Nilpotence of a class of commutative power-associative nilalgebras

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Abstract

Let A be a commutative algebra over a field F of characteristic ≠ 2, 3. In [M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices II, Duke Math. J. 27 (1960) 21–31], M. Gerstenhaber proved that if A is a nilalgebra of bounded index t and the characteristic of F is zero (or greater than 2t − 3), then the right multiplication R1 is nilpotent and R2t−3 = 0 for all x ∈ A. In this work, we prove that this result is also valid for commutative power-associative algebras of characteristic t. In Section 3, we prove that when A is a power-associative nilalgebra of dimension ≤ 6, then A is nilpotent or (A2)2 = 0. In Section 4, we prove that every power-associative nilalgebra A of dimension n and nilindex t ≥ n − 1 is either nilpotent of index t or isomorphic to the Suttles’ example.

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1. Nilalgebras

Let $A$ be an algebra (not necessarily finite-dimensional or associative) over a field $F$. An element $a \in A$ is nil of index $\leq t$ if all possible products of at least $t$ factors, each of them equal to $a$, vanish. We will use notations and definitions from [3]. Thus, $P$ is the free commutative non-associative polynomial ring in two generators $x$ and $y$ over a field $F$. For every $\alpha_1, \ldots, \alpha_r \in P$, the operator linearization $\delta(\alpha_1, \ldots, \alpha_r)$ can be defined as follows: if $p(x, y)$ is a monomial in $P$, then $p(x, y)\delta(\alpha_1, \ldots, \alpha_r)$ is obtained by making all the possible replacements of $r$ of the $k$ identical arguments $x$ by $\alpha_1, \ldots, \alpha_r$ and summing the resulting terms if $x$-degree of $p(x, y)$ is $\geq r$, and is equal to zero in other case. We will denote by $\Delta$ the vector space over $F$ of all linear combinations of operators of the form $\delta(\alpha_1, \ldots, \alpha_r)$. From [3, Lemma 3] we have the following.

Lemma 1. For all $\alpha \in P$, and for all non-negative integers $k$ and $n$,

$$\delta(\alpha R^n_x, x^2 : k) = \delta(x^2 : k)\delta(\alpha R^n_x) - 2k\delta(\alpha R^{n+1}_x, x^2 : k - 1).$$

If the characteristic of $F$ is greater than $k$, then Lemma 1 reads

$$\delta(\alpha R^{n+1}_x, x^2 : k - 1) = \frac{1}{2k}[\delta(x^2 : k)\delta(\alpha R^n_x) - \delta(\alpha R^n_x, x^2 : k)].$$

We remark that, although M. Gerstenhaber considered in [3, Lemma 4] algebras over a field with characteristic zero, the proof of the lemma is also valid for algebras over fields with characteristic greater than $n$.

Lemma 2. Let $n$ and $k$ be non-negative integers with $k \leq n$, $F$ a field with characteristic zero or greater than $n$, and $\alpha$ an element in $P$. Then there exist homogeneous elements $D_r$, $r = 0, 1, \ldots, n + k$ in $\Delta$ of degree $r$ in $x$ and $0$ in $y$ such that

$$\delta(\alpha R^{n+k}_x) = \sum_{i=0}^{k-1} D_i \delta(\alpha R^{k-i}_x, x^2 : n) + \sum_{j=k}^{n+k} D_j \delta(\alpha, x^2 : n + k - j).$$

Proof. By [3, Lemma 4] the case $k = 0$ holds. Now, we proceed by induction and we assume the lemma for all the pairs $(k, n)$, with $0 \leq k \leq \min\{s - 1, n\}$, where $s$ is a positive integer. Considering $\alpha x$ in place of $\alpha$ the inductive assumption implies, for $n \geq s$:

$$\delta(\alpha R^{n+s}_x) = \sum_{i=0}^{s-2} E_i \delta(\alpha R^{s-i}_x, x^2 : n) + \sum_{j=s-1}^{n+s-1} E_j \delta(\alpha x, x^2 : n + s - j - 1)$$

$$= \sum_{i=0}^{s-2} E_i \delta(\alpha R^{s-i}_x, x^2 : n) + E_{s-1} \delta(\alpha x, x^2 : n)$$
and by Lemma 1, this is equal to

$$\sum_{i=0}^{s-1} E_i \delta(\alpha R_i, x^2 : n)$$

$$+ \sum_{j=s}^{n+s-1} \frac{1}{2(n+s-j)} E_j \left[ \delta(x^2 : n+s-j) \delta(\alpha) - \delta(\alpha, x^2 : n+s-j) \right].$$

Setting

$$D_i = E_i, \quad D_j = -\frac{1}{2(n+s-j)} E_j,$$

for \( i = 0, 1, \ldots, s-1 \) and \( j = s, \ldots, n+s-1 \), and

$$D_{n+s} = \sum_{r=s}^{n+s-1} \frac{1}{2(n+s-r)} E_r \delta(x^2 : n+s-r),$$

the lemma is proved. \( \square \)

Let \( F[X] \) be the free commutative power-associative algebra over \( F \) with countable set of generators \( X = \{x, y, \ldots\} \). If \( B \) is a subalgebra of a commutative power-associative algebra \( A \), then a polynomial \( f(x, y, \ldots, z) \in F[X] \) is called a \( B \)-identity of \( A \), if \( f(a_1, a_2, \ldots, a_n) \in B \) for all \( a_i \in A \). We denote by \( T(A, B) \) the set of all \( B \)-identities of \( A \). Observe that \( T(A, B) \) is a subalgebra of \( F[X] \) closed under the following operation:

\[ (\ast) \text{ if } f(x, y, \ldots, z) \in T(A, B) \text{ and } \alpha \in F[X], \text{ then } f(\alpha, y, \ldots, z), f(x, \alpha, \ldots, z), f(x, y, \ldots, \alpha) \in T(A, B). \]

Every subalgebra of \( F[X] \) closed under the operation \((\ast)\) is called \( T \)-subalgebra of \( F[X] \). A subset \( I \) of a \( T \)-subalgebra \( U \) is called a generator if \( U \) is the smallest \( T \)-subalgebra containing \( I \). The following lemma is analogous to a result given in [7, p. 13] for \( T \)-ideals.

