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Almost all triple systems with independent neighborhoods are semi-bipartite

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ABSTRACT

The neighborhood of a pair of vertices u, v in a triple system is the set of vertices w such that uvw is an edge. A triple system $\mathcal{H}$ is semi-bipartite if its vertex set contains a vertex subset X such that every edge of $\mathcal{H}$ intersects X in exactly two points. It is easy to see that if $\mathcal{H}$ is semi-bipartite, then the neighborhood of every pair of vertices in $\mathcal{H}$ is an independent set. We show a partial converse of this statement by proving that almost all triple systems with vertex sets $[n]$ and independent neighborhoods are semi-bipartite. Our result can be viewed as an extension of the Erdős–Kleitman–Rothschild theorem to triple systems.

The proof uses the Frankl–Rödl hypergraph regularity lemma, and stability theorems. Similar results have recently been proved for hypergraphs with various other local constraints.

1. Introduction

Let $[V]^k$ denote the collection of all k-element subsets of a set V (if $V = [n] = \{1, 2, \dots, n\}$, then we write $[n]^k$ instead of $[\![n]\!]^k$). Say that $\mathcal{H}$ is a k-uniform hypergraph (k-graph for short) with vertex set $V = V(\mathcal{H})$ if $\mathcal{H} \subset [V]^k$. If $k = 2$, then $\mathcal{H}$ is a graph. Let F be a k-graph. A k-graph is F-*free* if it contains no copy of F as a (not necessarily induced) subhypergraph.

This is the second in a sequence of our papers where we describe the global structure of typical k-graphs that satisfy certain local conditions. This line of research originated with the seminal result

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of Erdős, Kleitman and Rothschild [12] which proved that almost all triangle-free graphs with vertex set $[n]$ are bipartite. Our goal is to prove a hypergraph version of this theorem.

Subsequent to [12], there has been much work concerning the number and structure of F-free graphs with vertex set $[n]$ (see, e.g. [10,11,17,21,2–4,1,5,7]). The results essentially state that for a large class of graphs F, most of the F-free graphs with vertex set $[n]$ have a similar structure to the F-free graph with the maximum number of edges. Many of these results use the Szemerédi regularity lemma.

With the recent development of the hypergraph regularity lemma, one can prove similar theorems for hypergraphs. We often refer to a 3-graph as a triple system. The first result in this direction was due to Nagle and Rödl [19] who proved that the number of F-free triple systems (for fixed triple system F) on vertex set $[n]$ is

$$2^{\mathrm{ex}(n,F)+o(n^3)},$$

where $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free triple system on n vertices. Due to the absence of a general extremal result for hypergraphs in the vein of Turán's graph theorem, one cannot expect hypergraph results that completely parallel the graph case. Still, there has been recent progress on various specific examples. Person and Schacht [20] proved that almost all triple systems on $[n]$ not containing a Fano configuration are 2-colorable. The key property that they used was the linearity of the Fano plane, namely the fact that every two edges of the Fano plane share at most one vertex. This enabled them to apply the (weak) 3-graph regularity lemma, which is almost identical to Szemerédi's regularity lemma. They then proved an embedding lemma for linear hypergraphs essentially following ideas from Kohayakawa, Nagle, Rödl and Schacht [16].

It is well known that such an embedding lemma fails to hold for non-linear 3-graphs unless one uses the (strong) 3-graph regularity lemma, and operating in this environment is more complicated.

The first structural result for non-linear hypergraphs was due to the current authors [6]. It was proved in [6] that typical extended triangle-free triple systems are tripartite, where an extended triangle is $\{abc, abd, cde\}$. The corresponding extremal result, that the maximum number of triples on $[n]$ with no extended triangle is achieved by a complete tripartite triple system, was proved by Bollobás [9] and is the first extremal hypergraph result for a non-degenerate problem. In this paper we give a similar result for a different non-linear triple system.

The neighborhood of a $(k-1)$-set S of vertices in a k-graph is the set of vertices v whose union with S forms an edge. A set is *independent* if it contains no edge. We can rephrase Mantel's theorem about triangle-free graphs as follows: the maximum number of edges in an n vertex 2-graph with independent neighborhoods is $\lfloor n^2/4\rfloor$. This formulation can be generalized to $k > 2$ and there has been quite a lot of recent activity on this question [18,15,13,8].

Let us first observe that a triple system has independent neighborhoods if and only if it contains no copy of

$$T_5 = \{123, 124, 125, 345\}.$$

Say that a triple system is *semi-bipartite* if it has an (ordered) vertex partition (X, Y) such that every edge has exactly one point in Y. A short case analysis shows that all neighborhoods in a semi-bipartite triple system are independent (one can think of semi-bipartite triple systems as an analogue of bipartite graphs). Let $B^3(n)$ be the triple system with the maximum number of edges among all n vertex semi-bipartite triple systems. Note that

$$b^3(n) := \left|B^3(n)\right| = \max_a \binom{a}{2}(n-a) = \bigl(4/9 + o(1)\bigr)\binom{n}{3}$$

is achieved by choosing $a = \lfloor 2n/3\rfloor$ or $a = \lceil 2n/3\rceil$.

The second author and Rödl [18] conjectured, and Füredi, Pikhurko, and Simonovits [15] proved, that among all n vertex 3-graphs (n sufficiently large) containing no copy of T_5, the unique one with the maximum number of edges is $B^3(n)$.

Let $\mathcal{S}(n)$ be the set of semi-bipartite 3-graphs with vertex set $[n]$ and put $S(n) := |\mathcal{S}(n)|$. Let $I(n)$ be the number of 3-graphs with vertex set $[n]$ and independent neighborhoods, by which we mean that for every $x, y \in [n]$ there is no $e \in \mathcal{H}$ with $e \subset \{z\colon\ xyz \in \mathcal{H}\}$. Our main result, which is a possible extension of the Erdős–Kleitman–Rothschild theorem to triple systems, is the following:

Theorem 1. *Almost all triple systems with independent neighborhoods and vertex set $[n]$ are semi-bipartite. More precisely there is a constant C such that*

$$\bigl(1 + 2^{-4n}\bigr)S(n) < I(n) < \bigl(1 + C \cdot 2^{-n/10}\bigr)S(n). \tag{1}$$

2. Broad proof structure

The lower bound in Theorem 1 will be proved by constructing a large class of triple systems that are not semi-bipartite but yet have independent neighborhoods. This will be done in Section 3. The majority of the paper is devoted to proving the upper bound in Theorem 1. We will do this in two stages. First, we will prove that a large majority of triple systems with vertex set $[n]$ and independent neighborhoods are very close to being semi-bipartite. This is formalized in Theorem 2 below. After this, we can confine our attention to triple systems with independent neighborhoods that are close to being semi-bipartite. We will show (see Theorem 3) that most of these triple systems are semi-bipartite. Let us proceed more formally.

For a hypergraph F let $\mathrm{Forb}(n, F)$ denote the set of F-free hypergraphs on vertex set $[n]$. Let $P = (X, Y)$ be an ordered vertex partition of a 3-graph $\mathcal{H}$. Call an edge of $\mathcal{H}$ *consistent* with P if it has exactly two points in X, otherwise call it *inconsistent*. Let D_P be the set of inconsistent edges with P. A vertex partition P is *optimal* for $\mathcal{H}$ if it minimizes the number of inconsistent edges, and let $D = D_{\mathcal{H}}$ be the number of inconsistent edges in an optimal partition of $\mathcal{H}$. Define

$$\mathrm{Forb}(n, T_5, \eta) := \bigl\{\mathcal{H} \subset [n]^3\colon\ T_5 \not\subset \mathcal{H} \text{ and } D_{\mathcal{H}} \leqslant \eta n^3\bigr\}.$$

The proof of Theorem 1 can be separated into two parts: Theorem 2, proved in Sections 4 and 5 and Theorem 3, proved in Section 6. Note that the proof of Theorem 2 is independent from the rest of the results. However, both Theorems 1 and 3 are proved via induction on n: In the proof of the n-statement of Theorem 1 we use the n'-statement of Theorem 3 for every $n' \leqslant n$, and in the proof of the n-statement of Theorem 3 we use the n'-statement of Theorem 2 for every $n' < n$. This will be made more precise in Section 6.6.

Theorem 2. *For every $\eta > 0$, there exists $\nu > 0$ and n_0 such that if $n > n_0$, then*

$$\bigl|\mathrm{Forb}(n, T_5) - \mathrm{Forb}(n, T_5, \eta)\bigr| < 2^{(1-\nu)\frac{2n^3}{27}}.$$

We will use the hypergraph regularity lemma due to Frankl–Rödl to prove Theorem 2. In Section 4 we introduce the definitions needed to state this lemma.

Theorem 3. *For $\eta > 0$ sufficiently small there exists a C' such that*

$$\bigl|\mathrm{Forb}(n, T_5, \eta)\bigr| < \bigl(1 + C' 2^{-n/10}\bigr)S(n). \tag{2}$$

The proof of Theorem 3 uses many ideas from [2,3]: we prove in Section 6.3 that most $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta)$ have some lower-dense properties, in Section 6.4 that there are no vertices with many inconsistent edges, and in Section 6.5 we shall get rid of all the inconsistent edges. However many elements of the proof are new, like defining and using the concept of rich edges and shadow graphs.

3. Lower bound in Theorem 1

Let us prove the lower bound in (1), by constructing a set $\mathcal{NS}(n)$ of at least $2^{-4n}S(n)$ non-semi-bipartite T_5-free 3-graphs $\mathcal{H}$ with vertex set $[n]$. Indeed, this shows that $I(n) - S(n) \geqslant 2^{-4n}S(n)$ and it follows that $I(n) > (1 + 2^{-4n})S(n)$.

Let $s = s(n)$ be the maximum number of edges that a semi-bipartite 3-graph with vertex set $[n]$ can have, and suppose that this is achieved with class sizes $t = t(n)$ and $n - t$ (where $t \geqslant n - t$). Easy calculus shows that $t < 2n/3 + 2$. Then clearly

$$S(n) \leqslant 2^{n+s}.$$

Let $X = [t]$ and $Y = [n] - [t]$. Set

$$\mathcal{F} = \binom{\{1, 2, n-1, n\}}{3}.$$

Let $\mathcal{G}$ be the collection of triples e that simultaneously satisfy the following two conditions:

- $|e \cap X| = 2$,
- $|e \cap \{1, 2, n-1, n\}| \leqslant 1$. (*)

Let $\mathcal{NS}(n)$ be the collection of 3-graphs $\{\mathcal{F} \cup \mathcal{G}': \ \mathcal{G}' \subset \mathcal{G}\}$. We will now show that $\mathcal{NS}(n)$ comprises only non-semi-bipartite T_5-free 3-graphs. Pick an $\mathcal{H} \in \mathcal{NS}(n)$.

Since $\mathcal{F}$ is not semi-bipartite, $\mathcal{H}$ is also not semi-bipartite. Using (*), an easy case analysis shows that $T_5 \not\subset \mathcal{H}$. Finally, we must obtain a lower bound on $|\mathcal{NS}(n)| = 2^{|\mathcal{G}|}$. Recall that $s = \binom{t}{2}(n-t)$. Since we exclude all triples with two or more points in $\{1, 2, n-1, n\}$ when defining $\mathcal{G}$, and $t \leqslant 2n/3 + 2$,

$$|\mathcal{G}| = s - \big(n - t + 4(t-2) + 2\big) \geqslant -3n + s = -4n + n + s.$$

Consequently,

$$\big|\mathcal{NS}(n)\big| = 2^{|\mathcal{G}|} \geqslant 2^{-4n} 2^{n+s} \geqslant 2^{-4n} S(n)$$

and the proof is complete.

4. Hypergraph regularity

In this section, we quickly define the notions required to state the hypergraph regularity lemma. Further details can be found in [14] or [19].

Given a k-partite graph G with k-partition $V_1, \dots, V_k$, we write $G = \bigcup_{i<j} G^{ij}$, where $G^{ij} = G[V_i \cup V_j]$ is the bipartite subgraph of G with parts V_i and V_j. For $B \in [k]^3$, the 3-partite graph $G(B) = \bigcup_{\{i,j\} \in [B]^2} G^{ij}$ is called a *triad*. For a bipartite graph G, the *density* of the pair V_1, V_2 with respect to G is $d_G(V_1, V_2) = \frac{|G^{12}|}{|V_1||V_2|}$.

Given an integer $l > 0$ and real $\epsilon > 0$, a k-partite graph G is called an $(\epsilon, 1/l)$-*regular* k-partite graph if for every $i < j$, G^{ij} is ϵ-regular with density $(1/l)(1 \pm \epsilon)$. For a k-partite graph G, let $\mathcal{K}_3(G)$ denote the 3-graph with vertex set $V(G)$ whose edges correspond to triangles of G. An easy consequence of these definitions is the following fact.

