf{B} = 0 \quad \text{in} \quad B_0.
\]

Similarly to the microscopic problem, the macroscopic magneto-mechanical constitutive relations are conceptually identified with the help of an energy functional \( \Psi(\mathbf{F}, \mathbf{B}_l) \),

\[
\mathbf{P} - \mathbf{P}(\mathbf{F}, \mathbf{B}_l) = \frac{\partial \Psi(\mathbf{F}, \mathbf{B}_l)}{\partial \mathbf{F}}, \quad \mathbf{B}_l = \mathbf{B}_0(\mathbf{F}, \mathbf{B}_l) = -\frac{\partial \Psi(\mathbf{F}, \mathbf{B}_l)}{\partial \mathbf{B}_l}.
\]

The connection between the scales require volume averages of the macroscopic variables. In the homogenization framework we consider that the macroscopic variables are equal to the volume average of their microscopic counterparts over the undeformed RVE. Thus, using the divergence theorem, \( \mathbf{F}, \mathbf{P}, \mathbf{B}_l, \mathbf{B}_0 \) are given by Chatzigeorgiou et al. (2013)

\[
\mathbf{F} := \frac{1}{V_0} \int_{B_0} \mathbf{F} \mathrm{d}V, \quad \mathbf{p} := \frac{1}{V_0} \int_{B_0} \mathbf{p} \mathrm{d}V, \quad \mathbf{B}_l := \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V.
\]

where \( \mathbf{T} \) denotes the traction and \( \mathbf{N} \) the magnetic flux in Lagrangian configuration.

2.2. Macro-scale

In the macro-scale we consider a continuum body that in the material configuration occupies the space \( B_0 \) with boundary surface \( \partial B_0 \). The position vector \( \mathbf{x} \) of a point in the spatial configuration is described in terms of the position vector \( \mathbf{X} \) of the point in the material configuration by the nonlinear spatial motion map \( \mathbf{x} = \mathbf{Y}(\mathbf{X}) \) and the deformation is characterized by the spatial motion macroscopic deformation gradient \( \mathbf{F} \),

\[
\mathbf{F} = \text{Grad} \mathbf{Y}.
\]

In the absence of mechanical body forces, the macroscopic equilibrium equation is written in terms of the macroscopic Piola stress \( \mathbf{P} \) as

\[
\text{Div} \mathbf{P} = 0 \quad \text{in} \quad B_0,
\]

while the conservation of the macroscopic magnetic flux, in terms of the macroscopic magnetic induction \( \mathbf{B}_l \), is written

\[
\text{Div} \mathbf{B}_l = 0 \quad \text{in} \quad B_0.
\]

In this framework, \( \mathbf{Z} \) and \( \mathbf{z} \) are periodic, while the traction vector \( \mathbf{T} \) are anti-periodic. The average of the energy functional increment, \( \delta \Psi \), is thus determined from equations (5) as

\[
\frac{1}{V_0} \int_{B_0} \delta \psi \mathrm{d}V = \frac{1}{V_0} \int_{B_0} \mathbf{P} : \delta \mathbf{F} - \frac{1}{V_0} \int_{B_0} \mathbf{B}_l : \delta \mathbf{H} \mathrm{d}V
\]

\[
\quad + \mathbf{P} : \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V - \mathbf{P} : \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V - \mathbf{B}_l : \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V
\]

\[
= \mathbf{P} : \delta \mathbf{F} - \mathbf{P} : \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V
\]

\[
= \delta \Psi + \frac{1}{V_0} \int_{B_0} \mathbf{P} : \mathbf{B}_l \mathrm{d}V
\]

\[
- \frac{1}{V_0} \int_{B_0} \mathbf{B}_l \mathrm{d}V,
\]

where we used the definitions (10) for the macroscopic variables and the constitutive law (9). Using the divergence theorem and the equilibrium Eq. (3), we can write
\[
\int_{\Omega_0} \mathbf{P} \cdot \text{Grad} \delta \mathbf{Z} \, dV = \int_{\Omega_0} \text{Div}(\delta \mathbf{Z} \cdot \mathbf{P}) \, dV = \int_{\partial \Omega_0^p} \mathbf{T} \cdot \delta \mathbf{Z} \, dA = 0, \tag{13}
\]

since \(\delta \mathbf{Z}\) is periodic and \(\mathbf{T}\) is anti-periodic. Similarly, the magnetic flux conservation (4) and the divergence theorem lead to
\[
\int_{\Omega_0} \mathbf{B} \cdot \text{Grad} \delta \mathbf{Z} \, dV = \int_{\Omega_0} \text{Div}(\delta \mathbf{Z} \cdot \mathbf{B}) \, dV = \int_{\partial \Omega_0^m} \mathbf{T} \cdot \delta \mathbf{A} \, dA = 0, \tag{14}
\]

since \(\delta \mathbf{Z}\) is periodic and \(\mathbf{T}\) is anti-periodic. Thus, Eq. (12) states that
\[
\frac{1}{V_0} \int_{\Omega_0} \delta \mathbf{Y} \, dV = \delta \mathbf{\Psi}, \tag{15}
\]

which is the Hill–Mandel condition (the macroscopic energy functional increment is equal to the average of its microscopic counterpart). Eqs. (13) and (14) represent the weak form of the mechanical and magnetic micro-scale problems, respectively.\(^1\)

### 3. Numerical implementation

In this section we present the numerical implementation of the homogenization problem. Initially we develop a general FE formulation for solving the magneto-mechanical problem. Then we particularize this formulation for implementation in the homogenization framework.

#### 3.1. FE formulation

Ignoring inertia and body forces, the balance of linear momentum in weak form reads:\(^2\)
\[
\int_{\Omega_0} \mathbf{P} \cdot \text{Grad} \mathbf{Y} \, dV - \int_{\partial \Omega_0^p} \delta \mathbf{Y} \cdot \mathbf{T} \, dA = 0 \quad \forall \delta \mathbf{Y} \in \mathbf{H}_0^1(\Omega_0), \tag{16}
\]

where \(\mathbf{T} := \mathbf{P} \cdot \mathbf{N}\) denotes the prescribed traction (Neumann boundary condition) on a portion of the surface \(\partial \Omega_0^m \subset \partial \Omega_0\) upon which the traction, and not the motion, is prescribed.

In a near-identical fashion, the weak form of the magneto-static problem can be derived by testing the local conservation of magnetic flux with a scalar test function \(\delta \mathbf{Y} \in \mathbf{H}_0^1(\Omega_0)\) and then integrating the result over the corresponding domains in the material configuration. The global weak form of the magnetic balance equation reads
\[
\int_{\Omega_0} \mathbf{B} \cdot \text{Grad} \delta \mathbf{Y} - \int_{\partial \Omega_0^m} \delta \mathbf{Y} \cdot \mathbf{T} \, dA = 0 \quad \forall \delta \mathbf{Y} \in \mathbf{H}_0^1(\Omega_0), \tag{17}
\]

where \(\mathbf{T} := \mathbf{B} \cdot \mathbf{N}\) denotes the prescribed normal magnetic flux on a portion of the surface \(\partial \Omega_0^m \subset \partial \Omega_0\) upon which the normal magnetic flux, and not magnetic scalar potential, is prescribed.