**Lemma 3.** If a \( T \)-subalgebra \( U \) is defined by a system \( I \) of homogeneous polynomials of degree \( \leq t \) in each of the variables and \( F \) is a field of characteristic zero or \( \geq t \), then the \( T \)-subalgebra \( U \) is homogeneous and hence \( U \) is invariant with respect to linearization operators.

**Theorem 1.** Let \( B \) be a subalgebra of a commutative power-associative algebra \( A \) over a field \( F \) of characteristic zero or greater than \( t \), such that for every \( a \in A, a' \in B \). Then \( c R_{a'}^{2t-3} \in B \) for all \( a, c \in A \).
Proof. Let $U$ be the $T$-subalgebra of $F[X]$ generated by $x^t$. It is clear that $x^t \in U$ for all $r \geq t$. By Lemma 2, we have that

$$yR^{2r-3}_x = x\delta(yR^{2r-3}_x) = \sum_{i=0}^{t-4} xD_i \delta(yR^{2r-3}_x, x^2 : t) + \sum_{j=t-3}^{2t-3} xD_j \delta(y, x^2 : 2t - j - 3).$$

Since the degree of $xD_i$ in $x$ is $i + 1$ and $\delta(yR^{2r-3}_x, x^2 : t)$ has $t + 1$ arguments, $xD_i \delta(yR^{2r-3}_x, x^2 : t)$ will vanish if $i = 0, 1, \ldots, t - 4$. Analogously, we can see that $xD_j \delta(a, x^2 : 2t - j - 3) = 0$ for $j = t - 3, t - 2$. Now, if $j \geq t - 1$, then $xD_j \in U$ and therefore, by Lemma 3, $xD_j \delta(y, x^2 : 2t - j - 3) \in U$. This proves the lemma.

Theorem 2. Let $A$ be a commutative nilalgebra of a bounded index $t$ over a field $F$ of characteristic zero or $\geq t$. Then $R_n^{2r-3} = 0$ for all $a \in A$ in the following cases:

(i) $A$ is power-associative algebra,
(ii) $t \leq 6$,
(iii) $A$ has at least $2t - 2$ elements.

Proof. Let $U$ be the $T$-ideal generated for the set of all monomials in $F[x]$ of degree $t$. It is easy to prove that in the cases (i) and (ii) every monomial in $F[x]$ of degree $\geq t$ belongs to $U$ and from [7, Corollary, p. 13] the $T$-ideal $U$ is homogeneous. Now, the theorem follows as in the above theorem, but using Lemma 2 with $n = t - 1$ and $k = t - 2$.

We remark that according to Albert program we have the following conjecture: there exists a commutative nilalgebra $A$ of characteristic zero such that each monomial in $F[x]$ of degree 7 is an identity in $A$ and $a^3a^4 \neq 0$ for some element $a \in A$. Thus, we see that the proof of the above theorem fails for commutative nilalgebras of nilindex 7 over the finite field $Z_7$.

2. Power-associative nilalgebras

In the following, $A$ will be a commutative power-associative algebra over a field $F$ with characteristic $\neq 2, 3$. We define inductively $B^1 = B$ and $B^{k+1} = B^kB + B^{k-1}B^2 + \cdots + B^3$ for all $k \geq 1$. If there exists an integer $n \geq 2$ such that $B^n = 0$ and $B^{n-1} \neq 0$, then $B$ is nilpotent of index $n$. The algebra $A$ is called solvable in case $A^{(k)} = 0$ for some integer $k$, where $A^{(1)} = A$ and $A^{(k+1)} = (A^{(k)})^2$ for all $k \geq 1$. If $a_1, \ldots, a_i \in A$, we will denote by $\langle a_1, \ldots, a_i \rangle$ the subspace generated by the elements $a_1, \ldots, a_i$ in $A$. Since the identities $x^4 = (x^2)^2$ and $(x^2 x^2)x = x^3 x^3$ are valid in $A$, then by linearization we get that the following identities are also valid in $A$:

$$2((yx)x)x + (yx^2)x + yx^3 = 4(yx)x^2, \quad (1)$$
$$2((yz)x)x + 2((yx)z)x + 2((yx)x)z + 2((zx)y)x + (yx^2)z + 2((zx)x)y + (yz^2)y = 4(yz)x^2 + 8(yx)(zx), \quad (2)$$
$$y^2x + 2((yx)y)x + 2((yx)x)y + (yx^2)y = 4(yx)^2 + 2y^2x^2, \quad (3)$$
$$4((yx)x^2)x + yx^4 = 2(yx)x^3 + (yx^2)x^2 + 2((yx)x)x^2. \quad (4)$$
Lemma 4. In a commutative power-associative algebra, the algebra generated by all $R_x^k$ is in fact generated by $R_x$ and $R_x^2$.