Lemma 4 *(Triangle counting lemma). For integer $l > 0$ and real $\theta > 0$, there exists $\epsilon > 0$ such that every $(\epsilon, 1/l)$-regular k-partite G with $|V_i| = m$ for all i satisfies*

$$\big|\mathcal{K}_3(G)\big| = (1 \pm \theta)\frac{m^3}{l^3}.$$

Consider a k-partite 3-graph $\mathcal{H}$ with k-partition $V_1, \dots, V_k$. Here k-partite means that every edge of $\mathcal{H}$ has at most one point in each V_i. Often we will say that these edges are *crossing*, and the edges that have at least two points in some V_i are *non-crossing*. Given a $B \in [k]^3$, let $\mathcal{H}(B) = \mathcal{H}[\bigcup_{i \in B} V_i]$. Given a k-partite graph G and a k-partite 3-graph $\mathcal{H}$ with the same vertex partition, say that G *underlies* $\mathcal{H}$ if $\mathcal{H} \subset \mathcal{K}_3(G)$. In other words, every edge of $\mathcal{H}$ is a triangle in G. Define the density $d_{\mathcal{H}}(G(B))$ of $\mathcal{H}$ with respect to $G(B)$ as follows:

$$d_{\mathcal{H}}\big(G(B)\big) = \frac{|\mathcal{H}(B)|}{|\mathcal{K}_3(G(B))|}$$

if $|\mathcal{K}_3(G(B))| > 0$ and 0 otherwise. Informally, $d_{\mathcal{H}}(G(B))$ is the proportion of triangles in $G(B)$ that are edges of $\mathcal{H}$.

This definition leads to the more complicated definition of $\mathcal{H}$ being (δ, r)-regular with respect to the triad $G(B)$, where $r > 0$ is an integer and $\delta > 0$. We will not state this definition here and it suffices to take this definition as a "black box" that will be used later.

If $\mathcal{H}$ is (δ, r)-regular with respect to $G(B)$ and $d_{\mathcal{H}}(G(B)) = \alpha \pm \delta$, then say that $\mathcal{H}$ is (α, δ, r)-*regular* with respect to $G(B)$.

For a vertex set V, an (l, t, γ, ϵ)-*partition* $\mathcal{P}$ is a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ together with a collection of edge-disjoint bipartite graphs P_a^{ij}, where $1 \leqslant i < j \leqslant t$, $0 \leqslant a \leqslant l_{ij} \leqslant l$ that satisfy the following properties:

(i) $|V_0| < t$ and $|V_i| = \lfloor \frac{n}{t} \rfloor := m$ for each $i > 0$,
(ii) $\bigcup_{a=0}^{l_{ij}} P_a^{ij} = K(V_i, V_j)$ for all $1 \leqslant i < j \leqslant t$, where $K(V_i, V_j)$ is the complete bipartite graph with parts V_i, V_j,
(iii) all but $\gamma\binom{t}{2}m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i$, $v_j \in V_j$, are edges of ϵ-regular bipartite graphs P_a^{ij}, and
(iv) for all but $\gamma\binom{t}{2}$ pairs $\{i, j\} \in [t]^2$, we have $|P_0^{ij}| \leqslant \gamma m^2$ and $d_{P_a^{ij}}(V_i, V_j) = (1 \pm \epsilon)\frac{1}{l}$ for all $a \in [l_{ij}]$.

Finally, suppose that $\mathcal{H} \subset [n]^3$ is a 3-graph and $\mathcal{P}$ is an (l, t, γ, ϵ)-partition. For $B = \{i, j, l\}$, say that $G(B) = P_{a_1}^{ij} \cup P_{a_2}^{jl} \cup P_{a_3}^{il}$ is a (δ, r)-*regular triad* of $\mathcal{P}$ if $\mathcal{H}$ is (δ, r)-*regular* with respect to $G(B)$. Then $\mathcal{P}$ is (δ, r)-regular if

$$\sum\bigl\{\bigl|\mathcal{K}_3\bigl(G(B)\bigr)\bigr|:\ G(B) \text{ is not a } (\delta, r)\text{-regular triad of } \mathcal{P}\bigr\} < \delta n^3.$$

We can now state the regularity lemma due to Frankl and Rödl [14].

Theorem 5 *(Regularity lemma). For every δ, γ with $0 < \gamma \leqslant 2\delta^4$, for all integers t_0, l_0 and for all integer-valued functions $r = r(t, l)$ and all functions $\epsilon(l)$, there exist T_0, L_0, N_0 such that every 3-graph $\mathcal{H} \subset [n]^3$ with $n \geqslant N_0$ admits a $(\delta, r(t, l))$-regular $(l, t, \gamma, \epsilon(l))$-partition for some t, l satisfying $t_0 \leqslant t < T_0$ and $l_0 \leqslant l < L_0$.*

To apply the regularity lemma above, we need to define a cluster hypergraph and state an accompanying embedding lemma, sometimes called the key lemma. Given a 3-graph $\mathcal{F}$, let $\partial\mathcal{F}$ be the set of pairs that lie in an edge of $\mathcal{F}$.

Cluster 3-graph. For given constants k, δ, l, r, ϵ and sets $\{\alpha_B\colon B \in [k]^3\}$ of non-negative reals, let $\mathcal{H}$ be a k-partite 3-graph with parts $V_1, \ldots, V_k$, each of size m. Let G be a graph, and $\mathcal{F} \subset [k]^3$ be a 3-graph such that the following conditions are satisfied.

(i) $G = \bigcup_{\{i,j\} \in \partial\mathcal{F}} G^{ij}$ underlies $\mathcal{H}$ and for all $\{i, j\} \in \partial\mathcal{F}$, G^{ij} is $(\epsilon, 1/l)$-regular.
(ii) For each $B \in \mathcal{F}$, $\mathcal{H}(B)$ is (α_B, δ, r)-regular with respect to the triad $G(B)$.

Then we say that $\mathcal{F}$ is a *cluster* 3-*graph* of $\mathcal{H}$.

Lemma 6 *(Embedding lemma). Let $k \geqslant 4$ be fixed. For all $\alpha > 0$, there exists $\delta > 0$ such that for all integers $l > \frac{1}{\delta}$, there exists an integer $r > 0$ and $\epsilon > 0$ such that the following holds: Suppose that $\mathcal{F}$ is a cluster 3-graph of $\mathcal{H}$ with underlying graph G and parameters k, δ, l, r, ϵ, $\{\alpha_B\colon B \in [k]^3\}$ where $\alpha_B \geqslant \alpha$ for all $B \in \mathcal{F}$. Then $\mathcal{F} \subset \mathcal{H}$.*

The embedding lemma is an easy consequence of the counting lemma, which finds not just one but many copies of $\mathcal{F}$ in $\mathcal{H}$. Though for our purposes we need only the weaker statement of the embedding lemma (for a proof of the embedding lemma, see [19]).

5. Proof of Theorem 2

In this section we prove Theorem 2. We will need the following stability result proved in [15].

Theorem 7 *(Füredi–Pikhurko–Simonovits). (See [15].) For every $\nu'' > 0$, there exist ν_1', t_2 such that every T_5-free 3-graph on $t > t_2$ vertices and at least $(1-2\nu_1')\frac{2t^3}{27}$ edges has an ordered partition for which the number of inconsistent edges is at most $\nu'' t^3$. Additionally, there exists t_3 such that $\mathrm{ex}(n, T_5) \leqslant \frac{2t^3}{27}$ for all $t \geqslant t_3$.*

Given $\eta > 0$, our constants will obey the following hierarchy:

$$\eta \gg \nu'' \gg \nu' \gg \nu \gg \sigma, \qquad \theta \gg \alpha_0, \qquad \frac{1}{t_0} \gg \delta \gg \gamma > \frac{1}{l_0} \gg \frac{1}{r}, \qquad \epsilon \gg \frac{1}{n_0}.$$

Before proceeding with further details regarding our constants, we define the *binary entropy function* $H(x) := -x\log_2 x - (1-x)\log_2(1-x)$. We use the following two facts about $H(x)$ that apply for n sufficiently large:

- for $0 < x < 0.5$ we have
$$\binom{n}{\lfloor xn \rfloor} < 2^{H(x)n},$$
- if x is sufficiently small then
$$\sum_{i=0}^{\lfloor xn \rfloor} \binom{n}{i} < 2^{H(x)n}. \tag{3}$$

Detailed definition of constants. Set

$$\nu'' = \left(\frac{\eta}{30}\right)^3 \tag{4}$$

and suppose that ν_1' and t_2 are the outputs of Theorem 7 with input ν''. Put

$$\nu' = \min\left\{\nu_1', \frac{\nu''}{2}, \frac{\eta}{7}\right\} \quad \text{and} \quad \nu = \left(\nu'\right)^4. \tag{5}$$

We choose

$$\theta = \frac{\nu}{4(1-\nu)}. \tag{6}$$

Choose σ_1 small enough so that

$$\left(1 - \frac{\nu}{2}\right)\frac{2n^3}{27} + o\left(n^3\right) + H(\sigma_1)n^3 \leqslant \left(1 - \frac{\nu}{3}\right)\frac{2n^3}{27} \tag{7}$$

holds for sufficiently large n. In fact the function denoted by $o(n^3)$ will actually be seen to be of order $O(n^2)$ so (7) will hold for sufficiently large n. Choose σ_2 small enough so that (3) holds for $x = \sigma_2$. Let

$$\sigma = \min\left\{\sigma_1, \sigma_2, \frac{\eta}{2}\right\}.$$

Next we consider the triangle counting lemma (Lemma 4) which provides an ϵ for each θ and l. Since θ is fixed, we may let $\epsilon_1 = \epsilon_1(l)$ be the output of Lemma 4 for each integer l.

For σ defined above, set

$$\delta_1 = \alpha_0 = \frac{\sigma}{100} \quad \text{and} \quad t_1 = \left\lceil \frac{1}{\delta_1} \right\rceil. \tag{8}$$

Let

$$t_0 = \max\{t_1, t_2, t_3\}.$$

Now consider the embedding lemma (Lemma 6) with inputs $k = 5$ and α_0 defined above. The embedding lemma gives $\delta_2 = \delta_2(\alpha_0)$, and we set

$$\delta = \min\{\delta_1, \delta_2\}, \qquad \gamma = \delta^4, \qquad l_0 = \frac{2}{\delta}. \tag{9}$$

For each integer $l > \frac{1}{\delta}$, let $r = r(l)$ and $\epsilon_2 = \epsilon_2(l)$ be the outputs of Lemma 6. Set

$$\epsilon = \epsilon(l) = \min\bigl\{\epsilon_1(l), \epsilon_2(l)\bigr\}. \tag{10}$$

With these constants, the regularity lemma (Theorem 5) outputs N_0. We choose n_0 such that $n_0 > N_0$ and every $n > n_0$ satisfies (3) and (7).

Proof of Theorem 2. We will prove that

$$\bigl|\mathrm{Forb}(n, T_5) - \mathrm{Forb}(n, T_5, \eta)\bigr| < 2^{(1-\frac{\nu}{3})\frac{2n^3}{27}}.$$

This is of course equivalent to Theorem 2. The initial part of the proof that follows is similar to the proof of [19], though there is a slight difference in how we define equivalence classes. Starting from Lemma 8 most of the ideas are new.

For each $\mathcal{G} \in \mathrm{Forb}(n, T_5) - \mathrm{Forb}(n, T_5, \eta)$, we use the hypergraph regularity lemma, Theorem 5, to obtain a (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P} = \mathcal{P}(\mathcal{G})$. The input constants for Theorem 5 are as defined above and then Theorem 5 guarantees constants T_0, L_0, N_0 so that every 3-graph $\mathcal{G}$ on $n > N_0$ vertices admits a (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P}$ where $t_0 \leqslant t \leqslant T_0$ and $l_0 \leqslant l \leqslant L_0$. We may assume that $\mathcal{P}$ has vertex partition $[n] = V_0 \cup V_1 \cup \cdots \cup V_t$, $|V_i| = m = \lfloor \frac{n}{t} \rfloor$ for all $i \geqslant 1$, and system of bipartite graphs P_a^{ij}, where $1 \leqslant i < j \leqslant t$, $0 \leqslant a \leqslant l_{ij} \leqslant l$.

Let $\mathcal{E}_0 \subset \mathcal{G}$ be the set of triples that either

(i) intersect V_0, or
(ii) have at least two points in some V_i, $i \geqslant 1$, or
(iii) contain a pair in P_0^{ij} for some i, j, or
(iv) contain a pair in some P_a^{ij} that is not ϵ-regular with density $\frac{1}{l}$.