Next the weak forms (16) and (17) are discretized. The spatial discretization of the problem domain is performed using the Bubnov–Galerkin finite element method. The reference domain \(\Omega_0\) is discretized into a set of bulk elements \(\mathbf{B}^{0j}_{ij} \approx \bigcup_{j=1}^{n_{\text{B}}^0} \mathbf{B}^{0j}_{ij}\), where \(n \sim \text{bel}\) denotes the number of bulk elements. The geometry and magnetic potential of the bulk are approximated as a function of the natural coordinates \(\xi \in [-1,1]^p\) assigned to the bulk with \(pD\) denoting the problem dimension. Standard interpolations according to the isoparametric concept are employed as follows:
\[
\mathbf{X}|_{\mathbf{B}^{0j}_{ij}} \approx \mathbf{X}^j(\xi) = \sum_{i=1}^{n_n} N_i(\xi) \mathbf{X}^i.
\]
\[
\mathbf{Y}|_{\mathbf{B}^{0j}_{ij}} \approx \mathbf{Y}^j(\xi) = \sum_{i=1}^{n_n} N_i(\xi) \mathbf{Y}^i,
\]

where the shape functions of the bulk elements at their local node \(i\) are denoted as \(N_i\). The bulk elements consist of \(n_n\) nodes. The discrete form of the balance Eqs. (16 and 17) are now obtained by applying the spatial approximation.\(^3\) The fully discrete forms of mechanical and magnetic residuals associated with the global node \(l\) are defined by
\[
\mathbf{R}_l^\mathbf{Y} := \int_{\Omega_0} \mathbf{P} \cdot \text{Grad} \mathbf{Y}^l \, dV \tag{18}
\]
\[
\mathbf{R}_l^\mathbf{B} := \int_{\Omega_0} \mathbf{B} \cdot \text{Grad} \mathbf{B}^l \, dV \tag{19}
\]

The total global residuals (18) and (19) must vanish. The global residual consisting of the mechanical residual and magnetic residual vectors, \(\mathbf{R}_l^\mathbf{Y}\) and \(\mathbf{R}_l^\mathbf{B}\), respectively, take the form
\[
\begin{pmatrix}
\mathbf{R}_l^\mathbf{Y} \\
\mathbf{R}_l^\mathbf{B}
\end{pmatrix}
\]

where \(n_n\) denotes the total number of nodes.

The fully-discrete coupled nonlinear system of governing equations can finally be stated as follows:
\[
\mathbf{R} = \mathbf{R}(\mathbf{d}) = 0, \tag{21}
\]

where \(\mathbf{d}\) is the unknown global vector of deformation maps \(\mathbf{Y}\) and scalar potential \(\mathbf{V}\) defined by
\[
\mathbf{d} = \begin{bmatrix}
\mathbf{d}_\mathbf{Y} \\
\mathbf{d}_\mathbf{V}
\end{bmatrix}
\]

In order to solve the system (21), i.e. to find \(\mathbf{d}\) such that \(\mathbf{R}\) vanishes, a Newton–Raphson scheme is utilized. The consistent linearization of the resulting system of equations yields
\[
\frac{\partial \mathbf{R}}{\partial \mathbf{d}} \frac{d\mathbf{d}}{d\mathbf{d}} \mathbf{d}_k \approx \mathbf{0} \quad \text{and} \quad \mathbf{d}_{k+1} = \mathbf{d}_k + \Delta \mathbf{d}_k, \tag{22}
\]

where \(k\) is the iteration step. Solving (22) yields the iterative increment \(\Delta \mathbf{d}_k\), and consequently \(\mathbf{d}_{k+1}\). The corresponding algorithmic tangent stiffness matrix is defined by
\[^1\text{The derivations from (9)–(15) proves that the Hill condition is satisfied. This condition states the energetic equivalence between the micro- and macro-scale.}
\[^2\text{In the development of the formulation we use the micro-scale notation for simplicity. In the macro-scale one should utilize the corresponding spaces, surfaces and variables.}
\[^3\text{In what follows, for the sake of brevity, the Neumann boundary conditions are omitted, keeping in mind that, in periodic homogenization, the terms inside the integrals over } \partial \Omega_0^p \text{ and } \partial \Omega_0^m \text{ vanish identically due to the anti-periodicity condition on tractions and magnetic fluxes.}
which can be decomposed into mechanical and magnetic contributions as follows

\[
K = \begin{bmatrix} K_{YY} & K_{YV} \\ K_{VY} & K_{VV} \end{bmatrix},
\]

where

\[
K_{YY} = \frac{\partial R_{YY}}{\partial Y_Y}, \quad K_{YV} = \frac{\partial R_{YY}}{\partial Y_V}, \quad K_{VY} = \frac{\partial R_{VV}}{\partial Y_Y}, \quad K_{VV} = \frac{\partial R_{VV}}{\partial Y_V}.
\] (23)

These sub-matrices are computed according to the results summarized in Table 1.

### 3.2. Homogenization procedure

A complete homogenization scheme requires the solution of the macro-scale and the micro-scale simultaneously. In Fig. 2 we present an iterative scheme for homogenization in the material configuration. From the macro-scale analysis the macroscopic deformation gradient and magnetic field are obtained, which are used in the RVE problem, in order to compute the micro-scale variables and the correct macroscopic Piola stress and magnetic induction. Moreover, the RVE problem provides information in order to compute the effective tangent moduli, which are used in the macro-scale analysis.

### 3.3. Microscale problem and effective tangent moduli

It is enlightening to elaborate on the numerical procedure to obtain effective tangent moduli required for the FE² framework. For further details regarding the computation of the macroscopic tangents for multiscale mechanical problems see Terada and Kikuchi (2001) and Temizer and Wriggers (2008) and references therein.