The following two theorems are proved in [1] and [5], respectively:

Theorem 3. Let $A$ be a nilalgebra of nilindex $n$ and dimension $n$. If $b$ is an element in $A$ such that $b^{n-1} \neq 0$ and $B = \langle b, \ldots, b^{n-1} \rangle$, then $A^k = B^k$ for all $k \geq 2$. Hence $A^n = B^n = 0$.

Theorem 4. If $A$ is a nilalgebra of nilindex 3 and dimension 4, then $A^3 = 0$.

If $B$ is a subalgebra of $A$, then $A$ is called $B$-nilalgebra of $B$-nilindex $t$ if $x' \in B$ for all $x \in A$ and $r \geq t$, and there exists $a \in A$ such that $a^{t-1} \notin B$. The following lemma is clear.

Lemma 5. Let $B$ be a subalgebra of $A$ such that $x' \in B$ for all $x \in A$. If the characteristic of $F$ is zero or greater than $t$, then $A$ has $B$-nilindex $\leq t$.

Lemma 6. Let $A$ be a nilalgebra and $B$ a subalgebra of codimension $m$. Then the $B$-nilindex of $A$ is $\leq 2^m$.

Proof. If the $B$-nilindex of $A$ is greater than $2^m$, then there exists $a \in A$ such that $a^{2^m} \notin B$. We can see that $a^i \notin B$ for $i = 0, 1, \ldots, m$ since $B$ is a power-associative subalgebra of $A$. Now, we will prove that $a, a^2, a^3, \ldots, a^{2^m}$ are linearly independent modulo $B$. Let $b := \sum_{i=0}^m \lambda_ia^{2^i} \in B$, with $\lambda_i \in F$. Because $A$ is a nilalgebra, we have a positive integer $r$ such that $a^r \notin B$ and $a^k \in B$ for all $k > r$. Then $b^r = \lambda_0a^r + c$, with $c \in \langle a^k : k > r \rangle \subset B$ so that $\lambda_0a^r = b^r - c \in B$ and this implies $\lambda_0 = 0$. Analogously, let $i$ be a positive integer with $1 \leq i \leq m$ such that $\lambda_0 = \lambda_1 = \cdots = \lambda_{i-1} = 0$. There exists $s$, a positive integer, such that $a^{2^s} \notin B$ and $a^{2^s} \in B$ for all $k > s$. Then $b^s = \lambda_ia^{2^s} + d$, with $d \in \langle a^{2^s} : k > s \rangle \subset B$. Thus, $\lambda_ia^{2^s} = b^s - d \in B$. This forces $\lambda_i = 0$. □

We have the following table:

<table>
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<th>$B$-nilindex</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>codim($B$)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

From Theorem 3 we have

Lemma 7. Let $A$ be a nilalgebra of nilindex $t$ and $b \in A$ such that $b^{t-1} \neq 0$. If $A^2 \subset B := \langle b, b^2, \ldots, b^{t-1} \rangle$, then $A^k = B^k$ for all $k \geq 2$. Hence $A^t = B^t = 0$.

Lemma 8. Let $A$ be a nilalgebra of dimension $n$ and nilindex $t$, and $B$ the subalgebra of $A$ generated by $b$, where $b$ is an element in $A$ such that $b^{t-1} \neq 0$. If $A$ is a $B$-nilalgebra of $B$-nilindex $\leq 4$, then we have the following properties for all $x, y \in A$:
3. Nilalgebras of nilindex 4

In this section, $A$ will be a commutative algebra over a field $F$ with characteristic $\neq 2, 3$ such that the identities $x^4 = 0$ and $x^2x^2 = 0$ are valid in $A$. Linearizing these identities, we obtain for all $x, y, z$ in $A$:

\[(yx)x^2 = 0, \quad 2((yx)x)x + (yx^2)x + yx^3 = 0, \tag{8}\]
\[(zy)x^2 + 2(zx)(yx) = 0, \quad y^2x^2 + 2(yx)^2 = 0, \tag{9}\]
\[4((yx)x)x = (yx^2)x^2 = -2(yx)x^3. \tag{10}\]
We remark that in this case $A$ is a power-associative algebra. From Theorem 2 we have that $R_4^5 \equiv 0$ for all $x \in A$. Now it is easy to prove the following two results.

**Lemma 9.** If $y$, $x$ are elements in $A$ and $yx = ay$, then $xy = 0$.

**Lemma 10.** The relation $(\cdots ((yx^{m_1})x^{m_2})\cdots)x^{m_t} = 0$ holds for all $x$, $y \in A$ and positive integers $m_i$ with $m_1 + m_2 + \cdots + m_t \geq 5$.

**Proof.** By Lemma 4 we can assume that $m_i \in \{1, 2\}$ for all $i$ and from Theorem 2 and the first identity of (8) we only need to consider elements of the form $yR_s^x, R_s^x$ with $s \geq 1$ and $r + 2s \geq 5$. Multiplying Eq. (10) with $x$ we have $(yx^2)x^2x = 0$.