By the properties of an (l, t, γ, ϵ)-partition

$$|\mathcal{E}_0| \leqslant tn^2 + t\left(\frac{n}{t}\right)^2 n + \gamma\binom{t}{2}m^2 n + 2\gamma\binom{t}{2}\left(\frac{n}{t}\right)^2 n.$$

Let $\mathcal{E}_1 \subset \mathcal{G} - \mathcal{E}_0$ be the set of triples $\{v_i, v_j, v_k\}$ such that either

- the three bipartite graphs of $\mathcal{P}$ associated with the pairs within the triple form a triad $G(B)$ that is not (δ, r)-regular with respect to $\mathcal{G}(\{i, j, k\})$, or
- the density $d_{\mathcal{G}}(G(B)) < \alpha_0$.

Then

$$|\mathcal{E}_1| \leqslant \delta n^3 + \alpha_0 n^3.$$

Let $\mathcal{E}_{\mathcal{G}} = \mathcal{E}_0 \cup \mathcal{E}_1$. Now (8) and (9) imply that

$$|\mathcal{E}_{\mathcal{G}}| \leqslant \sigma n^3.$$

Set $\mathcal{G}' = \mathcal{G} - \mathcal{E}_{\mathcal{G}}$. Next define $\mathcal{J} = \mathcal{J}(\mathcal{G}) \subset [t]^3 \times [l] \times [l] \times [l]$ as follows: For $1 \leqslant i < j < k \leqslant t$, $1 \leqslant a, b, c \leqslant l$, we have $(\{i, j, k\}, a, b, c) \in \mathcal{J}$ if and only if

- $G = P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}$ is $(\epsilon, 1/l)$-regular, and
- $\mathcal{G}'(\{i, j, k\})$ is $(\overline{\alpha}, \delta, r)$-regular with respect to G, where $\overline{\alpha} \geqslant \alpha_0$.

From now on we shall replace the cumbersome notation $(\{i, j, k\}, a, b, c)$ by (ijk, abc).

For each $\mathcal{G} \in \mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta)$, choose one (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P} = \mathcal{P}(\mathcal{G})$ guaranteed by Theorem 5, and let $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_p\}$ be the set of all such partitions over the family $\mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta)$. Note also that once we have defined $\mathcal{P}(\mathcal{G})$ we have also defined $\mathcal{J} = \mathcal{J}(\mathcal{G})$. Define an equivalence relation on $\mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta)$ by letting $\mathcal{G}_1 \sim \mathcal{G}_2$ iff

- $\mathcal{P}(\mathcal{G}_1) = \mathcal{P}(\mathcal{G}_2)$ and
- $\mathcal{J}(\mathcal{G}_1) = \mathcal{J}(\mathcal{G}_2)$.

The number of equivalence classes q is the number of partitions times the number of choices of $\mathcal{J} \subset [t]^3 \times [l] \times [l] \times [l]$. The number of partitions satisfies

$$p \leqslant \left(\sum_{t=t_0}^{T_0} t^n\right)\left(\binom{T_0+1}{2}\sum_{l=l_0}^{L_0}(l+1)\right)^{\binom{n}{2}}.$$

Consequently,

$$q \leqslant T_0^{n+1}\left(\binom{T_0+1}{2}(L_0+1)^2\right)^{\binom{n}{2}} 2^{\binom{T_0+1}{3}(L_0+1)^3} < 2^{O(n^2)}.$$

We will show that each equivalence class $C(\mathcal{P}_i, \mathcal{J})$ satisfies

$$\left|C(\mathcal{P}_i, \mathcal{J})\right| = 2^{(1-\frac{\nu}{2})\frac{2n^3}{27} + H(\sigma)n^3}. \tag{11}$$

Combined with the upper bound for q above and (7), we obtain

$$\left|\mathrm{Forb}(n, T_5, \eta)\right| \leqslant 2^{O(n^2)} 2^{(1-\frac{\nu}{2})\frac{2n^3}{27} + H(\sigma)n^3} \leqslant 2^{(1-\frac{\nu}{3})\frac{2n^3}{27}}.$$

For the rest of the proof, we fix an equivalence class $C = C(\mathcal{P}_i, \mathcal{J})$ and we will show the upper bound in (11).

We view $\mathcal{J}$ as a multiset of triples on $[t]$. For each $\phi : [t]^2 \to [l]$, let $\mathcal{J}_\phi \subset \mathcal{J}$ be the 3-graph on $[t]$ with edge set

$$\bigl\{\{i, j, k\}:\ \bigl(ijk, \phi(\{i, j\})\phi(\{j, k\})\phi(\{i, k\})\bigr) \in \mathcal{J}\bigr\}.$$

In other words, $\{i, j, k\} \in \mathcal{J}_\phi$ iff the triples of $\mathcal{G}$ that lie on top of the triangles of $P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}$, $a = \phi(ij)$, $b = \phi(jk)$, $c = \phi(ik)$, are $(\overline{\alpha}, \delta, r)$-regular and the underlying bipartite graphs P_a^{ij}, P_b^{jk}, P_c^{ik} are all $(\epsilon, 1/l)$-regular.

By our choice of the constants in (9) and (10), and by the construction of $\mathcal{J}$, for a fixed ϕ, any five vertex 3-graph $\mathcal{F} \subset \mathcal{J}_\phi$ is a cluster 3-graph for $\mathcal{G}$, and hence by the embedding lemma $\mathcal{F} \subset \mathcal{G}$. Since $T_5 \not\subset \mathcal{G}$, we conclude that $T_5 \not\subset \mathcal{J}_\phi$.

It was shown in [15] that for $t \geqslant t_3$, we have $\mathrm{ex}(t, T_5) \leqslant \frac{2t^3}{27}$. Since we know that $t \geqslant t_3$, we conclude that

$$|\mathcal{J}_\phi| \leqslant \frac{2t^3}{27} \tag{12}$$

for each $\phi : [t]^2 \to [l]$. Recall from (5) that $\nu' = \nu^{1/4}$.

Lemma 8. *Suppose that $|\mathcal{J}| > (1-\nu)\frac{2l^3t^3}{27}$. Then for at least $(1-\nu')l^{\binom{t}{2}}$ of the functions $\phi: [t]^2 \to [l]$ we have*

$$|\mathcal{J}_\phi| \geqslant (1-\nu')\frac{|\mathcal{J}|}{l^3}.$$

Proof. Form the following bipartite graph: the vertex partition is $\Phi \cup \mathcal{J}$, where

$$\Phi = \{\phi\colon [t]^2 \to [l]\}$$

and the edges are of the form $\{\phi, (ijk, abc)\}$ if and only if $\phi \in \Phi$, $(ijk, abc) \in \mathcal{J}$ where $\phi(\{i,j\}) = a$, $\phi(\{j,k\}) = b$, $\phi(\{i,k\}) = c$. Let E denote the number of edges in this bipartite graph. Since each $(ijk, abc) \in \mathcal{J}$ has degree precisely $l^{\binom{t}{2}-3}$, we have

$$E = |\mathcal{J}| l^{\binom{t}{2}-3}.$$

Note that the degree of ϕ is $|\mathcal{J}_\phi|$. Suppose for contradiction that the number of ϕ for which $|\mathcal{J}_\phi| \geqslant (1-\nu')\frac{|\mathcal{J}|}{l^3}$ is less than $(1-\nu')l^{\binom{t}{2}}$. Then since $|\mathcal{J}_\xi| \leqslant \frac{2t^3}{27}$ for each $\xi \in \Phi$, we obtain the upper bound

$$E \leqslant (1-\nu')l^{\binom{t}{2}}\frac{2t^3}{27} + \nu' l^{\binom{t}{2}}(1-\nu')\frac{|\mathcal{J}|}{l^3}.$$

Dividing by $l^{\binom{t}{2}-3}$ it yields

$$|\mathcal{J}| \leqslant (1-\nu')l^3\frac{2t^3}{27} + \nu'(1-\nu')|\mathcal{J}|.$$

After simplifying it we obtain

$$\bigl(1-\nu'(1-\nu')\bigr)|\mathcal{J}| \leqslant (1-\nu')l^3\frac{2t^3}{27}.$$

The lower bound $|\mathcal{J}| > (1-\nu)\frac{2l^3t^3}{27}$ then gives

$$\bigl(1-\nu'(1-\nu')\bigr)(1-\nu) < 1-\nu'.$$

Since $\nu' = \nu^{1/4}$, the left-hand side expands to

$$1-\nu'+\nu^{1/2}-\nu+\nu^{5/4}-\nu^{3/2} > 1-\nu'.$$

This contradiction completes the proof. $\square$

Using Lemma 8 we will prove the following claim.

Claim 1.

$$|\mathcal{J}| \leqslant (1-\nu)\frac{2l^3t^3}{27}.$$

Once we have proved Claim 1, the proof of Theorem 2 is completed by the following argument which is very similar to that in [19]. Define

$$S^C = \bigcup_{(ijk,abc)\in\mathcal{J}} \mathcal{K}_3\bigl(P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}\bigr).$$

The triangle counting lemma implies that for each $(ijk, abc) \in \mathcal{J}$, $|\mathcal{K}_3(P_a^{ij} \cup P_b^{jk} \cup P_c^{ik})| < \frac{m^3}{l^3}(1+\theta)$. Now Claim 1 and (6) give

$$|S^C| \leqslant \frac{m}{l^3}(1+\theta)|\mathcal{J}| \leqslant m^3(1+\theta)(1-\nu)\frac{2t^3}{27} < m^3\frac{2t^3}{27}\left(1-\frac{\nu}{2}\right) \leqslant \frac{2n^3}{27}\left(1-\frac{\nu}{2}\right).$$

Since $\mathcal{G}' \subset S^C$ for every $\mathcal{G} \in C$,

$$\left|\left\{\mathcal{G}' \colon \mathcal{G} \in C\right\}\right| \leqslant 2^{(1-\frac{\nu}{2})\frac{2n^3}{27}}.$$

Each $\mathcal{G} \in C$ can be written as $\mathcal{G} = \mathcal{G}' \cup \mathcal{E}_{\mathcal{G}}$. In view of (3) and $|\mathcal{E}_{\mathcal{G}}| \leqslant \sigma n^3$, the number of $\mathcal{E}_{\mathcal{G}}$ with $\mathcal{G} \in C$ is at most $\sum_{i \leqslant \sigma n^3} \binom{n^3}{i} \leqslant 2^{H(\sigma)n^3}$. Consequently,

$$|C| \leqslant 2^{(1-\frac{\nu}{2})\frac{2n^3}{27}+H(\sigma)n^3}$$

and we are done proving (11). In the remaining part of this section our only goal is to prove Claim 1.

Proof of Claim 1. Suppose to the contrary that $|\mathcal{J}| > (1-\nu)\frac{2l^3t^3}{27}$. We apply Lemma 8 and conclude that for most functions ϕ the corresponding triple system $\mathcal{J}_\phi$ satisfies

$$|\mathcal{J}_\phi| \geqslant \left(1-\nu'\right)\frac{|\mathcal{J}|}{l^3} > \left(1-\nu'\right)(1-\nu)\frac{2t^3}{27} > \left(1-2\nu'\right)\frac{2t^3}{27}.$$

By Theorem 7, we conclude that for all of these ϕ, the triple system $\mathcal{J}_\phi$ has an ordered partition where the number of inconsistent edges is at most $\nu'' t^3$. Fix one such ϕ and let the optimal partition of $\mathcal{J}_\phi$ be $Q_\phi = (X, Y)$.