For the RVE problem we utilize the decomposition of \( Y \) and \( Y \) introduced in Eqs. (11). Since the macro-scale deformation gradient \( F \) and magnetic field \( H \) are provided from the macro-scale analysis, they are considered constant on the RVE problem. Thus in Eq. (22) the unknown increment \( \Delta d \) is substituted by \( \Delta d \), where

\[
\Delta d = \begin{bmatrix} \Delta d_{y} \\ \Delta d_{z} \end{bmatrix}
\]

with \( \Delta d_{y} \) = \( \frac{\partial Z^{1}}{\partial Y_Y}, \Delta d_{z} = \frac{\partial Z^{2}}{\partial Y_V} \), \( \Delta d_{z} = \frac{\partial Z^{3}}{\partial Y_V} \), and \( \Delta d_{z} = \frac{\partial Z^{4}}{\partial Y_Y} \). Once the Newton–Raphson scheme converges, then for a specific macroscopic deformation gradient \( F \) and magnetic field \( H \) we obtain the deformation gradient \( F \), the Piola stress \( P \) and the tangent moduli \( \frac{\partial R_{YY}}{\partial Y_Y} \) and \( \frac{\partial R_{YY}}{\partial Y_V} \) at each Gauss point of the discretized RVE. The macroscopic Piola stress \( P \) and the macroscopic magnetic induction \( H \) are then computed from equations (10), and (10) respectively.

With regard to the effective tangent moduli we follow the approach of Terada and Kikuchi (2001). When the Newton–Raphson scheme converges, the residual \( R \) is assumed identically zero and we consider that the increments of the macroscopic deformation gradient and the magnetic induction are given quantities. Thus, Eq. (22) is linear with respect to the increments of the fluctuation functions and has a solution of the form

\[
\Delta Z = \left[ u_{pq} A_{pq} F_{pq} + [u_{0}]_{pq} A_{pq} [F_{pq}] \right], \quad \Delta Z = \left[ u_{pq} A_{pq} [F_{pq}] + [u_{0}]_{pq} A_{pq} [F_{pq}] \right],
\] (24)

In the above expressions \( u_{pq} \) and \( [u_{0}]_{pq} \) are periodic vector valued functions, while \( [u_{pq}] \) and \( [u_{0}]_{pq} \) are periodic scalar valued functions.

Taking into account that the macroscopic deformation gradient and magnetic field do not depend on the micro-coordinates, for arbitrary \( \Delta F \) and \( \Delta H \) we have the system of equations

\[
\left[ \frac{\partial R_{YY}}{\partial Y_Y} \right]_{pq} + K \cdot [\gamma]_{pq} = 0,
\] (25)

where \( K \) is the total stiffness matrix of the last iteration step of the RVE problem,\(^4\)

\[
[\gamma]_{pq} = \left[ u_{pq} \right]_{pq}, \quad [\gamma]_{pq} = \left[ u_{0} \right]_{pq}, \quad [D]_{pq} = \left[ D_{pq} \right]_{pq},
\]

and \( [D]_{pq} = \left[ D_{pq} \right]_{pq} \),

with

\[
\frac{\partial R_{YY}}{\partial Y_Y} := \int_{\Omega} \frac{\partial P}{\partial \left[ \frac{\partial F_{pq}}{\partial Y_Y} \right]} \cdot \text{Grad} N \, dV, \quad \frac{\partial R_{YY}}{\partial Y_V} := \int_{\Omega} \frac{\partial P}{\partial \left[ \frac{\partial F_{pq}}{\partial Y_Y} \right]} \cdot \text{Grad} \, dV.
\]

From the definitions of the macroscopic Piola stress and magnetic induction (10), the decompositions (24), the relations (1), (2), and the constitutive relations (9) we have

\[
\Delta P = \frac{1}{V_0} \int_{\Omega} \Delta P \, dV = \frac{1}{V_0} \int_{\Omega} \frac{\partial P}{\partial \left[ \frac{\partial F}{\partial Y} \right]} \cdot \Delta \frac{\partial F}{\partial Y} \frac{dV}{\Delta \frac{\partial F}{\partial Y}} = \Delta P = \frac{\partial P}{\partial \left[ \frac{\partial F}{\partial Y} \right]} \cdot \Delta \frac{\partial F}{\partial Y}.
\]

\(^4\) We note that, in order to keep the same stiffness matrix \( K \) between the microscale problem and the effective tangent moduli calculations, the assembly of \( [D]_{pq} \) and \( [D]_{pq} \) in the finite element implementation needs to be written in the same format as the residual \( R \).

---

**Table 1**

| Consistent linearization of the residual and the components of the algorithmic tangent, \( \delta \) and \( \delta \) vanish due to the fact that the mechanical deformation map \( Y \) and the magnetic scalar potential \( Y \) are independent. The symbol \( \cdot \) denotes the non-standard double contraction of a fourth-order tensor \( D \) and a second-order tensor \( A \) with components \( D_{ij} A_{ij} - D_{ij} A_{ij} \),

\[
\frac{\partial R_{YY}}{\partial Y_Y} = \int_{\Omega} \frac{\partial P}{\partial Y_Y} \cdot \text{Grad} N \, dV = \int_{\Omega} \frac{\partial P}{\partial Y_Y} \cdot \text{Grad} N \, dV
\]

\[
\frac{\partial R_{YY}}{\partial Y_V} = \int_{\Omega} \frac{\partial P}{\partial Y_V} \cdot \text{Grad} N \, dV = \int_{\Omega} \frac{\partial P}{\partial Y_V} \cdot \text{Grad} N \, dV
\]

\[
\frac{\partial R_{YY}}{\partial Y_Y} = \int_{\Omega} \frac{\partial P}{\partial Y_V} \cdot \text{Grad} N \, dV
\]

---

**Fig. 2.** Computational homogenization scheme.
Table 2  
Numerically obtained effective tangent moduli.

\[
\begin{align*}
\frac{\Delta P}{\Delta F} & = - \frac{1}{V_0} \int_{V_0} \left[ \nabla \cdot \mathbf{w}_1 + \frac{\partial}{\partial F} \nabla \cdot \mathbf{w}_2 \right] dV \\
\frac{\Delta P}{\Delta F} & = - \frac{1}{V_0} \int_{V_0} \left[ \nabla \cdot \mathbf{w}_1 + \frac{\partial}{\partial F} \nabla \cdot \mathbf{w}_2 \right] dV \\
\frac{\Delta T}{\Delta F} & = - \frac{1}{V_0} \int_{V_0} \left[ \nabla \cdot \mathbf{w}_1 + \frac{\partial}{\partial F} \nabla \cdot \mathbf{w}_2 \right] dV \\
\frac{\Delta T}{\Delta F} & = - \frac{1}{V_0} \int_{V_0} \left[ \nabla \cdot \mathbf{w}_1 + \frac{\partial}{\partial F} \nabla \cdot \mathbf{w}_2 \right] dV
\end{align*}
\]

where the tangents are given in Table 2. Thus the macroscopic tangent moduli can be approximated as

\[
\frac{\partial P}{\partial F} \approx \frac{\Delta P}{\Delta F}, \quad \frac{\partial T}{\partial F} \approx \frac{\Delta T}{\Delta F}, \quad \frac{\partial B}{\partial F} \approx \frac{\Delta B}{\Delta F}, \quad \frac{\partial \theta}{\partial F} \approx \frac{\Delta \theta}{\Delta F}.
\]  