Replacing $z$ by $x$ and $x$ by $x^2$ in the first identity of (9) and using (8) we get

$$0 = (yx^2)x^3 = -((yx^2)x^2)x - 2(((yx^2)x)x)x = -2(((yx^2)x)x)x.$$ 

Finally, we may use (10) replacing $y$ by $yx^2$ to obtain a relation which may be combined with $(((yx^2)x)x)x = 0$ to yield $((yx^2)x)x^2 = 0$. This proves the lemma. □

**Lemma 11.** If there exist $a, b \in A$ such that $aR^4_b \neq 0$, then $\dim(A) \geq 9$.

**Proof.** Suppose that there exist elements $a$, $b$ in $A$ such that $aR^4_b \neq 0$. By (10) we have that $(ab^2)b^2 \neq 0$ and $b^3 \neq 0$. Let $B = \langle b, b^2, b^3 \rangle$. The elements $(ab^2)b^2$ and $b^3$ are linearly independent. In fact, if $(ab^2)b^2 = \lambda b^3$, then by (10) we get that $(ab)b^3 = -(\lambda/2)b^3$. Using Lemma 9, we obtain that $\lambda = 0$, which is a contradiction. Now, we will prove that $aR^0_b, aR^1_b, aR^2_b, aR^3_b, b, b^2, b^3$ and $ab^2$ are linearly independent. Let

$$\sum_{i=0}^4 \alpha_i aR^i_b + \sum_{j=1}^3 \beta_j b^j + \gamma ab^2 = 0.$$ 

Multiplying by $b^2$ and using (8), we get that

$$\alpha_0 ab^2 + \beta_1 b^3 + \gamma (ab^2)b^2 = 0.$$ 

Now, using Lemma 10, we have

$$0 = [\alpha_0 ab^2 + \beta_0 b^3 + \gamma (ab^2)b^2]b^2 = \alpha_0 (ab^2)b^2$$

and this implies $\alpha_0 = 0$. Since $b^3$ and $(ab^2)b^2$ are linearly independent, we get also that $\beta_1 = \gamma = 0$. Hence we have that

$$\sum_{i=1}^4 \alpha_i aR^i_b + \beta_2 b^2 + \beta_3 b^3 = 0.$$
Proof. By Lemma 11 we can assume that $\alpha_1 = \alpha_2 = 0$. Multiplying by $b$ we have that

$$0 = \alpha_3 a R_b^4 + \beta_2 b^3 = \frac{1}{4} \alpha_3 (ab^2)b^2 + \beta_2 b^3,$$

which gives $\alpha_3 = \beta_2 = 0$. Now

$$0 = \alpha_4 a R_b^4 + \beta_3 b^3 = \frac{1}{4} \alpha_4 (ab^2)b^2 + \beta_3 b^3$$

implies $\alpha_4 = \beta_3 = 0$. This proves the lemma. \qed

By constructing examples, it is not difficult to show that the relation given in the above lemma cannot be improved. Let $\mathfrak{A}$ be a commutative algebra of dimension 9 with a basis $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and a non-zero multiplication given by: $v_1 v_6 = v_2, v_1 v_7 = v_9, v_1 v_8 =- (2 + \beta) v_4 - \gamma v_5, v_2 v_6 = v_3, v_2 v_8 = -2 v_5, v_3 v_6 = v_4, v_4 v_6 = v_5, v_6^2 = v_7, v_8 v_7 = v_8, v_8 v_9 = \beta v_4 + \gamma v_5, v_7 v_9 = 4 v_5$. It is easy to prove that $\mathfrak{A}$ is a power-associative nilalgebra of nilindex 4, where $((v_1 v_6) v_6) v_6 = v_5$.

**Lemma 12.** If $R_i^4 \equiv 0$ is an identity in $A$, then $(yx^2)^2 = 0$ and $(yx)^3 = 0$, for all $x, y \in A$.

**Proof.** In view of (9) we have that $(xy^2)^2 = 0$. We know that the relation $(yx^2)x^2 = 0$ is valid in $A$, so that linearizing this identity we get the new identity

$$(11) \quad ((zx)y)x^2 + (zx)(yx^2) = 0.$$ 

Replacing $z$ by $y$ and $y$ by $y^2$ in the above identity and using (8), we see that $0 = ((xy) y^2)x^2 + (xy)(y^2 x^2) = (xy)(y^2 x^2) = -2(yx)^3$. This proves the lemma. \qed

**Theorem 5.** If $(A^2)^2 \neq 0$, then $\dim(A) \geq 6$.

**Proof.** By Lemma 11 we can assume that $R_i^4 \equiv 0$ is an identity in $A$. Since $(A^2)^2 \neq 0$, there exist elements $a, b$ in $A$ such that $a^2 b^2 \neq 0$. We will prove that $a, b, a^2, b^2, ab, a^2 b^2$ are linearly independent. Let $\alpha_1 a + \beta_1 b + \alpha_2 a^2 + \beta_2 b^2 + \gamma_1 ab + \gamma_2 a^2 b^2 = 0$. Then, using (8), (9) and the above lemma, we see that

$$(\alpha_1 a + \beta_1 b)^2 = (\alpha_2 a^2 + \beta_2 b^2 + \gamma_1 ab + \gamma_2 a^2 b^2)^2 = \left(2 \alpha_2 \beta_2 - \frac{\gamma_1^2}{2}\right)a^2 b^2.$$ 

Multiplying by $b^2$, we get that $\alpha_1 = 0$ and similarly we prove that $\beta_1 = 0$. Now it is easy to prove that $\alpha_2 a^2 + \beta_2 b^2 + \gamma_1 ab + \gamma_2 a^2 b^2 = 0$ implies $\alpha_2 = \beta_2 = \gamma_1 = \gamma_2 = 0$, and thus $a, b, a^2, b^2, ab, a^2 b^2$ are linearly independent. \qed

**Theorem 6.** If $A$ is of dimension 6 and $(A^2)^2 \neq 0$, then $A$ is a Jordan algebra.
Proof. By Theorem 5 we know that there exist elements $a, b$ in $A$ such that $\{a, b, a^2, b^2, ab, a^2b^2\}$ is a basis of $A$. We will consider the subspace $D = (a^2, b^2, ab, a^2b^2)$. We observe that $D^2 = (a^2b^2)$ and $D^3 = 0$, so that $D$ is a nilpotent subalgebra of $A$ with index 3. We will prove that $A^2 = D$ and that $A^3 = D^2 = (a^2b^2) = ann(A)$.