Let $\mathcal{L}$ be the set of consistent edges of $\mathcal{J}_\phi$ and let $\mathcal{B}$ be the set of inconsistent edges of J_ϕ. Write $\mathcal{M}$ for the set of 3-element sets that are consistent with $Q_\phi = (X, Y)$ but are not edges of $\mathcal{J}_\phi$. Then $\mathcal{L} \cup \mathcal{M}$ is semi-bipartite, so

$$|\mathcal{L}| + |\mathcal{M}| \leqslant \max_{1 \leqslant a \leqslant t} \binom{a}{2}(t-a) \leqslant \frac{2t^3}{27}.$$

We also have $|\mathcal{L}| + |\mathcal{B}| = |\mathcal{J}_\phi| \geqslant (1-2\nu')\frac{2t^3}{27}$ and $|\mathcal{B}| \leqslant \nu'' t^3$. Consequently,

$$|\mathcal{M}| \leqslant \frac{2t^3}{27} - |\mathcal{L}| \leqslant \left(\frac{4\nu'}{27} + \nu''\right)t^3 < 2\nu'' t^3. \tag{13}$$

Since $|\mathcal{J}_\phi| \geqslant (1-2\nu')\frac{2t^3}{27}$ and $|D_{Q_\phi}| \leqslant \nu'' t^3$, we obtain

$$|X| = \left(1 \pm \sqrt{\nu''}\right)\frac{2t}{3} \quad \text{and} \quad |Y| = \left(1 \pm 2\sqrt{\nu''}\right)\frac{t}{3}.$$

Indeed, otherwise a short calculation using (5) gives the contradiction

$$|\mathcal{J}_\phi| \leqslant \binom{|X|}{2}|Y| + |D_{Q_\phi}| \leqslant \binom{(1-\sqrt{\nu''})\frac{2t}{3}}{2}\left(1+2\sqrt{\nu''}\right)\frac{t}{3} + \nu'' t^3 < \left(1-2\nu'\right)\frac{2t^3}{27}.$$

Let $Q = (V_X, V_Y)$ be the corresponding vertex partition of $[n] - V_0$. In other words,

$$V_X = \bigcup_{i \in X} V_i \quad \text{and} \quad V_Y = \bigcup_{i \in Y} V_i.$$

Let Q' be the partition obtained from Q by arbitrarily distributing the vertices of V_0 into the two parts of Q. We will show that Q' is a partition of $[n]$ where the number of inconsistent edges $|D_{Q'}|$ is fewer than ηn^3. This will contradict the fact that $\mathcal{G} \in \mathrm{Forb}(n, T_5) - \mathrm{Forb}(n, T_5, \eta)$ and complete the proof of Claim 1. $\square$

We have argued earlier that $|\mathcal{E}_{\mathcal{G}}| \leqslant \sigma n^3 \leqslant \frac{\eta}{2}n^3$. The number of edges of $\mathcal{G}$ that intersect V_0 is at most $|V_0|n^2 \leqslant tn^2$, so

$$|D_{Q'} - \mathcal{E}_{\mathcal{G}}| \leqslant |D_Q - \mathcal{E}_{\mathcal{G}}| + tn^2.$$

Consequently, it suffices to prove that

$$|D_Q - \mathcal{E}_{\mathcal{G}}| + tn^2 \leqslant \frac{\eta}{2} n^3. \tag{14}$$

For each $\xi : [t]^2 \to [l]$, define

$$\mathcal{G}_\xi = \mathcal{G}' \cap \bigcup \bigl\{ \mathcal{K}_3\bigl(P^{ij}_{\xi(\{i,j\})} \cup P^{jk}_{\xi(\{j,k\})} \cup P^{ik}_{\xi(\{i,k\})}\bigr) \colon \{i,j,k\} \in \mathcal{J}_\xi \bigr\}.$$

In other words, $\mathcal{G}_\xi$ is the union, over all $\{i,j,k\} \in \mathcal{J}_\xi$, of the edges of $\mathcal{G}$ that lie on top of the triangles in $P^{ij}_{\xi(\{i,j\})} \cup P^{jk}_{\xi(\{j,k\})} \cup P^{ik}_{\xi(\{i,k\})}$.

Let D_ξ be the set of edges in $\mathcal{G}_\xi$ that are non-crossing with respect to $Q = (V_X, V_Y)$. We will estimate $|D_Q - \mathcal{E}_{\mathcal{G}}|$ by summing $|D_\xi|$ over all ξ. Please note that each $e \in D_Q - \mathcal{E}_{\mathcal{G}}$ lies in exactly $l^{\binom{t}{2}-3}$ different D_ξ due to the definition of $\mathcal{J}$. Call a $\xi : [t]^2 \to [l]$ *good* if it satisfies the conclusion of Lemma 8, otherwise call it *bad*. In other words, ξ is good iff

$$|\mathcal{J}_\xi| \geqslant \bigl(1 - \nu'\bigr) \frac{|\mathcal{J}|}{l^3}.$$

Summing over all ξ gives

$$l^{\binom{t}{2}-3} |D_Q - \mathcal{E}_{\mathcal{G}}| = \sum_{\xi : [t]^2 \to [l]} |D_\xi| = \sum_{\xi \text{ good}} |D_\xi| + \sum_{\xi \text{ bad}} |D_\xi|.$$

Note that for a given $\{i,j,k\} \in \mathcal{J}_\xi$ the number of edges in $\mathcal{G}_\xi$ corresponding to $\{i,j,k\}$ is the number of edges in $V_i \cup V_j \cup V_k$ on top of triangles formed by the three bipartite graphs, each of which is ϵ-regular of density $(1/l)(1 \pm \epsilon)$. By the triangle counting lemma, the total number of such triangles is at most

$$2|V_i||V_j||V_k| \left(\frac{1}{l}\right)^3 < 2\left(\frac{n}{t}\right)^3 \left(\frac{1}{l}\right)^3 := R.$$

By Lemma 8, the number of bad ξ is at most $\nu' l^{\binom{t}{2}}$. So we have

$$\sum_{\xi \text{ bad}} |D_\xi| \leqslant \nu' l^{\binom{t}{2}} \binom{t}{3} R < \nu' l^{\binom{t}{2}-3} n^3.$$

It remains to estimate $\sum_{\xi \text{ good}} |D_\xi|$.

Fix a good ξ and let the optimal partition of $\mathcal{J}_\xi$ be $Q_\xi = (A, B)$ (recall that we may assume $|D_{Q_\xi}| \leqslant \nu'' t^3$, $A = (1 \pm \sqrt{\nu''})\frac{2t}{3}$, $B = (1 \pm 2\sqrt{\nu''})\frac{t}{3}$).

Claim 2. *The number of consistent edges of $\mathcal{J}_\xi$ with Q_ξ that are inconsistent edges of $\mathcal{J}_\phi$ with Q_ϕ is at most $4(\nu'')^{1/3} t^3$.*

Suppose that Claim 2 is true. Then

$$\sum_{\xi \text{ good}} |D_\xi| \leqslant l^{\binom{t}{2}} \bigl[4\bigl(\nu''\bigr)^{1/3} t^3 R + \nu'' t^3 R\bigr] = l^{\binom{t}{2}-3} \bigl[10\bigl(\nu''\bigr)^{1/3} n^3\bigr].$$

Explanation: We consider the contribution from the inconsistent edges of Q_ϕ that are (i) consistent edges of Q_ξ and (ii) inconsistent edges of Q_ξ. We do not need to consider the contribution from the consistent edges of Q_ϕ since by definition, these do not give rise to edges of D_Q.

Altogether, using (4) and (5) we obtain

$$|D_Q - \mathcal{E}_{\mathcal{G}}| + tn^2 \leqslant \bigl(10\bigl(\nu''\bigr)^{1/3} + \nu'\bigr) n^3 + tn^2 < \frac{\eta}{2} n^3$$

which proves (14) and completes the proof of Claim 1. In the remaining part of this section, we prove Claim 2, which is all that is left to prove Claim 1.

Proof of Claim 2. First we argue that for every $A' \subset A$, $B' \subset B$ with $\min\{|A'|, |B'|\} \geqslant 3(\nu'')^{1/3}t$, the number of edges in $\mathcal{J}_\xi$ with two points in A' and one point in B' is at least $10\nu''t^3$. Indeed, $\binom{|A'|}{2}|B'| > 12\nu''t^3$, and the number of triples with two points in A' and one point in B' that are not edges of J_ξ is at most $2\nu''t^3$ by (13). The remaining triples are edges in $\mathcal{J}_\xi$ with two points in A' and one point in B' as desired.

Now suppose that $A' = A \cap Y$ and $B' = B \cap Y$ satisfy $\min\{|A'|, |B'|\} \geqslant 3(\nu'')^{1/3}t$. Then we have at least $10\nu''t^3$ edges $e \in \mathcal{J}_\xi$ with $|e \cap A'| = 2$ and $|e \cap B'| = 1$. For each such edge $e = \{k, k', k''\} \subset Y$, and each $\{i, j\} \in \binom{X}{2}$, consider three distinct triples $f = \{i, j, k\}$, $f' = \{i, j, k'\}$, $f'' = \{i, j, k''\}$ that are consistent with Q_ϕ. If $f, f', f'' \in J_\phi$ then consider the following ten bipartite graphs:

$$\begin{aligned}
&G^{ij} = Q^{ij}_{\phi(\{i,j\})}, \qquad G^{jk} = Q^{jk}_{\phi(\{j,k\})}, \qquad G^{ik} = Q^{ik}_{\phi(\{i,k\})},\\
&G^{ik'} = Q^{ik'}_{\phi(\{i,k'\})}, \qquad G^{jk'} = Q^{jk'}_{\phi(\{j,k'\})}, \quad G^{ik''} = Q^{ik''}_{\phi(\{i,k''\})}, \qquad G^{jk''} = Q^{jk''}_{\phi(\{j,k''\})},\\
&G^{kk'} = Q^{kk'}_{\xi(\{k,k'\})}, \qquad G^{k'k''} = P^{k'k''}_{\xi(\{k',k''\})}, \qquad G^{kk''} = Q^{kk''}_{\xi(\{k,k''\})}.
\end{aligned}$$

Set $G = \bigcup G^{uv}$ where the union is over the ten bipartite graphs defined above. Since $\{e, f, f', f''\} \subset \mathcal{J}_\phi \cup \mathcal{J}_\xi$, the 3-graph $J = \{e, f, f', f''\}$ associated with the 5-partite graph G and 3-graph $\mathcal{G}(\{i, j, k, k', k''\})$ is a cluster 3-graph. By our choice of constants in (9), we may apply the embedding lemma. As $J \cong T_5$, we obtain the contradiction $T_5 \subset \mathcal{G}$. We conclude that $g \notin \mathcal{J}_\phi$ for some $g \in \{f, f', f''\}$. Each e gives rise to at least $\binom{|X|}{2} > \frac{t^2}{5}$ such g and each g is counted by at most $|Y|^2 < \frac{t^2}{8}$ different e. Altogether we obtain at least

$$\frac{10\nu''t^3 \times \frac{t^2}{5}}{\frac{t^2}{8}} > 2\nu''t^3$$

distinct triples g that are consistent with Q_ϕ but are not edges of $\mathcal{J}_\phi$. This contradicts (13) and we may therefore suppose that either $|A \cap Y| < 3(\nu'')^{1/3}t$ or $|B \cap Y| < 3(\nu'')^{1/3}t$.

Next suppose that $A' = A \cap X$ and $B' = B \cap X$ satisfy $\min\{|A'|, |B'|\} \geqslant 3(\nu'')^{1/3}t$. Then we have at least $10\nu''t^3$ edges $e \in \mathcal{J}_\xi$ with $|e \cap A'| = 2$ and $|e \cap B'| = 1$. For each such edge $e = \{k, k', k''\} \subset X$, and each $(i, j) \in (X - e) \times Y$, consider three distinct triples $f = \{i, j, k\}$, $f' = \{i, j, k'\}$, $f'' = \{i, j, k''\}$ that are consistent with P_ϕ. If $f, f', f'' \in J_\phi$ then consider the ten bipartite graphs defined above. Set $G = \bigcup G^{uv}$ where the union is over these ten bipartite graphs. Since $\{e, f, f', f''\} \subset \mathcal{J}_\phi \cup \mathcal{J}_\xi$, the 3-graph $J = \{e, f, f', f''\}$ associated with the 5-partite graph G and 3-graph $\mathcal{H}(\{i, j, k, k', k''\})$ is a cluster 3-graph. Again, by the embedding lemma we obtain the contradiction $T_5 \subset \mathcal{H}$. We conclude that $g \notin \mathcal{J}_\phi$ for some $g \in \{f, f', f''\}$. Each e gives rise to at least $(|X| - 3)|Y| > \frac{t^2}{5}$ such g and each g is counted by at most $|X|^2 < \frac{t^2}{2}$ different e. Altogether we obtain at least

$$\frac{10\nu''t^3 \times \frac{t^2}{5}}{\frac{t^2}{2}} > 2\nu''t^3$$

distinct triples g that are consistent with Q_ϕ but are not edges of $\mathcal{J}_\phi$. This contradicts (13).

We may therefore suppose that

(i) $|A \cap Y| < 3(\nu'')^{1/3}t$ or $|B \cap Y| < 3(\nu'')^{1/3}t$, and
(ii) $|A \cap X| < 3(\nu'')^{1/3}t$ or $|B \cap X| < 3(\nu'')^{1/3}t$.

Let us now show that (i) and (ii) imply that

$$|A \cap Y| + |B \cap X| < 6\big(\nu''\big)^{1/3}t. \tag{15}$$

If $|A \cap Y| \geqslant 3(\nu'')^{1/3}t$, then by (i) we have $|B \cap Y| < 3(\nu'')^{1/3}t$. Consequently,

$$|A \cap X| = |A - Y| \geqslant |A| - |Y| \geqslant \big(1 - \sqrt{\nu''}\big)\frac{2t}{3} - \big(1 + 2\sqrt{\nu''}\big)\frac{t}{3} > 3\big(\nu''\big)^{1/3}t$$

and also

$$|B \cap X| = \big|B - (B \cap Y)\big| \geqslant \big(1 - 2\sqrt{\nu''}\big)\frac{t}{3} - 3\big(\nu''\big)^{1/3}t > 3\big(\nu''\big)^{1/3}t.$$

This contradicts (ii) so we may assume that $|A \cap Y| < 3(\nu'')^{1/3}t$.