(26)

4. Magneto-mechanical constitutive model

Before proceeding to the numerical examples it is essential to choose appropriate constitutive laws for connecting the microscopic magneto-mechanical variables. In this work we assume an isotropic elastic material with linear and isotropic magnetic response. The energy potential for such material can be decomposed into a mechanical and a magnetic part,

\[
\Psi(F, H) = \Psi_{\text{mech}}(F) + \Psi_{\text{mag}}(F, H) = \frac{1}{2} \lambda_1 [F : F - 3 - 2 \ln J] + \frac{1}{2} \lambda_2 [J - 1] F : H - C^{-1} : H,
\]

(27)

where \(\lambda_1\) and \(\lambda_2\) are the Lamé parameters and \(\mu\) is the magnetic permeability.\(^5\) Also, \(C = F : F\) denotes the right Cauchy-Green deformation tensor. From Eqs. (5) 1 the Piola stress \(P\) reads

\[
P = \frac{\partial \Psi_{\text{mech}}(F)}{\partial F} + \frac{\partial \Psi_{\text{mag}}(F, H)}{\partial F}
= \frac{\partial}{\partial F} \left( \frac{1}{2} J \left[ \lambda_1 F : F - 3 - 2 \ln J \right] + \frac{1}{2} J \left[ \lambda_2 J - 1 \right] F : H - C^{-1} : H \right)
= \lambda_1 F + \left[ \lambda_2 J - \lambda_1 F^{-1} - \frac{1}{2} J [H \otimes H] : M \right],
\]

(28)

with \(M\) defined as

\[
M := C^{-1} \otimes F^{-1} + \frac{\partial C^{-1}}{\partial F}
= [F^{-1} \otimes F^{-1}] \otimes F^{-1} + [F^{-1} \otimes F^{-1}] \cdot F^{-1} - [F^{-1} \otimes F^{-1}].
\]

(29)

The symbols \(\otimes\) and \(\otimes\) denote the two non-standard tensor products of two second-order tensors \(A\) and \(B\) with components \(A \otimes B = |A|B\) and \(A \otimes B = |A|B\) respectively. Note that the tensor \(M\) has a minor symmetry property \(|M|_{ijkl} = |M|_{ijlk}\). Furthermore, the magnetic induction \(B\) can be computed from (5) as

\[
B = - \frac{\partial \Psi_{\text{mag}}(F, H)}{\partial H} = - \frac{\partial}{\partial H} \left( \frac{1}{2} J [H \otimes H] \cdot C^{-1} \cdot H \right)
= \frac{1}{2} J [C^{-1} \cdot H \otimes H + H \otimes C^{-1}] \quad \text{and due to the symmetry of } C
= \mu C^{-1} \cdot H
\]

(30)

Note that using standard transformations between spatial and material magnetic variables

\[
b = [\text{Det}(F)^{-1} \otimes F] \cdot b, \quad h = H \cdot F^{-1},
\]

(31)

where \(b\) and \(h\) are the Eulerian counterparts of \(B\) and \(H\), the following classical relation holds

\[
b = \mu h.
\]

(32)

For the selected energy potential the tangent moduli \(\partial P / \partial F, \partial P / \partial H, \partial B / \partial F\) and \(\partial B / \partial H\), which are necessary for the numerical analysis, are obtained in analytical form as follows

\[
\frac{\partial P}{\partial F} = \left( \frac{\partial}{\partial F} \left( \lambda_2 \ln J - \lambda_1 F^{-1} + \lambda_1 F + \frac{\partial}{\partial F} \left( \frac{1}{2} \mu [H \otimes H] : M \right) \right) \right)
= \lambda_2 F^{-1} \otimes F^{-1} + \lambda_1 \frac{\partial}{\partial F} \left( \frac{1}{2} \mu [H \otimes H] : M \right)
+ \lambda_1 \left[ J - 1 \right] F^{-1} \otimes F^{-1} - \frac{1}{2} \mu [H \otimes H] : M
\]

(33)

with

\[
\left[ \frac{\partial M}{\partial F} \right]_{ijkl} = - \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} - \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} - \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} + \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} + \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} + \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} + \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl} + \left[ F^{-1} \otimes F^{-1} \otimes F^{-1} \right]_{ijkl}.
\]

(34)

\[
\frac{\partial P}{\partial H} = \frac{\partial}{\partial H} \left( \frac{1}{2} \mu [H \otimes H] : M \right) = - \mu [M^t \otimes H],
\]

(35)

\[
\frac{\partial B}{\partial F} = \frac{\partial}{\partial F} (\mu C^{-1} \cdot H) = \mu \left[ C^{-1} \cdot H \otimes F^{-1} + H \otimes \frac{\partial C^{-1}}{\partial F} \right]
= \mu \left[ [F^{-1} \otimes F^{-1} \otimes F^{-1} \otimes F^{-1}] \cdot F^{-1} + [F^{-1} \otimes F^{-1} \otimes F^{-1} \otimes F^{-1}] \right],
\]

(36)

\[
\frac{\partial B}{\partial H} = \frac{\partial}{\partial H} (\mu C^{-1} \cdot H) = \mu C^{-1}.
\]

(37)

The symbol \(\cdot\) denotes the non-standard single contraction of a vector \(a\) and a fourth-order tensor \(D\) with components \(a \cdot D_{ijkl} = a_i D_{ijkl}\). Moreover, \(M^t\) denotes the fourth order tensor with components \(|M^t|_{ijkl} = |M|_{ijlk}\).

5. Numerical examples

The objective of this section is to elucidate the theory presented in the previous sections. In particular, the role of the magnetic field \(H\) on the overall response of the microstructure is studied. Two-dimensional examples for the case of simple-extension and simple-shear are presented in Sections 5.1 and 5.2. Attention is paid to the influence of the mechanical as well as the magnetic material parameters of the inclusion compared to the matrix. Furthermore,
the geometrically nonlinear behavior of the overall response of the material, without loss of generality, is shown for the case of simple-extension. Chatzigeorgiou et al. (2013) show analytically that for periodic boundary conditions the use of the magnetic field instead of the magnetic induction potential provides a more appropriate homogenization process in which averaging over the RVE in the material and spatial descriptions renders equivalent counterparts. That is, space averages in the material and spatial configurations become meaningful. This issue is observed numerically which in turn confirms the efficiency and accuracy of the finite element code. Finally, it should be emphasized that neither the theory presented in this manuscript nor the proposed finite element framework is limited to two dimensions. In Section 5.4 three-dimensional numerical examples, in nature similar to the two-dimensional ones, are briefly studied for the sake of completeness. All examples are solved using our in-house finite element code in the C++ syntax. The solution procedure is robust and for all examples shows asymptotically the quadratic rate of convergence associated with the Newton-Raphson scheme.