Let $a^3 = a_1a + a_2b + a_3a^2 + a_4b^2 + a_5ab + a_6a^2b^2$. Then $a_1^2a^2 + 2a_1a_2ab + a_2^2a^2 = 2a_3a^3b = (a^3 - a_1a - a_2b) \in D^2 = (a^2b^2)$. Multiplying by $b^2$, we get that $a_1 = 0$. Clearly also $a_2 = 0$. Therefore $a^3 = a_3a^2 + a_4b^2 + a_5ab + a_6a^2b^2 \in D$. Multiplying by $a^2$ we get $a_1 = 0$ and by $ab$ that $a_5 = 0$. Now, we have that $(a - a_6b^2)a^2 = a^2a^2$ and hence $a_6 = 0$. Thus, we prove that $a^3 \in (a^2b^2)$ and similarly we obtain that $b^3 \in (a^2b^2)$.

Let $ab^3 = \beta_1a + \beta_2b + \beta_3a^2 + \beta_4b^2 + \beta_5ab + \beta_6a^2b^2$. Now, $0 = (ab^2)b^2 \in \beta_1a^2b^2 + D = (\beta_1^2a + \beta_1\beta_2b + D$ and hence $\beta_1 = 0$. We observe that using (9) we have that $(ab^2)(ab) = -(1/2)a^2b^3 + \beta_2ab + \beta_3b^2 + \beta_4\beta_2b^2$ and hence $\beta_2 = 0$. We have that $a^2(\beta_2^2b^2) = \beta_3a^2b^2 + \beta_5ab + \beta_6a^2b^2 \in D^2 = (a^2b^2)$ and hence $\beta_3 = 0$. Finally, $0 = (ab^2)b^2 = \beta_4a^2b^2 + \beta_5ab + \beta_6a^2b^2 \in D^2$ and hence $\beta_4 = 0$. The relation $ba^2 \in D^2$ can be proved similarly.

Let $(ab)b = y_1a + y_2b + y_3a^2 + y_4b^2 + y_5ab + y_6a^2b^2$. Replacing $y$ and $z$ by $b$, and $y$ by $a$ in (11), we get that $(a(ab)b)a^2 = -(a(ab)b) \in (ab)bD^2 = 0$. A similar way we can prove that $(ab)a^2b^2 = 0$. Now, using (9) we see that $(ab)b(a)^2 = -(1/2)((ab)a)(a)^2 = 0$ and $(ab)b(a)^2 = 0$. Using above relations we see that $(y_1a + y_2b)^2 = (ab)b - y_1a^2 - y_2b^2 - y_5ab - y_6a^2b^2$ belongs to $D^2$ so that $y_1 = y_2 = 0$. Now, the relations $(a(ab)b)^2 = 0$ and $(a(ab)b) = 0$ force $y_3 = 0$ and $y_5 = 0$. Next, because $(ab)b(a)^2 = -2(ab)(a)(ab) = 0$, we have that $y_4 = 0$. Therefore $(ab)b \in D^2$, and similarly we can prove that $(ab)a \in D^2$.

It remains to verify the products $aD^2$ and $bD^2$. To prove $aD^2 = 0$ we use the relations $a^2, ab^2 \in D^2$ and the second identity of (8) with $x = a$ and $y = b^2$ to obtain $a(a^2b^2) = -a^3b^2 = -(1/2)(a(ab)b)^2 = 0$. An analogous argument shows that $b(a^2b^2) = 0$. Analogously, we can prove that $b(a^2b^2) = 0$.

Therefore, we have proved that $A^3 = A^4 = A^5 = (a^2b^2)$, $A^j = 0$, for all $j \geq 5$, and clearly $A$ is a Jordan algebra. $\square$

Corollary 1. If $A$ is of dimension 6 and $A$ is not nilpotent, then $(A^2)^2 = 0$.