If $|B \cap X| \geqslant 3(\nu'')^{1/3}t$, then by (ii), we have $|A \cap X| < 3(\nu'')^{1/3}t$. This yields the contradiction

$$|X| = |A \cap X| + |B \cap X| < 3\big(\nu''\big)^{1/3}t + |B| < 3\big(\nu''\big)^{1/3}t + \big(1 + 2\sqrt{\nu''}\big)\frac{t}{3} < \big(1 - \sqrt{\nu''}\big)\frac{2t}{3}.$$

We may therefore also assume that $|B \cap X| < 3(\nu'')^{1/3}t$ and now (15) follows.

A consistent edge of Q_ξ that is inconsistent with Q_ϕ must have a point in $(A \cap Y) \cup (B \cap X)$, hence the number of such edges is at most $6(\nu'')^{1/3}t\binom{t}{2} < 4(\nu'')^{1/3}t^3$. This completes the proof of Claim 2. $\square$

6. Proof of Theorem 1

6.1. Preliminaries

Recall that the binary entropy function $H(x) := x\log_2 1/x + (1 - x)\log_2 1/(1 - x)$. We shall use Chernoff's inequality in the form below:

Theorem 9. *Let $X_1, \dots, X_m$ be independent $\{0, 1\}$ random variables with $P(X_i = 1) = p$ for each i. Let $S = \sum_i X_i$. Then the following inequality holds for $a > 0$:*

$$P(S < \mathbb{E}S - a) < \exp\big(-a^2/(2pm)\big).$$

We shall also need the following easy statement.

Lemma 10. *Every graph G with n vertices contains a matching of size at least $\frac{|G|}{2n}$.*

6.2. Estimates on $S(n)$

In this section we give some estimates on $S(n)$.

Lemma 11.

(i) $$\log_2\big(S(n)\big) \geqslant \frac{2}{27}n^3 - \frac{1}{9}n^2 - \frac{1}{9}n.$$

(ii) *For n large enough*:

$$S(n) \geqslant S(n-1) \cdot 2^{(2n^2-5n+1)/9} \geqslant S(n-2) \cdot 2^{(4n^2-14n+9)/9} \geqslant S(n-3) \cdot 2^{(6n^2-27n+28)/9}.$$

Proof. (i) We generate many semi-bipartite 3-graphs as follows: Partition $[n]$ into classes of sizes $t = \lceil 2n/3 \rceil$ and $n - t = \lfloor n/3 \rfloor$, and add any collection of consistent edges. A short calculation shows that

$$\binom{t}{2}(n - t) \geqslant \frac{2}{27}n^3 - \frac{1}{9}n^2 - \frac{1}{9}n$$

and the result follows.

(ii) It is sufficient to prove the first inequality. Given a semi-bipartite 3-graph on $[n - 1]$ with partition (X, Y), add n to Y if $|Y| < n/3$ otherwise to X, and decide about each consistent edge containing n to be added to the 3-graph or not. If $|Y| < 2n/3$ then careful calculation shows that for a given partition there are at least $2^{(2n^2-5n+2)/9}$ ways to add consistent edges containing n. However, if $|Y| \geqslant 2n/3$ then we do not generate too many 3-graphs, indeed in this case the number of possible consistent edges is at most

$$\binom{|X|}{2}|Y| \leqslant \binom{n/3}{2}\frac{2n}{3} \leqslant \frac{n^3}{27}.$$

Consequently, the number of semi-bipartite 3-graphs with vertex set $[n-1]$ and $|Y| \geqslant 2n/3$ is at most $2^{n+n^3/27} < S(n-1)\cdot(1-2^{-1/9})$ for n large enough by part (i). Therefore

$$S(n) > \bigl(S(n-1) - 2^{n+n^3/27}\bigr)2^{(2n^2-5n+2)/9} > S(n-1)\cdot 2^{(2n^2-5n+1)/9}. \qquad \Box$$

6.3. *Lower-density*

Our goal in this section is twofold: First to define a subset $\mathrm{Forb}(n, T_5, \eta, \mu) \subset \mathrm{Forb}(n, T_5, \eta)$ which comprises 3-graphs with ordered partitions (X, Y) that have a collection of useful properties. Second, to prove that most 3-graphs in $\mathrm{Forb}(n, T_5, \eta)$ are in $\mathrm{Forb}(n, T_5, \eta, \mu)$.

Let $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta)$ and let (X, Y) be an ordered partition of the vertices of $\mathcal{H}$ which minimizes the number of inconsistent edges. We call such a partition *optimal*. For a vertex x let $L_{X,X}(x)$ be the set of edges containing x, and having the other two vertices in X, and let $L_{X,Y}(x)$ and $L_{Y,Y}(x)$ be similarly defined. Sometimes, trusting that it will not cause confusion, we refer to $L_{X,X}(x)$ as the link graph of x on X. As before, we often associate a graph or hypergraph with its edge set.

Definition 12. An ordered partition (X, Y) is *μ-lower-dense* if each of the following is satisfied:

(i) For every matching $G_1 \subset \binom{X}{2}$ and every graph $G_2 \subset X \times Y$ with $|G_1| > \mu n$, $|G_2| > \mu n^2$ the following holds:

$$\bigl|\bigl\{(ab, uv)\colon\ ab \in G_2,\ uv \in G_1,\ abu, abv \in \mathcal{H}\bigr\}\bigr| > \frac{|G_1||G_2|}{72}.$$

(ii) For every graph $G_1 \subset \binom{X}{2}$ and every matching $G_2 \subset \binom{Y}{2}$ with $|G_1| > \mu n^2$, $|G_2| > \mu n$ the following holds:

$$\bigl|\bigl\{(ab, uv)\colon\ ab \in G_2,\ uv \in G_1,\ auv, buv \in \mathcal{H}\bigr\}\bigr| > \frac{|G_1||G_2|}{8}.$$

(iii) For every $A_X \subset X$, $A_Y \subset Y$ with $|A_X|, |A_Y| \geqslant \mu n$ the following holds:

$$\bigl|\{E \in \mathcal{H}\colon\ |E \cap A_X| = 2,\ |E \cap A_Y| = 1\}\bigr| > \frac{|A_X|^2|A_Y|}{8}.$$

(iv) Let $Y' \subset Y$ with $|Y'| \geqslant 2\mu n$, and suppose that for every $y \in Y'$ we have an $X_y \subset X$ with $|X_y| > 200\mu n$. Then

$$\bigl|\bigl\{E \in \mathcal{H}\colon\ \exists y \in Y' \text{ s.t. } |E \cap X_y| = 2,\ y \in E\bigr\}\bigr| > 10\,000\mu^3 n^3.$$

(v) $\bigl||Y| - n/3\bigr| < \mu n$.

We say that an $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta)$ is *μ-lower-dense* if each of its optimal partitions satisfies conditions (i)–(v). Let $\mathrm{Forb}(n, T_5, \eta, \mu) \subset \mathrm{Forb}(n, T_5, \eta)$ be the collection μ-lower-dense hypergraphs.

Lemma 13. *Let $1000H(\eta) < \mu^3$ and μ be sufficiently small. Then for n sufficiently large*

$$\bigl|\mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta, \mu)\bigr| < 2^{n^3(\frac{2}{27} - \frac{\mu^3}{500})}.$$

Proof. We count the number of hypergraphs $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta, \mu)$ violating conditions (i)–(v) separately: We shall use the following estimates in many of the cases. The number of ways to choose an ordered partition of $\mathcal{H}$ is at most 2^n. In what follows let us assume that we are given such a partition (X,Y). The number of ways the at most ηn^3 inconsistent edges could be placed is at most $2^{H(\eta)n^3}$, the number of ways a subset of vertices could be chosen is at most 2^n, the number of ways a matching (of graph edges) could be chosen is at most $2^{n\log n}$, and the number of ways a graph could be chosen is at most 2^{n^2}. The number of ways the consistent edges could be chosen is at most $2^{\frac{|X|^2}{2}|Y|} \leqslant 2^{2n^3/27}$. For this last bound, we will give some improvements using the fact that $\mathcal{H}$ is not μ-lower-dense.

For a fixed partition of the vertex set, we may view the consistent edges as a probability space, where we choose each of them, independently, with probability 1/2. We use Chernoff's inequality to show that the probability that a particular condition of the definition of μ-lower-density is violated is low, yielding an upper bound on the number of ways of choosing the consistent edges of $\mathcal{H}$.

(i) Given the choice of G_1 and G_2, there are $|G_1||G_2| \geqslant \mu^2 n^3$ possible pairs of edges to be included mentioned in the condition. However not all the edges are distinct, for example if u_1v_1, u_2v_2 are edges in G_1 and u_1b, u_2b are edges in G_2 then the triple u_1u_2b is considered for two pairs of edges: (u_1b, u_2v_2) and (u_2b, u_1v_1). In order to avoid this overcounting (which manifests itself as a lack of independence in a probability calculation) we shall choose subgraphs $G_1' \subset G_1$, $G_2' \subset G_2$, such that G_1', G_2' are vertex disjoint, and $|G_1'| \geqslant \mu n/3$, $|G_2'| \geqslant \mu n^2/3$.

We prove the existence of such G_1' and G_2' by randomly picking each edge of the matching G_1 with probability 1/2, where these choices are independent for distinct edges. Let H_1 be the (random) set of edges that were picked. Let H_2 be the (random) set of edges of G_2 that are disjoint from all edges of H_1. Then $|H_1|$ is a binomial random variable with parameters $|G_1|$ and 1/2 and $|H_2|$ dominates a binomial random variable with parameters $|G_2|$ and 1/2. The reason for this is that for $e \in G_2$, the probability that $e \in H_2$ is 1/2 or 1, depending on whether e is incident to an edge of G_1 or not. So by Chernoff's inequality,

$$P\big(|H_i| < |G_i|/3\big) = P\big(|H_i| < |G_i|/2 - |G_i|/6\big) < \exp\big(-|G_i|/36\big) < \frac{1}{2}.$$

Consequently,

$$P\big(|H_1| \geqslant |G_1|/3 \text{ and } |H_2| \geqslant |G_2|/3\big) > 0$$

and there exist G_1' and G_2' as above.

For each $uv \in G_1'$ and $ab \in G_2'$ let $X_{ab,uv}$ be the random variable that is 1 if both $abu, abv \in \mathcal{H}$ and 0 otherwise. Then $P(X_{ab,uv} = 1) = 1/4$, and since G_1' and G_2' are vertex disjoint, these random variables are independent. We apply Chernoff's inequality to these $m = |G_1'||G_2'|$ random variables with $a = m/8$ and $p = 1/4$. For $S = \sum_{uv\in G_1',\, ab\in G_2'} X_{ab,uv}$ this gives

$$P\left(S \leqslant \frac{|G_1||G_2|}{72}\right) \leqslant P(S \leqslant m/8) \leqslant \exp\left(-\frac{(m/8)^2}{(m/2)}\right) = \exp(-m/32) < \exp\left(-\frac{\mu^2 n^3}{9\cdot 32}\right).$$

Using this upper bound we obtain that the number of hypergraphs that violate condition (i) is upper bounded by

$$2^{n+H(\eta)n^3+n\log n+n^2+2n^3/27}\exp\big(-\mu^2n^3/(9\cdot 32)\big) < 2^{2n^3/27-\mu^2n^3/300}.$$

(ii) Given the choice of G_1 and G_2, there are $|G_1||G_2| \geqslant \mu^2 n^3$ possible pairs of edges to be included mentioned in the condition. Unlike in case (i), here all the edges are distinct so we do not need to construct G_i'.

For each $uv \in G_1$ and $ab \in G_2$ let $X_{ab,uv}$ be the random variable that is 1 if both $uva, uvb \in \mathcal{H}$ and 0 otherwise. Then $P(X_{ab,uv} = 1) = 1/4$, and these random variables are independent. We apply Chernoff's inequality to these $m = |G_1||G_2|$ random variables with $a = m/8$ and $p = 1/4$. For $S = \sum_{uv\in G_1,\, ab\in G_2} X_{ab,uv}$ this gives

$$P\left(S \leqslant \frac{|G_1||G_2|}{8}\right) \leqslant P(S \leqslant m/8) \leqslant \exp\left(-\frac{(m/8)^2}{(m/2)}\right) = \exp(-m/32) \leqslant \exp\left(-\frac{\mu^2 n^3}{32}\right).$$

Using this upper bound we obtain that the number of hypergraphs that violate condition (ii) is upper bounded by

$$2^{2n^3/27-\mu^2 n^3/32}.$$

(iii) Given the choice of A_X and A_Y, there are $|A_X|(|A_X|-1)|A_Y|/2 \geqslant \mu^3 n^3/3 =: m$ possible edges of $\mathcal{H}$ with two vertices in A_X and one in A_Y. Using Chernoff's inequality (with $p = 1/2$) we obtain that the number of hypergraphs violating condition (iii) is at most

$$2^{3n+H(\eta)n^3+2n^3/27}\exp\bigl(-\mu^3 n^3/24\bigr) < 2^{2n^3/27-\mu^3 n^3/24}.$$

(iv) Given the ordered 2-partition, there are at most 2^n choices for each of X_y and of Y'. Also

$$\left|\left\{E \in \binom{[n]}{3}\colon \exists y \in Y' \text{ s.t. } |E \cap X_y| = 2,\ y \in E\right\}\right| \geqslant 2\mu n\binom{200\mu n}{2} > 35\,000\mu^3 n^3.$$

By Chernoff's inequality we obtain that the number of hypergraphs violating condition (iv) is at most

$$2^{2n^2+H(\eta)n^3+2n^3/27}\exp\bigl(-\mu^3 n^3\bigr) < 2^{2n^3/27-\mu^3 n^3}.$$

Note that in the computation above we used $1000H(\eta) < \mu^3$ and n is sufficiently large.