### 5.1. Fiber composite under stretching and magnetic field

Consider a composite consisting of a matrix material and long fibers distributed aligned and uniformly, in tetragonal arrangement, inside the matrix. In this case the microstructure can be represented with a two-dimensional unit cell illustrated in Fig. 3.

The aim of this specific example is to illustrate the computational efficiency and robustness of the finite element code and to perform parametric studies that demonstrate the influence of the applied magnetic field on the mechanical response of the composite. For the foregoing parametric studies the material parameters of the matrix are assumed to be:

- shear modulus $\mu_{\text{mat}} = 8$.
- Poisson’s ratio $\nu_{\text{mat}} = 0.3$.
- magnetic permeability $\mu_{\text{mat}} = 0.001$.

For the fiber properties the same Poisson’s ratio as for the matrix is chosen, and three cases for the shear modulus ($\lambda_{\text{fib}} = 0.8$, $8.8$, $80$) and three cases for the magnetic permeability ($\mu_{\text{fib}} = 0.0001$, $0.001$, $0.01$) are examined. All the nine possible combinations are gathered in Table 3. Periodic boundary conditions for displacements and magnetic potentials are imposed on the sides of the unit cell together with anti-periodic tractions and magnetic fluxes.

Next, for all cases, a uniaxial stretching (simple-extension) in the $x$-direction as macroscopic deformation together with a magnetic field in the $x$-direction are imposed. The macroscopic deformation gradient tensor and the magnetic field vector are therefore:

$$\mathbf{F} = \begin{bmatrix} 1.05 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}.$$  

The unit cell is discretized using 720 bi-quadratic quadrilateral (9-Node) finite elements. The mesh quality is shown in Fig. 3. The total number of nodes was 2929. The radius of the fiber is 0.2 resulting in a fiber volume fraction of ca. 12.5%.

In Fig. 4, for each fiber type, the numerical results for the microscopic Piola stress, normalized by the macroscopic (averaged) stress are illustrated. We observe that the fiber inclusion takes its maximum deformation in the case of mechanically weak and magnetically strong fiber, compared to the matrix. Additionally, when the fiber is mechanically equivalent or stronger compared to the matrix, the macroscopic stress tends to increase with the increase of the magnetic fiber properties. When the fiber is mechanically weaker than the matrix, the macroscopic stress takes its smaller value for magnetically strong fiber. Another interesting point is that the microscopic stress at the center of a, both mechanically and magnetically, strong fiber is almost twice the macroscopic stress, while the microscopic stress at the center of a, both mechanically and magnetically, weak fiber is almost 17% of its macroscopic counterpart.

Fig. 5 shows the obtained microscopic magnetic induction in the $x$ direction, normalized by the macroscopic (averaged) magnetic induction. We observe that the macroscopic magnetic induction tends to increase with the increase of the magnetic fiber properties, independently of the fiber mechanical behavior. Moreover, the microscopic magnetic induction at the center of a, both mechanically and magnetically, strong fiber is 70% higher than the macroscopic magnetic induction, while the microscopic magnetic induction at the center of a, both mechanically and magnetically, weak fiber is almost 17% of its macroscopic counterpart.

Fig. 6 shows the distribution of the scalar magnetic potential. We observe that a magnetically weak fiber “attracts” the magnetic lines, while a magnetically strong fiber “repels” the magnetic lines. In order to better understand this phenomena, consider a similar mechanical problem. When the inclusion is mechanically stronger than the matrix, the inclusion deforms less than the matrix and therefore, the majority of the deformation is stored in the matrix. Note that although a magnetically strong inclusion repels the magnetic lines, the resultant magnetic induction is mainly dominated within the inclusion.

Recall that the magnetic field vector is the gradient of the scalar magnetic potential. That is, the magnetic field vectors are normal to the magnetic potential lines (level sets) and point towards the increase of the magnetic potential. This fact is illustrated in Fig. 7 for the case with the inclusion, both mechanically and magnetically, stronger than the matrix, i.e. the bottom-right of the Fig. 6. Clearly, the density of the magnetic field vectors is higher in the inclusion due to the fact that the inclusion is ten times more magnetic than the matrix.

#### 5.1.1. Illustration of the geometrically nonlinear overall response

In this example we consider the case of a fiber which is ten times stronger than the matrix, both mechanically and magnetically, i.e. bottom-right of the Table 3. The evolutions of the macroscopic stress and the magnetic induction versus the macroscopic deformation for various values of the macroscopic magnetic field are presented in Fig. 8. The macroscopic deformation gradient tensor is

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{xx} & 0 \\ 0 & 1 \end{bmatrix}.$$
in which $F_{10}$ is the stretch. The stretch varies from the case of no stretch ($F_{10} = 1.0$) up to $F_{10} = 2.0$. It can be observed that the increase of the applied magnetic field causes an increase in stress levels and a slight decrease in the effective stiffness of the composite. On the other hand at high applied magnetic fields the magnetic induction depends strongly on the deformation level. In particular, the geometrical nonlinearity of the overall responses can be observed.

Fig. 8 presents the macroscopic (nonlinear) behavior of the composite for the boundary conditions and geometrical characteristics of the associated unit cell and can be used to identify an empirical effective constitutive law for this specific case. Repeating this procedure for different unit cells with different volume fractions and several sets of boundary conditions could render a phenomenological (macroscopic) constitutive law. Nevertheless, this purpose requires many more graphs like the one presented in Fig. 8 and exceeds the scope of this contribution.

5.2. Fiber composite under shearing and magnetic field

In this section we present the microscopic behavior of fiber reinforced composites, with the same mechanical and magnetic properties for the matrix and the fibers as those of the previous example. The composite is subject to simple-shear deformation in the $xy$-plane and a magnetic field in the $x$-direction. The macroscopic deformation gradient tensor and the magnetic field vector are thus

$$\mathbf{F} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}.$$
The distributions of the microscopic shear stress, normalized by its macroscopic (averaged) counterpart, are presented in Fig. 9. The fiber presents its maximum deformation in the case of mechanically weak and magnetically strong fiber, compared to the matrix. Moreover the macroscopic shear stress decreases with the increase of the fiber magnetic properties for all types of fiber mechanical behavior.

Fig. 10 shows, for each fiber case, the obtained microscopic magnetic induction in the x direction, normalized by the macroscopic (averaged) magnetic induction. We observe that the macroscopic magnetic induction tends to increase with the increase of the magnetic fiber properties, independently of the fiber mechanical behavior.

Finally, Fig. 11 shows the distribution of the scalar magnetic potential. Again a magnetically weak fiber “attracts” the magnetic lines, while a magnetically strong fiber “repels” the magnetic lines.