4. The main theorem

In this section, $A$ will be a commutative power-associative nilalgebra of nilindex $n - 1$ and dimension $n$ over a field $F$ with characteristic $\neq 2, 3$. Moreover, $b$ is an element in $A$ such that $b^{n-2} \neq 0$ and $B := (b, \ldots, b^{n-2})$. We will assume that $A^2 \subset B$, because in the other case, $A^k = B^k$ for all $k \geq 2$ and hence $A$ is nilpotent of index $n - 1$. Then $x^4 \in B$ for all $x \in A$. Because the nilpotent index of $x$ is less than $n$, we have that $x^4 \in B^3$ for all $x \in A$. We can define for every $x \in A$, with $x B \subset B$, a linear transformation $R_x$ in the quotient vector space $\tilde{A} = A/B$ as follows: $(y + B)R_x = x(y + B)$. Every endomorphism in the linear space $M := [R_x : x \in A, x B \subset B]$ is nilpotent. In view of [4], the dimension of $M$ is either 0 or 1. Since we are assuming that $A^2 \not\subset B$, it follows that $\dim M = 1$. Thus, there is a
non-zero element \(a + B\) in the quotient linear space \(A/B\) such that \(f(y + B) \in (a + B)\), \(f(a + B) = 0 + B\) for all \(f \in M\) and \(y + B \in A/B\). This implies that \(aB \subset B\) and therefore \(\overline{K_a} \in M\), so that \(a^2 \in B\) since \(\overline{K_a}\) is nilpotent. This makes \(C := \langle a \rangle + B\) a subalgebra of codimension 1. From Lemmas 6 and 8 it follows that each subalgebra of codimension 1 are linearly independent. Let \(\alpha_a\) is in fact generated by \(\alpha_a\) and \(\alpha_a\) will prove that \(\alpha_a\) is a non-zero element.

Lemma 13. We have that \(A^2 \subset C\) and \(C^k = B^k\) for every \(k \geq 2\).

Because, in a commutative power-associative algebra, the algebra generated by all \(R_{ak}\) is in fact generated by \(R_a\) and \(R_{a^2}\), we obtain the following.

Corollary 2. We have that \(A^2B^k\), \(AC^{k+2} \in B^{k+1}\), for every positive integer \(k\).

Lemma 14. If \(a^2 \not\in B\), then \(\{a, a^2, b, \ldots, b^{n-2}\}\) is a basis of \(A\) and

\[A^2 = \langle a^2, b^2, b^3, \ldots, b^{n-2} \rangle.\]

Proof. Let \(a\) be an element in \(A\) such that \(a^2 \not\in B\). We will prove that \(a, a^2, b, \ldots, b^{n-2}\) are linearly independent. Let \(\alpha a + \beta a^2 \in B\). If \(k\) is a positive integer, \(2 \leq k \leq 3\), such that \(a^k \not\in B\) and \(a^r \in B\) for all \(r > k\), then the relation \((\alpha a + \beta a^2)^k \in B\) implies \(a^k a^k \in B\), and so \(\alpha = 0\). This yields \(\beta = 0\), and hence \(\{a, a^2, b, \ldots, b^{n-2}\}\) is a basis of \(A\). Now we will prove that \(A^2 = \langle a^2, b^2, \ldots, b^{n-2} \rangle\). Because \(C = \langle a^2, b^2, \ldots, b^{n-2} \rangle\), it is sufficient to show that \(a^3\) and \(ab^m\), for every integer \(m \geq 1\), are elements in \(\langle a^2, b^2, \ldots, b^{n-2} \rangle\). If \(a^3 = \beta a^2 + \alpha b + \cdots + \alpha_{n-2} b^{n-2}\), then \(0 = (a^3 - \beta a^2)^{n-2} = (\alpha_1 b + \cdots + \alpha_{n-2} b^{n-2}) b^{n-2} = a^{n-2} b^{n-2}\) so that \(\alpha_1 = 0\). This implies \(a^3 \in \langle a^2, b^2, \ldots, b^{n-2} \rangle\). We now consider the element \(ab^m\) with \(m \geq 1\). Since \(ab^m \in C\), we have a decomposition of the form \(ab^m = \lambda a^2 + \epsilon b + u\) with \(u \in B^2\). By (5) and (6) we have that \((\epsilon b + u)^2 = (ab^m - \lambda a^2)^2 = (ab^m)^2 - 2\lambda(ab^m) a^2 + \lambda^2 a^4 \in B^3\), since \(a^2(b^m)^2 \in C B^{2m} = B^{2m+1}\). Therefore, \(\epsilon = 0\). This proves the lemma.

Lemma 15. If \(B^2\) is an ideal of \(A\), then \(B^k\) is an ideal of \(A\) for all \(k \geq 2\). If \(B^2\) is not an ideal of \(A\), then there exist elements \(u, v\) in \(A\) such that \(\{u, uv^2, v, \ldots, v^{n-2}\}\) is a basis of \(A\) with \(uv = 0\) and \(B = \langle v, \ldots, v^{n-2} \rangle\). Moreover,

\[
\begin{align*}
uv^3 &= -(uv^2)v, & uv^4 &= (uv^2)v^2, & ((uv^2)v^3)v &= 0, \\
(uv^2)v^3 + (uv^4)v &= -2(uv^3)v^2, \\
3uv^5 + (uv^4)v &= 4(uv^3)v^2, & uv^5 + (uv^4)v &= (uv^2)v^3, \\
uv^6 &= (uv^2)v^4 = (uv^4)v^2 &= -(uv^5)v = -2(uv^3)v^3;
\end{align*}
\]

Furthermore, if \(F\) has more than 5 elements, then

\[
(uv^2)v^3 = 0, \quad (uv^6)v = 0.
\]
Proof. If $y \in A$ and $yb^2 \in B^2$, then from Lemma 13 and identity (2) for $z = b^{k-2}$ and $x = b$, we obtain inductively that $yb^k \in B^k \subset B^2$ for $k = 3, \ldots, n - 2$. Therefore, $B^2$ is an ideal of $A$ if and only if $B^k$ is an ideal, for all $k \geq 2$; and if $B^2$ is not an ideal, then there exists $a \in A$ such that $ab^2 \notin B^2$.