(v) In this case we show that if the ratio of the parts of the ordered partition differ too much from 2, then the number of ways to place the consistent edges decreases exponentially. This is simply because the number of consistent edges is small. More precisely, if $||Y| - n/3| \geqslant \mu n$ then the number of possible consistent edges is at most $(2/27 - \mu^2/2 + \mu^3/2)n^3 < (2/27 - \mu^2/3)n^3$. This implies that the number of such hypergraphs is at most

$$2^n \cdot 2^{n^3(2/27+H(\eta)-\mu^2/3)} < 2^{n^3(2/27-\mu^2/6)}.$$

Summing up the number of 3-graphs in cases (i)–(v) gives

$$\begin{aligned}
&\bigl|\mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta, \mu)\bigr| \\
&\quad \leqslant 2^{2n^3/27}\bigl(2^{-\mu^2 n^3/300} + 2^{-\mu^2 n^3/32} + 2^{-\mu^3 n^3/24} + 2^{-\mu^3 n^3} + 2^{-\mu^2/6}\bigr) \\
&\quad < 2^{2n^3/27-\mu^3 n^3/500}.
\end{aligned}$$

This completes the proof of the lemma. □

6.4. *Getting rid of bad vertices*

From now on we shall have the following hierarchy of constants: $1 \gg \alpha \gg \beta \gg \mu \gg \eta$. More precisely we will assume

$$\begin{aligned}
&0.01 > H(\alpha), \qquad \alpha^2 > 100\bigl(H(\beta) + H(2\mu) + \mu^2\bigr), \\
&\beta > 100H(2\mu), \qquad \mu^3 \geqslant 1000H(\eta).
\end{aligned} \tag{16}$$

In this section we prove additional properties of hypergraphs in $\mathrm{Forb}(n, T_5, \eta, \mu)$ which involve the link graph of vertices.

Lemma 14. *Let $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$ with an optimal ordered partition (X, Y). Then the following hold.*

(i) *For $x \in X$ we have $|L_{X,X}(x)| \leqslant 2\mu n^2$.*
(ii) *For $y \in Y$ we have $|L_{X,Y}(y)| \leqslant 2\mu n^2$.*
(iii) *For $y \in Y$ we have $\min\{|L_{X,X}(y)|, |L_{Y,Y}(y)|\} < 2\mu n^2$.*

We remark here that the lack of similar bounds for $x \in X$ on $|L_{X,Y}(x)|$ makes the proof of the main result complicated.

Proof. (i) Assume that for some $x \in X$ we have $|L_{X,X}(x)| > 2\mu n^2$. By the optimality of the partition we have $|L_{X,Y}(x)| > 2\mu n^2$ as well. By Lemma 10 $L_{X,X}(x)$ contains a matching G_1 of size at least μn. With $G_2 = L_{X,Y}(x)$, using property (i) of the definition of μ-lower-density, there exists an $ab \in G_2$ and $uv \in G_1$ such that $abu, abv \in \mathcal{H}$. Together with abx and uvx, we obtain T_5 in $\mathcal{H}$, a contradiction.

(ii) Assume that for some $y \in Y$ we have $|L_{X,Y}(y)| > 2\mu n^2$. By the optimality of the partition we have $|L_{X,X}(y)| > 2\mu n^2$ as well. By Lemma 10 $L_{X,X}(y)$ contains a matching G_1 of size at least μn. With $G_2 = L_{X,Y}(y)$, using property (i) of the definition of μ-lower-density, there exists an $ab \in G_2$ and $uv \in G_1$ such that $abu, abv \in \mathcal{H}$. Together with aby and uvy we obtain a T_5 in $\mathcal{H}$, a contradiction.

(iii) Assume that for some $y \in Y$ we have $|L_{X,X}(y)|, |L_{Y,Y}(y)| > 2\mu n^2$. By Lemma 10 $L_{Y,Y}(y)$ contains a matching G_2 of size at least μn. With $G_1 = L_{X,X}(y)$, using property (ii) of the definition of μ-lower-density, there exists an $ab \in G_2$ and $uv \in G_1$ such that $auv, buv \in \mathcal{H}$. Together with aby and uvy we obtain a T_5 in $\mathcal{H}$, a contradiction. $\square$

For a set $S \subset [n]$ of size two and for $A \subset [n]$, we define $L_A(S)$ to be the set of vertices $v \in A$ such that $\{v\} \cup S \in \mathcal{H}$. We call an edge $xyz \in \mathcal{H}$ *α-rich* with respect to an optimal partition (X, Y) of $\mathcal{H}$ if $x \in X$, $y, z \in Y$ and $\max\{|L_X(x, y)|, |L_X(x, z)|\} > \alpha n$. The vertex z is the *poor* vertex of a rich edge if $|L_X(x, y)| \geqslant |L_X(x, z)|$; in case of a tie we can decide arbitrarily.

Lemma 15. *Let (X, Y) be an optimal ordered partition of an $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$. For $\alpha \geqslant 200\mu$ the following holds*:

(i) *The number of distinct poor vertices of the α-rich edges of $\mathcal{H}$ is at most $2\mu n$.*
(ii) *For any vertex $x \in X$ the number of α-rich edges containing x is at most $2\mu n^2$.*

Proof. (i) Assume not, i.e., let $\{x_i y_i z_i\}$ be α-rich edges for $i \in [\lceil 2\mu n \rceil]$, where $x_i \in X$ and $x_i y_i z_i$ has poor vertex z_i and the z_i's are different vertices. Let $Y' = \{z_1, \dots, z_{\lceil 2\mu n \rceil}\}$ and $X_{z_i} = L_X(x_i, y_i)$. As y_i is not the poor vertex of the rich edge $x_i y_i z_i$, we have $|X_{z_i}| > 200\mu n$. By condition (iv) of the definition of μ-lower-density there is an i such that for some $a, b \in L_X(x_i, y_i)$, $abz_i \in \mathcal{H}$. But then $x_i y_i z_i$, $x_i y_i a$, $x_i y_i b$, abz_i form a T_5 in $\mathcal{H}$, a contradiction.

(ii) The number of rich edges containing a vertex $z \in Y$ and x is at most n, hence if (ii) was false, then there would be at least $2\mu n$ poor vertices in Y, contradicting (i). $\square$

6.5. Getting rid of the inconsistent edges

In this section we estimate the number of 3-graphs $\mathcal{H}$ from $\mathrm{Forb}(n, T_5, \eta, \mu)$ which violate one of the conditions below. Note that if an $\mathcal{H}$ does not violate any of the conditions below then $\mathcal{H} \in \mathcal{S}(n)$.

(1) In every optimal partition (X, Y) of $\mathcal{H}$ and for every $x \in X$ we have $|L_{Y,Y}(x)| < \beta n^2$.
(2) In every optimal partition (X, Y) of $\mathcal{H}$ every $y \in Y$ satisfies $|L_{Y,Y}(y)| < 2\mu n^2$.
(3) No optimal partition (X, Y) of $\mathcal{H}$ contains an α-rich edge.
(4) No optimal partition (X, Y) of $\mathcal{H}$ has an inconsistent edge xyz with $|\{x, y, z\} \cap X| \in \{0, 3\}$.
(5) No optimal partition (X, Y) of $\mathcal{H}$ has an inconsistent edge xyz with $|\{x, y, z\} \cap X| = 1$.

Our goal is to prove the following result, which will be completed in the next section.

Theorem 16. *There is a C_1 such that the number of $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$ not satisfying any of the conditions* (1)–(5) *is at most $C_1 \cdot 2^{-n/10} S(n)$.*

Before proceeding we state and prove the following lemma. For integers $a < b$, let $[a, b] = \{a, a + 1, \dots, b\}$.

Lemma 17. *Fix a matching M with m edges, say $\{1,2\},\dots,\{2m-1,2m\}$. The number of graphs on $[N]$, where M is a maximum matching is less than*

$$2^{2m^2-2m}\left(N-2m+2^{N-2m+1}\right)^m.$$

Proof. We allow complete freedom to include edges on $[2m]$ yielding $2^{\binom{2m}{2}-m}=2^{2m^2-2m}$ ways to choose these edges. There is no edge inside $[2m+1,N]$ by the maximality of M. Consider an edge $\{2i-1,2i\}\in M$. If for $j_1,j_2\in[2m+1,N]$ both $\{j_1,2i-1\}$ and $\{j_2,2i\}$ are edges then again by maximality of M, we have $j_1=j_2$. So either there is a vertex in $[2m+1,N]$ with edges to both $2i-1$ and $2i$, or one of $2i-1$ or $2i$ has no edge to any vertex in $[2m+1,N]$. For each i we obtain $N-2m+2^{N-2m+1}$ possibilities for the set of edges incident to $\{2i-1,2i\}$, thereby completing the proof. $\square$

In the next five subsections, we will let n be sufficiently large as needed.

6.5.1. 3-graphs violating (1)

In this section we prove the following lemma.

Lemma 18. *The number of $\mathcal{H}\in\mathrm{Forb}(n,T_5,\eta,\mu)$ violating condition* (1) *is at most*

$$\left|\mathrm{Forb}(n-1,T_5)\right|2^{\frac{2n^2}{9}-\frac{\beta n^2}{5}}.$$

Proof. First we fix an optimal partition (X,Y) of $\mathcal{H}$, which can be chosen in at most 2^n ways. Choose an $x\in X$, which can be done in at most n ways. Assume that $|L_{Y,Y}(x)|\geqslant\beta n^2$. Let

$$B:=\{z\in Y\colon\ \exists y\in Y\ \text{s.t.}\ xyz\ \text{is}\ \alpha\text{-rich, where}\ z\ \text{is the poor vertex of}\ xyz\}.$$

By Lemma 15 (i) we have $|B|\leqslant 2\mu n$. So $Y-B$ does not contain both y and z from an α-rich edge xyz. Let $M\subset\binom{Y-B}{2}$ be a maximum matching in $L_{Y,Y}(x)$. Since $|Y|<n/2$ and $\beta>10\mu$, we have

$$|M|\geqslant\left(\left|L_{Y,Y}(x)\right|-2\mu n^2\right)/2|Y|\geqslant\beta n/2.$$

Denote the vertex set of the matching M by A, and let $m=|M|$.

The number of choices for A is at most 2^n, and the number of choices for M is at most $2^{n\log n}$. For every $y\in A$ we have $|L_X(x,y)|<\alpha n^2$. The number of choices for $\mathcal{H}-x$ is at most $|\mathrm{Forb}(n-1,T_5)|$, and by Lemma 14 part (i) the number of choices for $L_{X,X}(x)$ is at most $\sum_{i\leqslant 2\mu n^2}\binom{n^2}{i}\leqslant 2^{H(2\mu)n^2}$. The number of choices for the edges of $L_{Y,Y}(x)$ intersecting B is at most $2^{|B||Y|}<2^{\mu n^2}$. Using Lemma 17, given M, the number of ways the rest of $L_{Y,Y}(x)$ can be chosen is at most

$$2^{2m^2-2m}\left(|Y|-2m+2^{|Y|-2m+1}\right)^m<2^{2m^2-2m}\left(2^{|Y|-2m+2}\right)^m=2^{|Y|m}.$$

Since $|Y|\leqslant n/3+\mu n$, the number of ways the consistent edges containing x could be chosen is at most $2^{|X||Y|}<2^{2n^2/9+\mu n^2}$. Our goal is to improve this bound by using the fact that x violates condition (1). Specifically, we write $Y=A\cup(Y-A)$ and replace $2^{\frac{2n^2}{9}+\mu n^2}$ by $2^{\frac{2n^2}{9}+\mu n^2}\cdot 2^{-2m|X|}\cdot\ell$, where ℓ is the number of ways to add edges of the form $xab\in\mathcal{H}$ with $a\in A$, $b\in X$.