Remark. In the aforementioned examples, for both simple-extension and simple-shear loading, in the case where the magnetic and mechanical properties of the inclusion and the matrix are identical, i.e. in the center of nine combinations, the overall response of the RVE is the same as the uniform microscopic response itself.
Therefore, one can insert the prescribed macroscopic quantities $F$ and $\Pi$ in the constitutive Eqs. (28) and (30) instead of their microscopic counterparts in order to calculate precisely the macroscopic Piola stress and magnetic induction. This test has been performed and the numerical results are indeed in perfect agreement with the analytical solution which re-assures the efficiency of the finite element framework.

5.3. A fully coupled multi-level analysis

Consider a fiber composite magnetomechanical plate under plane strain condition subjected to simple extension and magnetic field boundary conditions as shown in Fig. 12. In certain cases one may need a complete FE$^2$ framework to predict the material behavior. Here, we present a fully coupled multi-level analysis and in
particular, study the convergence of the proposed scheme. The tangents are computed according to Section 3.3.

Fig. 12 shows the fiber composite of interest together with the applied macroscopic boundary conditions and material properties of the microstructure. On top and the bottom of the specimen (at the macroscale) magnetically and mechanically homogeneous Neumann-type boundary conditions are applied by setting the fluxes to zero. The specimen is fully constrained on the left side via homogeneous Dirichlet-type boundary conditions. On the right edge a displacement of 10\% in the $x$-direction is prescribed while in the $y$-direction nodes are allowed to move freely, i.e. a homogeneous Neumann boundary condition. Also, a magnetic scalar potential of 1 is prescribed on the right edge which shall be understood as a non-homogeneous Dirichlet-type boundary condition.

Fig. 13 shows the results of a fully-coupled FE$^2$ analysis together with the response of the microscale at four different points. Each point corresponds to the Gauss point located at the bottom right corner of the element and each element possesses four Gauss points. Table 4 summarizes the convergence behavior of the
(macro-) problem for different number of elements. Furthermore, Table 4 includes the convergence behavior of an analogous purely mechanical problem whereby this has been carried out by setting the magnetic permeability (at the microscale) to zero. The micro-problem is discretized by 320 bilinear quadrilateral elements and 389 nodes in total. The mesh quality can be seen in Fig. 13. From Table 4 it can be seen that firstly, one obtains the asymptotically quadratic rate of convergence associated with the Newton–Raphson scheme and secondly, the problem with only elasticity shows a better convergence behavior than the magnetoelastic one at very small residuals. This can be explained by the fact that the magnetic permeability assumes very small values compared to the elastic parameters. This leads the solver to become less sensitive with respect to the magnetic field and hence loses the precision. This is similar to the case of thermoelasticity where the thermal expansion coefficient, in general, assumes very small values compared to the mechanical parameters resulting in a less well-conditioned tangent matrix Javili and Steinmann (2011); Temizer and Wriggers (2011). In such cases one could improve the convergence behavior by precoditioners, however, for the purpose of the current work the obtained convergence is enough and studying this (purely numerical) issue is out of the scope of this paper. Also, note that

Fig. 11. Macroscopic shearing and magnetic field: Distribution of scalar magnetic potential for all combinations of mechanical and magnetic fiber properties. The undeformed configuration is indicated using a dashed (grey) line.

Fig. 12. Geometry and boundary conditions the fiber composite of interest. The fiber is magnetically and mechanically stronger than the matrix and material properties correspond to the case 9 of the example in Section 5.1.
the magnetomechanical problem has one more dimension than the mechanical one.

Table 5 summarizes the convergence behavior of the microscale problem. One could again see that the asymptotically quadratic rate of convergence is obtained at the microscale, too. Furthermore, we need to emphasize that the accuracy of the macroscale problem is generally less than the accuracy of the microscale one due to the accumulation of the round-off errors at the microscale. For this reason we stop the iterations by reaching the residual norm of 1.0ε−6 for the macroscale and 1.0ε−9 at the microscale.\footnote{In finite element computations it is customary to work with the normalized residuals, however here we provide the residuals without normalizing them so that one could use this example as a benchmark.}

5.4. Particle composite under magnetic field

The two-dimensional numerical examples of Sections 5.1 and 5.2 were designed to illustrate the effect of the magnetic field on the microscopic and consequently on the overall macroscopic response of the materials. Nevertheless, the analysis provided here is not limited to two dimensions. For the sake of completeness, we include briefly a three-dimensional example of a particle composite.

Consider a composite consisting of a matrix material and particles distributed uniformly inside, in hexahedral arrangement. In this case the microstructure can be represented with a three-dimensional unit cell illustrated in Fig. 14. The radius of the inclusion is 0.2 resulting in the inclusion volume fraction of ca. 3.35%. The elastomer has approximately the elastic modulus and Poisson’s ratio of 75 GPa and 0.5, respectively. At high values of ε, ε/, which corresponds to the case 9 of Table 3, the distribution of microscopic Piola stresses as well as magnetic inductions for two cases of simple-extension and simple-shear subject to the same macroscopic mechanical one.

The elastic modulus and Poisson’s ratio of the iron particles are approximately 211 GPa and 0.29, respectively, which correspond to Lamé constants κ = 81.78 GPa and μ = 112.94 GPa. The magnetic permeability of the inclusion is about five thousand times that of the free space, hence μ = 20000π × 10−7 N/A².

5.4. Particle composite under magnetic field

The macroscopic deformation gradient tensors for the simple-extension and simple-shear subject to the same macroscopic mechanical one.

The macroscopic deformation gradient tensors for the simple-extension and simple-shear case as well as the macroscopic magnetic field vector are assumed to be

\[
\mathbf{F}_{\text{extension}} = \begin{bmatrix} 1.05 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}_{\text{shear}} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}
\]

Since the particle is magnetically ten times stronger than the matrix, the magnetic induction assumes a higher value within the particle compared to the matrix.

6. Polymer matrix with magnetic iron particles

Equipped with the computational micro-to-macro transition framework proposed and numerically tested in the previous sections, a physically motivated three-dimensional example with realistic material parameters is studied next.

The composite of interest consists of an elastomer matrix (Elastosil LR 3003/03) and carbonyl iron particles (CIP-SQ BASF). Such a composite is often called magnetoreological elastomer, since it presents viscoelastic response. In this example we neglect any viscous effects and we concentrate our attention on the magnetoeelastic responses. It is assumed that the particles are distributed uniformly, in hexahedral arrangement, inside the matrix. In this case the microstructure can be represented with a three-dimensional unit cell illustrated in Fig. 14. The particles volume fraction for this example is ca 3.35%. The elastomer has approximately the elastic modulus of 10 MPa and Poisson’s ratio of 0.45 to allow for compressibility. This set of material parameters results in \(\lambda_{1}=3.45\) MPa and \(\lambda_{2}=31\) MPa. The magnetic permeability of the matrix is similar to that of the free space, hence \(\mu_{\text{mat}}=4\pi \times 10^{-7} \text{N/A}^2\). The elastic modulus and Poisson’s ratio of the iron particles are approximately 211 GPa and 0.29, respectively, which correspond to Lamé constants \(\lambda_{\text{part}}=81.78\) GPa and \(\lambda_{\text{part}}=112.94\) GPa. The magnetic permeability of the inclusion is about five thousand times that of the free space, hence \(\mu_{\text{part}}=20000\pi \times 10^{-7} \text{N/A}^2\).