Now, we will assume that $B^2$ is not an ideal of $A$ and let $a$ be an element of $A$ such that $ab^2 \notin B^2$. By Lemma 14, we know that $A^2 = (ab, b^2, \ldots, b^{n-2})$. Thus, there exist scalars $\alpha_i \in F$ such that $ab = \alpha_1 ab^2 + \alpha_2 b^2 + \cdots + \alpha_{n-2} b^{n-2}$. We note that, if $v = b - \alpha_1 b^2$, then $av \in B^2$. It is clear that $B = \langle v, \ldots, v^{n-2} \rangle$, and thus $av = \beta_2 v^2 + \cdots + \beta_{n-2} v^{n-2}$.

Now if $u = u - (\beta_2 v + \cdots + \beta_{n-2} v^{n-3})$, then $uv = 0$. Using Corollary 2, we see that $uv^2 - ab^2 \in B^2$, which implies that $uv^2 \notin B^2$. This shows that $\{u, uv, \ldots, v^{n-2}\}$ is a basis of $A$ with $uv = 0$ and $B = \langle v, v^2, \ldots, v^{n-2} \rangle$. Now, setting $x = v$ and $y = u, uv^2$ in (1) and (4) immediately yield relations $(uv^2)v = -uv^3, uv^4 = (uv^2)v^2, (uv^2)v^3 = (uv^4)v^2$ and (13). From (13), taking $y = u, x = v$ and $z = v^4$ in (2), we have the first identity of (14), and taking $y = x$ and $x = v + \lambda v^2$, with $\lambda \in F$, in (4), we see that $uv^5 + (uv^4)v = (uv^2)v^3$. Replacing $y = u$ and $x = v^2$ in (1) gives $uv^6 = (uv^4)v^2$. Next, Eq. (2) for $x = v, y = u$ and $z = v^4$ gives $uv^6 = -(uv^2)v$. Multiplying (14) and (13) with $v$, we get $uv^6 = -2(uv^3)v^3$ and $(uv^2)v^3 = 0$. Finally, if $A$ has at least 7 elements, then linearizing $x^3 x^3 = (x^2 x^2)x^2$, we get the identity $2((xy)x)x^3 + 4((yx)x)x^3 = 4((yx)x)x^2 + 2((yx)x)x^4$, and taking $y = u, uv^2$ and $x = v$ we obtain $(uv^2)v^3 = 0$ and $(uv^4)v^2 = 0$. \qed

Corollary 3. If $B^3$ is an ideal of $A$, then $AB^k \subset B^{k+1}$ for all $k \geq 3$.

Proof. Assume that $B^3$ is an ideal of $A$. First of all we will prove that $B^k$ is an ideal of $A$ for all $k \geq 3$. From the above lemma, we can assume that $B^2$ is not an ideal of $A$ and there exist $u, v \in A$ such that $\{u, uv, v, v^2, \ldots, v^{n-2}\}$ is a basis of $A$ with $uv = 0$ and $B = \langle v, v^2, \ldots, v^{n-2} \rangle$. Now, multiplying (13) by $v$ we get $uv^4 \in B^4$. This shows that $(uv^2)v = -uv^3 \in B^3, (uv^2)v^2 = uv^4 \in B^4$ and hence, from Lemma 4, we have that $(uv^2)v^k, uv^{k+2} \in B^{k+2}$ for all $k \geq 1$. Therefore $B^k$ is an ideal of $A$ for all $k \geq 3$.

Finally, taking $y = v^k$ in (1) gives $xv^k \in B^{k+1}$ for all $x \in A$ and $k \geq 3$. \qed

Theorem 7. If $n \neq 5$, then $B^3$ is an ideal of $A$.

Proof. By Theorem 4, if $n \leq 4$, then $B^3 \subset A^3 = 0$ and hence $B^3$ is trivially an ideal of $A$. Let now $n$ be greater than 4 and suppose that $B^3$ is not an ideal of $A$. By the above lemma, we know that there exists a basis $\{u, uv, v, v^2, \ldots, v^{n-2}\}$ of $A$ such that $uv = 0$. Moreover, $A^3 = \langle uv, v^2, \ldots, v^{n-2} \rangle, B = \langle v, \ldots, v^{n-2} \rangle$ and $C = \langle uv, v, \ldots, v^{n-2} \rangle$. Since $(uv^2)B^3 \subset C B^3 \subset C^4 = B^4$, and $B^3$ is not an ideal of $A$, then by Corollary 2, we have that there exists $\lambda \neq 0$ such that $uv^3 - \lambda v^3 \in B^3$. We can assume that $\lambda = 1$. Since $uv^4 = (uv^2)v^2 \in C B^2 \subset C^3$ there exists $\beta \in F$ such that $(uv^2)v^2 - \beta v^3 \in B^4$. Replacing $x$ by $v^2$ and $y$ by $v$ in (3) we obtain that $4(uv^2)^2 - 3\beta v^2 \in B^3$. But relation (6) implies $(uv^2)^2 \in B^3$, hence $\beta = 0$, and so $uv^4, (uv^2)v^2 \in B^4$. Multiplying (13) by $v$ we get that $-2v^3 \in B^4$ and hence $v^3 = 0$. Finally, the first equation of (14) implies that $4v^4 \in B^5$, so that $v^4 = 0$. This proves the theorem. \qed

Theorem 8. If $B^3$ is an ideal of $A$, then $A$ is nilpotent of index $n - 1$.
Proof. It suffices to prove the theorem for the case $A^2 \not\subset B$ since otherwise, by Lemma 7, $A^j = B^j$ for all $j \geq 2$ and hence $A$ is nilpotent with index $n - 1$. Also, by Theorem 4 we can assume that $n \geq 5$.