The number of ways to choose the (consistent) edges of the form $xab\in\mathcal{H}$ with $a\in A$, $b\in X$ is

$$\ell\leqslant\left(\sum_{i\leqslant\alpha n}\binom{|X|}{i}\right)^{2m}<2^{2H(\alpha)mn}.$$

Here we use the fact that $\{a,x\}$ is not subset of any α-rich inconsistent edge, so for given a the number of choices for b is at most αn. To summarize, the number of 3-graphs for given m violating (1) is at most

$$n2^n2^n2^{n\log n}\left|\mathrm{Forb}(n-1,T_5)\right|2^{H(2\mu)n^2}2^{\mu n^2}2^{-2m|X|}2^{|Y|m}2^{\frac{2n^2}{9}+\mu n^2}2^{2H(\alpha)mn}. \tag{17}$$

The coefficient of m in the exponent above is

$$-2|X| + |Y| + 2H(\alpha)n < -n/2.$$

Therefore, viewing (17) as a function of m, it is maximized when m is minimized, i.e. $m = \beta n/2$. Fixing m at this value, for the coefficient of n^2 in the exponent, since $100H(2\mu) < \beta \ll 1$, we have

$$\frac{2\log n}{n^2} + \frac{2+\log n}{n} + H(2\mu) + 2\mu - \beta/4 < -\beta/5.$$

Since there are at most n choices for m, we conclude that the number of 3-graphs violating (1) is bounded above by

$$\left|\mathrm{Forb}(n-1, T_5)\right| 2^{\frac{2n^2}{9} - \frac{\beta n^2}{5}} \tag{18}$$

as required. □

6.5.2. 3-graphs violating (2)

In this section we prove the following lemma.

Lemma 19. *The number of $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$ violating condition (2) is at most*

$$\left|\mathrm{Forb}(n-1, T_5)\right| \cdot 2^{\frac{n^2}{17}}.$$

Proof. First fix an optimal partition (X, Y), which can be chosen at most 2^n ways. Given an optimal partition (X, Y), assume that there is a $y \in Y$ such that $|L_{Y,Y}(y)| \geqslant 2\mu n^2$. Then by Lemma 14 (iii) we have $|L_{X,X}(y)| < 2\mu n^2$, and by optimality of the partition (X, Y) we have $|L_{X,Y}(y)| \leqslant 2\mu n^2$. So the number of 3-graphs having such a vertex y is at most

$$n2^n\left|\mathrm{Forb}(n-1, T_5)\right| \cdot 2^{|Y|^2/2 + 2H(2\mu)n^2} < \left|\mathrm{Forb}(n-1, T_5)\right| \cdot 2^{\frac{n^2}{17}}, \tag{19}$$

where we used condition (iv) of Definition 12. □

6.5.3. 3-graphs satisfying (1) and (2) but violating (3)

In this section we prove the following lemma.

Lemma 20. *The number of $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$ satisfying conditions (1) and (2) but violating condition (3) is at most*

$$\left|\mathrm{Forb}(n-3, T_5)\right| 2^{\frac{2n^2}{3} - \frac{\alpha^2 n^2}{3}}.$$

Proof. Assume that (X, Y) is an optimal partition of $\mathcal{H}$ and xyz is an α-rich edge with $x \in X$, $y, z \in Y$ and $|L_X(x, y)| \geqslant |L_X(x, z)|$. The edge xyz could be chosen in at most n^3 ways and $L_X(x, y)$ can be chosen in at most 2^n ways. Given these choices, we can choose $\mathcal{H} - \{x, y, z\}$ in at most $|\mathrm{Forb}(n-3, T_5)|$ ways. By Lemma 14 (i) and the fact that $\mathcal{H}$ satisfies condition (1), the number of ways the inconsistent edges containing x can be chosen is at most $2^{H(2\mu)n^2 + H(\beta)n^2}$. By Lemma 14 (ii) and the fact that $\mathcal{H}$ satisfies condition (2), the number of ways of having the inconsistent edges intersecting y or z is at most $2^{4H(2\mu)n^2}$. The number of ways the consistent edges containing x or y could be chosen is at most $2^{|X|\cdot|Y| + |X|^2/2}$. The number of ways the consistent edges containing z could be chosen is at most $2^{\frac{|X|^2}{2} - \binom{|L_X(x,y)|}{2}}$, as for $a, b \in L_X(x, y)$, edge abz together with xyz, xya, xyb forms a copy of T_5. Since xyz is an α-rich $|L_X(x, y)| \geqslant \alpha n$. So the number of 3-graphs satisfying (1) and (2) but violating (3) is at most

$$2^n n^3\left|\mathrm{Forb}(n-3, T_5)\right| 2^{H(2\mu)n^2 + H(\beta)n^2 + 4H(2\mu)n^2} \cdot 2^{|X|\cdot|Y| + \frac{|X|^2}{2} + \frac{|X|^2}{2} - \binom{|L_X(x,y)|}{2}}. \tag{20}$$

Since $\alpha^2 > 100(H(\beta) + H(2\mu) + \mu^2)$, $|X| \leqslant 2n/3 + \mu n$, $|X||Y| + |X|^2 = |X|n$ and

$$\frac{n + 3\log n}{n^2} + 6H(2\mu) + H(\beta) + \mu - \alpha^2/2 + \frac{\alpha}{n} < -\frac{\alpha^2}{3},$$

we conclude that (20) is at most

$$\bigl|\mathrm{Forb}(n-3, T_5)\bigr| 2^{\frac{2n^2}{3} - \frac{\alpha^2 n^2}{3}}, \tag{21}$$

thereby completing the proof. □

6.5.4. 3-graphs satisfying (1), (2) and (3) but violating (4)

In this section we prove the following lemma.

Lemma 21. *The number of $\mathcal{H} \in \mathrm{Forb}(n, T_5, \eta, \mu)$ satisfying conditions* (1) *and* (2) *and* (3) *but violating condition* (4) *is at most*

$$\bigl|\mathrm{Forb}(n-3, T_5)\bigr| 2^{\frac{7n^2}{11}}.$$

Proof. First fix an optimal partition (X, Y), which can be chosen at most 2^n ways. Given an optimal partition (X, Y), an inconsistent edge xyz could be chosen in at most n^3 ways. We can choose $\mathcal{H} - \{x, y, z\}$ in at most $|\mathrm{Forb}(n-3, T_5)|$ ways. The number of edges having at least two of x, y, z is at most $3n$, giving at most 2^{3n} ways to place them.

Now consider the case that $x, y, z \in X$. There are two types of inconsistent edges e containing one of $\{x, y, z\}$, either $e \subset X$, or $e - \{x, y, z\} \subset Y$. In the first case Lemma 14 (i) implies that there are at most $3 \cdot 2\mu n^2$ such edges, and in the second case, since $\mathcal{H}$ satisfies condition (1) there are at most $3 \cdot \beta n^2$ such edges. So the number of ways the inconsistent edges intersecting $\{x, y, z\}$ can be chosen is at most

$$2^{(3H(\beta)+3H(2\mu))n^2}.$$

The number of ways that the consistent edges containing any of x, y, z can be chosen is restricted as follows: For any $a \in X$, $b \in Y$ out of the 8 possibilities including edges abx, aby, abz only 7 can occur (all of them cannot be chosen at the same time), so the number of possible connections is at most $7^{|X||Y|}$.

Consider now the other case when $x, y, z \in Y$. There are two types of inconsistent edges e: Either $e \subset Y$ or $e \cap X \neq 0$. In the first case, since $\mathcal{H}$ satisfies condition (2), that there are at most $3 \cdot 2\mu n^2$ such e, and in the second case Lemma 14 (ii) implies that there are at most $3 \cdot 2\mu n^2$ such e. So the number of ways to choose those edges is at most $2^{6H(2\mu)n^2} < 2^{(3H(\beta)+3H(2\mu))n^2}$. Now let us bound the number of ways the consistent edges intersecting $\{x, y, z\}$ can be chosen. Since for any pair $a, b \in X$, we cannot have $\{abx, aby, abz\} \subset \mathcal{H}$, the number of ways to place these type of edges is at most $7^{|X|^2/2}$.

Altogether the number of 3-graphs satisfying (1), (2) and (3) but violating (4) is bounded by

$$2^{n+1} n^3 2^{3n} 2^{(3H(\beta)+3H(2\mu))n^2} \bigl|\mathrm{Forb}(n-3, T_5)\bigr| \bigl(7^{|X||Y|} + 7^{\frac{|X|^2}{2}}\bigr). \tag{22}$$

Since $\log_2 7 < 2.81$, $\max\{|X||Y|, |X|^2/2\} \leqslant (2/9 + \mu)n^2 - 1$, and

$$\frac{n + 1 + 3\log n + 3n}{n^2} + 3H(\beta) + 3H(2\mu) + \frac{1}{n^2} < \frac{1}{100},$$

(22) is upper bounded by

$$\bigl|\mathrm{Forb}(n-3, T_5)\bigr| 2^{\frac{7n^2}{11}}$$

as required. □

6.5.5. 3-graphs satisfying (1), (2), (3) and (4) but violating (5)

Let us denote the 3-graphs $\mathcal{H}$ described in the title of this section by $\mathrm{Forb}^{(1)}(n, T_5, \eta, \mu)$. Our goal in this section is to prove the following lemma.

Lemma 22. *The number of $\mathcal{H} \in \mathrm{Forb}^{(1)}(n, T_5, \eta, \mu)$ is at most*

$$\big(2^{-\alpha n^3} + 2^{-n/10}\big) S(n).$$

Lemma 22 will be proved in several steps. First we need some more definitions. Let $\mathcal{H} \in \mathrm{Forb}^{(1)}(n, T_5, \eta, \mu)$ and (X, Y) be an optimal partition of $\mathcal{H}$. The *shadow graph* of the inconsistent edges with respect (X, Y) is

$$G := G_{\mathcal{H}}(X, Y) := \bigcup_{y \in Y} L_{X,Y}(y).$$

Let $\mathrm{Forb}(n, T_5, \eta, \mu, \alpha) \subset \mathrm{Forb}^{(1)}(n, T_5, \eta, \mu)$ be the collection of 3-graphs $\mathcal{H}$ whose every optimal partition (X, Y) satisfies $|G_{\mathcal{H}}(X, Y)| < 100\alpha n^2$.

Lemma 23. *For n sufficiently large*

$$\big|\mathrm{Forb}^{(1)}(n, T_5, \eta, \mu) - \mathrm{Forb}(n, T_5, \eta, \mu, \alpha)\big| < 2^{-\alpha n^3} S(n). \tag{23}$$

Proof. Let us count the number of $\mathcal{H} \in \mathrm{Forb}^{(1)}(n, T_5, \eta, \mu) - \mathrm{Forb}(n, T_5, \eta, \mu, \alpha)$. We can fix an optimal partition in at most 2^n ways, and a shadow graph G in at most 2^{n^2} ways. As $\mathcal{H}$ satisfies condition (3), there is no α-rich edge of $\mathcal{H}$. Since for any edge $xy \in G$ there is a $z \in Y$ such that $xyz \in \mathcal{H}$, hence there are at most $2^{H(\alpha)n}$ ways to choose $L_X(x, y)$. Given G, the number of inconsistent edges is at most $|G||Y|/2$ (each is counted twice). The number of consistent triples that are not edges is at least $|G|(|X| - \alpha n)/2$ for the following reason: for each edge $xy \in G$, there is a vertex $z \in Y$ with $xyz \in \mathcal{H}$. Since there is no α-rich edge, e.g. xyz is not an α-rich edge, $|L_X(x, y)| \leqslant \alpha n$, and so the number of consistent triples containing x and y that are not edges is at least $|X| - \alpha n$. The factor two arises as these triples are counted at most twice. Since $\binom{|X|}{2}|Y| \leqslant 2n^2/9$, we conclude that the number of vertex triplets which could be consistent edge, is at most

$$\frac{2n^3}{27} - \frac{|G|}{2}\big(|X| - \alpha n\big) \leqslant \frac{2n^3}{27} - \frac{|G||X|}{2} + \alpha n^3.$$

Each of these could either be included in $\mathcal{H}$ or not. Altogether we obtain

$$\begin{aligned}\big|\mathrm{Forb}^{(1)}(n, T_5, \eta, \mu) - \mathrm{Forb}(n, T_5, \eta, \mu, \alpha)\big| &< 2^n 2^{n^2} 2^{H(\alpha)n|G|} 2^{\frac{|G||Y|}{2}} 2^{\frac{2n^3}{27} - \frac{|G||X|}{2} + \alpha n^3} \\ &= 2^{\frac{2n^3}{27} - \frac{|G|(|X| - |Y| - 2H(\alpha)n)}{2} + \alpha n^3 + n^2 + n} \\ &< 2^{\frac{2n^3}{27} - 2\alpha n^3}\end{aligned}$$

where the last inequality follows from $|G| \geqslant 100\alpha n^2$, $|X| - |Y| > n/4$ and $H(\alpha) < 0.01$. The lower bound on $S(n)$ from Lemma 11, and n sufficiently large gives $S(n) > 2^{\frac{2n^3}{27} - \alpha n^3}$. Consequently,

$$\big|\mathrm{Forb}^{(1)}(n, T_5, \eta, \mu) - \mathrm{Forb}(n, T_5, \eta, \mu, \alpha)\big| \leqslant 2^{-\alpha n^3} S(n)$$

and the proof is complete. □

Now we shall show that the number of non-semi-bipartite 3-graphs in $\mathrm{Forb}(n, T_5, \eta, \mu, \alpha)$ is much smaller than the number of semi-bipartite 3-graphs. First we partition $\mathrm{Forb}(n, T_5, \eta, \mu, \alpha)$ into $O(n^2)$ classes, and for each class we construct a bipartite graph B_i. One part of B_i will be the elements of a class $\mathcal{C}$, and the other part of B_i will be the set of semi-bipartite 3-graphs $\mathcal{S}(n)$. B_i will have the property that the degree of the vertices in $\mathcal{C}$ will be exponentially larger than the degrees in $\mathcal{S}(n)$. This approach will allow us to prove the following lemma. Clearly Lemma 23 and Lemma 24 immediately imply Lemma 22.