6.1. Simple-extension test

In the proceeding analysis we examine the effective response of the composite under simple-extension and uniaxial magnetic field.

The macroscopic deformation gradient tensor and the macroscopic magnetic field vector are assumed to be

\[
\mathbf{F} = \begin{bmatrix} F_{xx} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} H_{xx} \\ 0 \end{bmatrix}
\]

The stretch \(F_{xx}\) varies from 1.0 to 1.5. We consider three cases of macroscopic deformation associated with

\[
F_{xx} = 1.0, \quad F_{xx} = 1.1 \quad \text{and} \quad F_{xx} = 1.5,
\]

in combination with four cases of macroscopic magnetic field (in A/m)

\[
|H_{xx}| = 0, \quad |H_{xx}| = 50000, \quad |H_{xx}| = 250000 \quad \text{and} \quad |H_{xx}| = 500000
\]

resulting in twelve different cases.

Figs. 16 and 17 show the distribution of the microscopic stress and magnetic field over the unit cell. The stress and magnetic field are calculated on the undeformed configuration of the RVE. In Fig. 16 we observe that under no macroscopic mechanical deformation, the macroscopic stress has a quadratic relation with the macroscopic magnetic field, as it was expected for the specific constitutive law of the constituents. On the other hand, the relative distribution of the microscopic magnetic stress is not influenced by the level of \(|H_{xx}|\). At 10% macroscopic extension, a ten times greater macroscopic magnetic field increases the macroscopic stress by approximately 5%. At high values of \(|H_{xx}|\), the microscopic stress does not remain uniform inside the iron particle. At 50% macroscopic extension the macroscopic magnetic field almost does not affect the microscopic stress and consequently its macroscopic counterpart.

Fig. 17 shows that the linear connection, in spatial description, between the magnetic induction and the magnetic field of the individual constituents leads to a nearly linear relation between the macroscopic counterparts at each macroscopic deformation case. When the RVE is subject to nonzero macroscopic magnetic field, the iron particles contribute a uniform magnetic induction. As expected, according to the first row of the Fig. 17, when no macroscopic magnetic field is applied, increasing the stretch will not produce any magnetic induction. This shall be compared with the case that no stretch is applied but, increasing the magnetic field will result in a macroscopic Piola stress, i.e. the first column of Fig. 16.

6.2. Simple-shear test

Finally, the effective response of the composite under simple-shear and uniaxial magnetic field is studied. The macroscopic deformation gradient tensor and the macroscopic magnetic field vector are assumed to be
A fiber composite magnetomechanical plate under plane strain condition subjected to simple extension and magnetic field boundary conditions at the macroscale and its microstructure. Deformation gradient, magnetic field, Piola stress and magnetic induction are given. Magnetic induction is multiplied by a factor of 1.0e3.

The shear component \( F_{|xz} \) varies from 1.0 to 1.5. We consider three cases of macroscopic deformation associated with \( F_{|xz} = 0.0, \ F_{|xz} = 0.1 \) and \( F_{|xz} = 0.5 \).
the residual on each iteration for different number of elements.

Convergence behavior for the magnetomechanical micro-problem for various points shown in Fig. 13. The numbers in each column indicate the \((L^2)\) norm of the residual on each iteration.

<table>
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<th>iteration</th>
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<th>4 el.</th>
<th>9 el.</th>
<th>16 el.</th>
<th>25 el.</th>
<th>36 el.</th>
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<td>6.90e−08</td>
<td>1.95e−07</td>
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</table>

**Table 4**

Convergence behavior for the magneto-mechanical problem in comparison with a purely mechanical analogous problem. The numbers in each column indicate the \((L^2)\) norm of the residual on each iteration.

Table 5

Convergence behavior for the magnetomechanical micro-problem for various points shown in Fig. 13. The numbers in each column indicate the \((L^2)\) norm of the residual on each iteration.

<table>
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<tr>
<th>iteration</th>
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<th>point B</th>
<th>point C</th>
<th>point D</th>
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</tr>
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<tr>
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</tr>
<tr>
<td>6</td>
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<td>3.69e+13</td>
<td>3.57e+13</td>
<td>3.76e+13</td>
</tr>
</tbody>
</table>

In combination with four cases of macroscopic magnetic field (in A/m)

\[|\mathbf{B}|_x = 0.0, \quad |\mathbf{B}|_y = 50000, \quad |\mathbf{B}|_z = 250000 \quad \text{and} \quad |\mathbf{B}|_x = 500000,\]

resulting in twelve different cases.

Figs. 18 and 19 show the distribution of the microscopic stress and microscopic magnetic induction respectively for the twelve combinations of magneto-mechanical loading. All the figures correspond to the undeformed configuration of the RVE.

Looking at the first column of the Fig. 18, it is clear that in the absence of macroscopic mechanical deformation, i.e. \(\mathbf{F} = \mathbf{I}\), the increase of the magnetic field does not induce a macroscopic Piola stress in \(xz\)-direction. This shall be compared with the first column of the Fig. 18.

In both the extension and the shear test the macroscopic magnetic induction shows almost linear relation with the (macroscopic) magnetic field. It should be emphasized that this behavior is not imposed according to the constitutive law (30).

Even though several observations agree with those of the extension test, one should note that not all the conclusions are identical. In particular, we observe that the high magnetic field influences the macroscopic stress state even under large shear deformation, i.e. the last column of Fig. 18. This observation plays an important role for the experimental procedure because this type of loading is often applied for rheometer tests.

**Fig. 14.** Three-dimensional unit cell of a particle composite.

**Fig. 15.** Macroscopic deformation and magnetic field: The first line refers to the simple-extension and applied magnetic field conditions. The second line refers to the simple-shear and applied magnetic field conditions. The undeformed configuration is indicated using a dashed line.
Fig. 16. Microstructure of magnetorheological elastomer under simple-extension and magnetic field. Stress distribution for several macroscopic stretching and magnetic field values. All the profiles correspond to the undeformed configuration. The units for the macroscopic Piola stress \( \frac{1}{2} \mathbf{P} \) and the magnetic field \( \frac{1}{2} \mathbf{H} \) are Pa and A/m, respectively.