If $B^2$ is an ideal of $A$, then $AB^k \subset B^{k+1}$ for all $k \geq 2$. Let $a \in A$. If $a^2 \in B^2$, then $a^3 \in B^3$. In the other case, $\{a, a^2, b, b^2\}$ is a basis of $A/B^3$, where $a = a + B^3$ and $b = b + B^3$. Since $a^3 \in (A/B^3)^2 = (a^2, b^2)$, we get that $a^3 = a\alpha^2 + \beta b^2$, with $\alpha, \beta \in F$. Now, $a^4 = a\alpha a^2 + \beta a\alpha b^2 = a\alpha a^2$, and hence $\alpha = 0$ and $a^3 = \beta b^2 + w$ with $w \in B^3$. If $r$ is a positive integer such that $2r < n - 1 \leq 3r$, then the identity $0 = (a^3)^r = (\beta b^2 + w)^r$ implies $\beta = 0$. Consequently, $A/B^3$ is a nilalgebra of nilindex 3, and by Theorem 4, $A^3 \subset B^3$. On the other hand, $A/B^4$ is a nilalgebra of nilindex 4 and hence, by Theorem 5, we have $((A/B^4)^2)^2 = 0$ so that $A^2 A^2 \subset B^4$. Now, it is easy to establish inductively, first that $A^2 B^k \subset B^{k+2}$, and next that $A^2 A^k, A^k^2 \subset B^{k+2}$, for all $k \geq 2$. Consequently, we have that $A^{n-1} = 0$ if $B^2$ is an ideal of $A$.

We will now assume that $B^2$ is not an ideal of $A$. By Lemma 15 there exist $u, v \in A$ such that $\{u, uv^2, v^2, \ldots, v^{n-2}\}$ is a basis of $A$, with $uv = 0$. Since $B^3$ is an ideal of $A$, Corollary 3 implies $(uv^2)v = -uv^3 \in B^4$ and $(uv^2)v^2 = uv^4 \in B^2$ and hence, from Lemma 4, $(uv^2)v^k \in B^{k+3}$ for all positive integers $k$. This forces that $(A^2)^2 v^2$ and $A^2 v^2$ are subsets of $B^4$. Now, relation (3) for $x = v, v^2$ and $y = u$ implies that $(uv^2)u = 2u^2 v^2 = (u^2 v)v \in B^4$ and $(uv^2)v^2 \in B^6$. Finally, we get that

\[
A^3 = AA^2 = A(uv^2, v^2, \ldots, v^{n-2}) = \{uv^2, v^3, \ldots, v^{n-2}\},
\]

\[
A^4 = A^3 A + (A^2)^2 = \{v^4, \ldots, v^{n-2}\} \subset B^4,
\]

\[
A^5 = A^4 A + A^3 A^2 \subset \{(uv^2)^2, v^5, \ldots, v^{n-2}\} \subset B^5,
\]

\[
A^6 = A^5 A + A^4 A^2 + A^3 A^3 = \{(uv^2)^2, (uv^2)^2 v, (uv^2)^2 u, v^6, \ldots, v^{n-2}\} \subset B^6,
\]

and $A^j \subset B^j$ for all $j \geq 6$. Thus, we have that $A^{n-1} = 0$. \hfill \Box

Corollary 4. If $n \neq 5$, then $A$ is nilpotent of index $n - 1$.

Finally using the above results it is possible to establish the following.

Corollary 5. Let $\mathfrak{A}$ be a commutative power-associative nilalgebra over a field $F$ with characteristic $\neq 2, 3$. If $\dim(\mathfrak{A}) \leq 6$, then $\mathfrak{A}$ is nilpotent or $(\mathfrak{A}^2)^2 = 0$.

Remark 1. We observe that Corollary 4 is not valid for $n = 5$. Let $\mathfrak{A}$ be a commutative algebra of dimension 5 with a basis $\{u_1, u_2, u_3, u_4, u_5\}$ and a non-zero multiplication given by $u_1 u_4 = u_3, u_1 u_5 = u_4, u_2^2 = u_4, u_2 u_3 = -u_4, u_2 u_4 = u_5$. We remark that $\mathfrak{A}$ is a power-associative nilalgebra of nilindex 4 such that $\mathfrak{A}^k = \langle u_3, u_4, u_5 \rangle$ for all $k \geq 2$. This example is given by D. Sutles in [6]. In [2, Theorem 3.3], we prove that this algebra is the unique commutative power-associative nilalgebra of nilindex 4 and dimension 5 which is not a Jordan algebra. Finally, we shall prove that every commutative Jordan nilalgebra $A$ of dimension 5 and nilindex 4 is nilpotent of index 4. In view of Lemma 15 and Theorem 8, we can assume that $A$ has a basis of the form $\{u, uv^2, v, v^2, v^3\}$, where $uv = 0$ and the
identities (12)–(15) hold. Since $A$ is a Jordan algebra, we have $(uv^2)v = v^2(uv) = 0$ and
from relation (12) we see that $(uv^2)v^2 = uv^4 = 0$ so that $(uv^2)v^3 = 0$. Now, using (12)
we obtain $uv^3 = -(uv^2)v = 0$. Consequently, we have proved that $v^3 \in \text{ann}(A)$ and hence
Theorem 8 gives that $A^4 = 0$.

References