Lemma 24. *For n sufficiently large*

$$\left|\mathrm{Forb}(n, T_5, \eta, \mu, \alpha) - \mathcal{S}(n)\right| < 2^{-n/10} S(n).$$

Proof. For $0 < i \leqslant 100\alpha n^2$ let $\mathcal{C}_i \subset \mathrm{Forb}(n, T_5, \eta, \mu, \alpha) - \mathcal{S}(n)$ be the collection of 3-graphs which have an optimal partition in which the shadow graph of inconsistent edges has exactly i edges. We construct a bipartite graph B_i with parts $\mathcal{C}_i$ and $\mathcal{S}(n)$. An $\mathcal{H} \in \mathcal{C}_i$ will be joined in B_i to the following set of semi-bipartite 3-graphs, denoted by $\Phi(\mathcal{H})$:

- Remove all edges which contain an edge of G (the shadow graph of $\mathcal{H}$) (so all the inconsistent edges will be removed).
- For every $xy \in G$ add some collection of edges axy to $\mathcal{H}$ where $a \in X$.

First we give a lower bound on the degree (in B_i) of a vertex $\mathcal{H} \in \mathcal{C}_i$. Here we have to give a lower bound on the number of edges of the form axy where $xy \in G$ (and say $y \in Y$). Each edge can be counted at most twice, so the number of edges that we must decide to add to $\mathcal{H}$ is at least $(|X| - 1)i/2$, therefore $\deg_{B_i}(\mathcal{H}) \geqslant 2^{(|X|-1)i/2}$.

Before proceeding further we need the following.

Claim. *Let $\mathcal{H} \in \mathcal{S}(n)$ such that $\Phi^{-1}(\mathcal{H}) \neq \emptyset$. Then the number of partitions of $[n]$ which are optimal partitions of $\Phi^{-1}(\mathcal{H})$ is at most*

$$2^{H(10\mu)n}.$$

Proof. If $\mathcal{F} \in \Phi^{-1}(\mathcal{H})$ then $\mathcal{F} \in \mathrm{Forb}(n, T_5, \eta, \mu, \alpha)$ so it has a partition with at most ηn^3 inconsistent edges. Let $\mathcal{F}_j \in \Phi^{-1}(\mathcal{H})$ have an optimal partition (X_j, Y_j) for $j = 1, 2$. We claim that $|X_1 \Delta X_2| < 10\mu n$. Indeed, otherwise w.l.o.g. $|X_1 - X_2| \geqslant 5\mu n$. Then by Definition 12 (v) we have $||Y_1| - n/3|, ||Y_2| - n/3| < \mu n$ so $|X_2 \cap Y_1| \geqslant 3\mu n$ and $|X_1 \cap X_2| > n/4$. This makes it possible to find many inconsistent edges inside X_2, as using Definition 12 (iii)

$$\left|\{abc \in \mathcal{H}\colon a, b \in X_1 \cap X_2,\ c \in X_2 \cap Y_1\}\right| \geqslant \frac{3}{16}\mu n^3 > \eta n^3.$$

This contradiction shows that the optimal partitions do not differ too much from each other. To complete the proof of the Claim, we may count the number of optimal (X_2, Y_2) by first picking the vertices of $|X_1 \Delta X_2|$ and observing that this determines (X_2, Y_2). □

Now we fix an $\mathcal{H} \in \mathcal{S}(n)$, and give an upper bound on its degree in the auxiliary graph. Recall that in forming $\mathcal{H}$ we did not change any of the consistent edges that did not contain any edge of G.

- The number of ways G could be chosen is at most $\binom{n^2}{i}$.
- Given (X, Y) and G, the number of ways the inconsistent edges could be added is at most $2^{i|Y|/2}$.
- Given G, and $xy \in G$, as xy arises from an inconsistent edge that is not α-rich, the number of consistent edges on xy in the source 3-graph is at most αn. This gives at most $\binom{n}{\alpha n}^i$ possibilities to choose the consistent edges that contain an edge of G.

By the Claim, the number of optimal partitions (X, Y) is at most $2^{H(10\mu)n}$. So for each $\mathcal{H} \in \mathcal{S}(n)$ we have

$$\deg_{B_i}(\mathcal{H}) \leqslant 2^{H(10\mu)n}\binom{n^2}{i}2^{i|Y|/2}\binom{n}{\alpha n}^i \leqslant \left(2^{H(10\mu)n + 6\log n + |Y|/2 + H(\alpha)n}\right)^i.$$

Trivially, $|\mathcal{C}_i|/|\mathcal{S}(n)|$ is at most the ratios of the bounds of the degrees, i.e.,

$$\frac{|\mathcal{C}_i|}{S(n)} \leqslant \left(2^{H(10\mu)n+6\log n+|Y|/2+H(\alpha)n-|X|/2+1/2}\right)^i.$$

Since $||Y|-n/3| \leqslant \mu n$, and μ is sufficiently small, $|X|-|Y| \geqslant n/3-2\mu n \geqslant n/4$. Consequently, the expression above is upper bounded by $2^{-in/9}$. We conclude that

$$\left|\mathrm{Forb}(n,T_5,\eta,\mu,\alpha)-\mathcal{S}(n)\right| \leqslant \sum_{i=1}^{100\alpha n^2} |\mathcal{C}_i| \leqslant n^2 S(n)2^{-n/9} < S(n)2^{-n/10}$$

and the proof is complete. □

6.6. Completing the proofs of Theorems 1, 3 and 16

In this section we will simultaneously prove Theorems 1, 3 and 16 by induction on n. Write Theorem $P(n)$ for the statement that Theorem P holds for n. Also, let Theorem $3(\eta,n)$ denote the statement that Theorem 3 holds for n with input parameter η.

Let us first choose $\eta > 0$ sufficiently small so that the hierarchy of the parameters in (16) holds and η is a valid input parameter for Theorem 3. The structure of the induction arguments in the three proofs is as follows:

$$\text{Theorem } 1(n-1) \to \text{Theorem } 16(n) \to \text{Theorem } 3(\eta,n) \to \text{Theorem } 1(n).$$

The above will prove that Theorems 1 and 16 hold, and that Theorem 3 holds with input η. Since this is proved for each $\eta > 0$ that is sufficiently small, it also proves Theorem 3.

With input parameter η, Theorem 2 outputs ν and n_0. Let $n_1 > n_0$ be sufficiently large such that for every $n > n_1$ Lemmas 13, 11, 18, 19, 20, 21 and 22 hold. We also require $1/n_1$ to be much smaller than all the fixed small constants in (16). Let $c > 100$ be chosen so that Theorem 16 holds with $C_1 = c$ for all $n \leqslant n_1$, Theorem 3 with input η holds with $C' = c$ for all $n \leqslant n_1$ and Theorem 1 holds with $C = c$ for all $n \leqslant n_1$. Now we fix

$$C = 2C' = 4C_1 = 4c > 400.$$

Proof of Theorem 16. We wish to prove Theorem $16(n)$, so as indicated above, we may assume Theorem $1(n')$ for $n' < n$. We recall that if $\mathcal{H} \in \mathrm{Forb}(n,T_5,\eta,\mu) - \mathcal{S}(n)$, then $\mathcal{H}$ violates one of the conditions (1)–(5). Consequently, an upper bound for $|\mathrm{Forb}(n,T_5,\eta,\mu)-\mathcal{S}(n)|$ is obtained by summing the bounds in Lemmas 18–22, which is

$$\begin{aligned}&\left|\mathrm{Forb}(n-1,T_5)\right|2^{2n^2/9-\beta n^2/5} + \left|\mathrm{Forb}(n-1,T_5)\right|\cdot 2^{n^2/17} + \left|\mathrm{Forb}(n-3,T_5)\right|2^{6n^2/9-\alpha^2n^2/3}\\ &\quad + \left|\mathrm{Forb}(n-3,T_5)\right|2^{7n^2/11} + \left(2^{-\alpha n^3}+2^{-n/10}\right)S(n).\end{aligned}$$

We may assume that Theorem $1(n')$ holds for all $n' < n$ with parameter C. Hence we can upper bound this expression by

$$\begin{aligned}&S(n-1)\left(C2^{-(n-1)/10}+1\right)\left(2^{2n^2/9-\beta n^2/5}+2^{n^2/17}\right)\\ &\quad + S(n-3)\left(C2^{-(n-3)/10}+1\right)\left(2^{6n^2/9-\alpha^2n^2/3}+2^{7n^2/11}\right) + S(n)\left(2^{-\alpha n^3}+2^{-n/10}\right).\end{aligned}$$

Let us upper bound the terms above separately. Since $n > n_1$, Lemma 11 (ii), yields $S(n-1) \leqslant S(n)2^{-(2n^2-5n+1)/9}$. As β is sufficiently small (by (16)), we also have $2^{2n^2/9-\beta n^2/5} > 2^{n^2/17}$. Therefore

$$S(n-1)\left(C2^{-(n-1)/10}+1\right)\left(2^{2n^2/9-\beta n^2/5}+2^{n^2/17}\right) < S(n)\left(C2^{-(n-1)/10}+1\right)2^{-\beta n^2/6}.$$

Similarly, using $S(n-3) \leqslant S(n)2^{-(6n^2-27n+28)/9}$ and $2^{6n^2/9-\alpha^2n^2/3} > 2^{7n^2/11}$ we obtain

$$S(n-3)\left(C2^{-(n-3)/10}+1\right)\left(2^{6n^2/9-\alpha^2n^2/3}+2^{7n^2/11}\right) < S(n)\left(C2^{-(n-3)/10}+1\right)2^{-\alpha^2n^2/4}.$$

Summing up these bounds, we conclude that $|\mathrm{Forb}(n, T_5, \eta, \mu) - \mathcal{S}(n)|$ is upper bounded by

$$S(n)\big[\big(C2^{-(n-1)/10}+1\big)2^{-\beta n^2/6}+\big(C2^{-(n-3)/10}+1\big)2^{-\alpha^2 n^2/4}+2^{-\alpha n^3}+2^{-n/10}\big].$$

After expanding the expression above, we see that each of the six summands is upper bounded by $\frac{C_1}{6}S(n)2^{-n/10}$ and we finally obtain

$$\big|\mathrm{Forb}(n, T_5, \eta, \mu) - \mathcal{S}(n)\big| \leqslant S(n)C_1 2^{-n/10}.$$

This completes the proof. □

Proof of Theorem 3. We wish to prove Theorem $3(\eta, n)$, so as indicated above, we may assume Theorem $16(n)$. We also use Lemmas 13, 11 (i) and $C' = 2C_1$:

$$\begin{aligned}\big|\mathrm{Forb}(n, T_5, \eta) - \mathcal{S}(n)\big| &\leqslant \big|\mathrm{Forb}(n, T_5, \eta) - \mathrm{Forb}(n, T_5, \eta, \mu)\big| + \big|\mathrm{Forb}(n, T_5, \eta, \mu) - \mathcal{S}(n)\big| \\ &\leqslant 2^{n^3(2/27-\mu^3/500)} + C_1 2^{-n/10}S(n) \\ &\leqslant C_1 2^{-n/10}S(n) + C_1 2^{-n/10}S(n) \\ &= C' 2^{-n/10}S(n). \qquad \square\end{aligned}$$

Proof of Theorem 1. We wish to prove Theorem $1(n)$, so as indicated above, we may assume Theorem $3(\eta, n)$. We also use Theorem 2, Lemma 11 (i) and $C = 2C'$:

$$\begin{aligned}\big|\mathrm{Forb}(n, T_5) - S(n)\big| &\leqslant \big|\mathrm{Forb}(n, T_5) - \mathrm{Forb}(n, T_5, \eta)\big| + \big|\mathrm{Forb}(n, T_5, \eta) - \mathcal{S}(n)\big| \\ &\leqslant 2^{(1-\nu)2n^3/27} + C' 2^{-n/10}S(n) \\ &\leqslant C' 2^{-n/10}S(n) + C' 2^{-n/10}S(n) \\ &= C 2^{-n/10}S(n). \qquad \square\end{aligned}$$

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