Fig. 17. Microstructure of magnetorheological elastomer under simple-extension and magnetic field. Magnetic induction distribution for several macroscopic shear and magnetic field values. All the profiles correspond to the undeformed configuration. The units for the macroscopic magnetic induction \( \mathbf{B} \) and the magnetic field \( \mathbf{H} \) are N/[Am] (Tesla) and A/m, respectively.
6.3. A phenomenological constitutive law for MRE

As we already described in the previous section, the complete homogenization procedure includes a simultaneous solution of macro- and microscopic problems (FE² framework) that requires also the computation of the effective tangent moduli as described in Section 3.3. In order to avoid the high computational cost, one could bypass this step by proposing appropriate macroscopic...
and employing the least squares method, we obtain the expressions for the polymer and the iron particles, described by the expressions

\[ \frac{\mu}{2} \hat{\mathbf{J}} \left( \mathbf{T} \otimes \mathbf{T} \right) : \mathbf{M} = \mathbf{M} \hat{\mathbf{J}} \mathbf{F}^{-1} \mathbf{F}^{-T} \cdot \mathbf{H}, \]

where \( \mathbf{M} \) is defined in analogous manner with \( \mathbf{M} \) of Eq. (29).

Using the results obtained in both simple-extension and simple-shear tests for the effective magnetic induction (Figs. 17, 19) and employing the least squares method, we obtain the results from the homogenization analysis and the lines represent the response using the empirical constitutive law. The units for the macroscopic Piola stress \( \mathbf{P}_{xx} \), the magnetic induction \( \mathbf{B} \), and the magnetic field \( \mathbf{H} \), are Pa, N/[Am] (Tesla) and A/m, respectively.

Fig. 20. Macroscopic response of the MRE for the simple-extension test: stress and magnetic induction versus deformation gradient for various magnetic fields. The points denote results from the homogenization analysis and the lines represent the response using the empirical constitutive law. The units for the macroscopic Piola stress \( \mathbf{P}_{xx} \), the magnetic induction \( \mathbf{B} \), and the magnetic field \( \mathbf{H} \), are Pa, N/[Am] (Tesla) and A/m, respectively.

Fig. 21. Macroscopic response of the MRE for the simple-shear test: stress and magnetic induction versus deformation gradient for various magnetic fields. The points denote results from the homogenization analysis and the lines represent the response using the empirical constitutive law. The units for the macroscopic Piola stress \( \mathbf{P}_{xx} \), the magnetic induction \( \mathbf{B} \), and the magnetic field \( \mathbf{H} \), are Pa, N/[Am] (Tesla) and A/m, respectively.

(phenomenological) constitutive magnetomechanical laws that describe quite accurately the overall response of the composite as described from the homogenization procedure. Such a task though is very difficult, if not impossible, especially when the material constituents are anisotropic or they present nonlinear material and/or geometric behavior. In highly nonlinear and anisotropic cases a complete FE framework would be unavoidable.

For the example presented here, based on the obtained effective response of the magnetorheological elastomer with the specific RVE under simple-extension and simple-shear loading conditions, we can propose an empirical phenomenological law for the effective medium that is able to reproduce quite accurately the same behavior. In small strain linear elasticity theory, an isotropic matrix with isotropic, uniformly distributed, particles behaves as an isotropic effective medium (Christensen, 1979). Based on this idea, we can assume that the effective medium for the MRE under investigation has similar constitutive magnetomechanical law with the polymer and the iron particles, described by the expressions

\[ \mathbf{P} = \lambda_1 \mathbf{F} + \left[ \lambda_2 \ln \mathbf{J} - \lambda_1 \right] \mathbf{F}^{-1} \frac{1}{2} \mu \mathbf{J} \left[ \mathbf{T} \otimes \mathbf{T} \right] : \mathbf{M}, \quad \mathbf{H} = \mu \mathbf{J} \mathbf{F}^{-1} \mathbf{F}^{-T} \cdot \mathbf{H}, \]

where \( \mathbf{M} \) is defined in analogous manner with \( \mathbf{M} \) of Eq. (29).

Using the results obtained in both simple-extension and simple-shear tests for the effective magnetic induction (Figs. 17, 19) and employing the least squares method, we obtain the effective normal stress results of Fig. 16 and the least squares method we compute \( \lambda_1 = 13.793 \times 10^7 \) N/A². From the effective shear stress results of Fig. 18 we can assume that the effective medium for the MRE under investigation has similar constitutive magnetomechanical law with the polymer and the iron particles, described by the expressions

\[ \mathbf{P} = \lambda_1 \mathbf{F} + \left[ \lambda_2 \ln \mathbf{J} - \lambda_1 \right] \mathbf{F}^{-1} \frac{1}{2} \mu \mathbf{J} \left[ \mathbf{T} \otimes \mathbf{T} \right] : \mathbf{M}, \quad \mathbf{H} = \mu \mathbf{J} \mathbf{F}^{-1} \mathbf{F}^{-T} \cdot \mathbf{H}, \]

where \( \mathbf{M} \) is defined in analogous manner with \( \mathbf{M} \) of Eq. (29).

using the least squares method we compute \( \lambda_1 = 3.732 \) MPa.

The effective normal stress results of Fig. 16 and the least squares method we compute \( \lambda_1 = 3.732 \) MPa.

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The effective normal stress results of Fig. 16 and the least squares method we compute \( \lambda_1 = 3.732 \) MPa.

the results of the homogenization analysis and the empirical constitutive law. As we observe, the empirical law can describe quite accurately the behavior of the effective medium computed from the unit cell problem.

We note that the proposed phenomenological isotropic magnetoelastic model is valid only for circular particulate composites with isotropic magnetoelastic constituents (which is the examined case). In the examples of Section 5.1 and 5.2, for instance, such model fails, since fiber composites even in the small strain theory behave anisotropically.

7. Conclusion

A geometrically nonlinear homogenization framework for composites consisting of constituents with magneto-mechanical behavior has been presented. The framework is based on the finite strain theory and therefore, the composite can undergo large deformation. It is shown that the choice of periodic boundary conditions for the RVE satisfies the Hill–Mandel condition. A finite element framework is developed to compute the effective properties of the microstructure as well as the response within the RVE itself. The numerical procedure is efficient, robust and shows asymptotically the quadratic rate of convergence associated with the Newton–Raphson scheme. Several numerical examples in two and three dimensions under simple-shear and simple-extension loading are studied. The finite element results indicate that the magnetic field at low to moderate deformation levels influences the mechanical response of a magnetorheological elastomer. It is also observed that for the magnetorheological elastomer of interest the macroscopic material magnetic induction represents a nearly
linear dependence on the applied (macroscopic) material magnetic field. Equipped with the presented framework, it would be possible to carry out the fully-coupled FE\textsuperscript{2} homogenization procedure. This work should be considered as the first half of a FE\textsuperscript{2} framework since we have described all the steps to link the microstructure to the macrostructure.

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References


