A Lie–Poisson bracket formulation of plasticity and the computations based on the Lie-group $SO(n)$

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ABSTRACT

In this paper we develop a generalized Hamiltonian formulation of a perfectly elastoplastic model, which is a typical dissipative system. On the cotangent bundle of the yield manifold, a Lie–Poisson bracket is used to construct the differential equations system. The stress trajectory is a coadjoint orbit on the Poisson manifold under a coadjoint action by the Lie-group $SO(n)$. The plastic differential equation is an affine non-linear system, of which a finite-dimensional Lie algebra can be constructed, and the superposition principle is available for this system. Accordingly, we can construct numerical schemes to automatically preserve the yield-surface for perfect plasticity, for isotropic hardening material, as well as for an anisotropic elastic–plastic model. Then, we describe an anisotropic elastic–plastic material model without entering the work-hardening range and deforming under a specified dissipation rate, which can be achieved through a stress-dependent feedback control law of strain rate.

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1. Introduction

The study of plastic behavior of solid materials and structural members under complicated mechanical loading environment is a very important issue for engineering science and industrial practice. In this study a substantial role has been the constitutive laws of elastoplasticity, to which many theoretical and experimental contributions have been made.

In general, the constitutive model of plasticity is specified separately into several ingredients and written in the form of rate equations. The usual approaches require numerically integrating the rate equations over a discrete sequence of steps. Because of the interwoven and non-linear nature of the model ingredients, the difficulty of exact integration involves. Due to its great consumption of computational time, the efficiency and accuracy of solutions of structural and mechanical problems were strongly determined by the efficiency and accuracy of the constitutive-equation solving schemes. Hence, it needs to compose all the ingredients together and derives a global theory of plasticity, which not only directly gives a local response of material in the time domain once an input is prescribed, but also facilitates us to handle the model behavior from a global view. To achieve this purpose several requirements need to be satisfied. The first is integrating the rate-type equations, which consist of separated but interwoven ingredients, specified on a high-dimensional space such as the states in the stress space and the paths in the plastic strain space and their product space. The second is solving the irreversibility parameters, such as the equivalent plastic strain, dissipation, and so on, and in general there exist complicated monotonic mappings between these parameters. The third as to be strongly emphasized here is studying the internal symmetry properties of the material model.

The differential form of plasticity laws has been discussed for a long time. The invariant yield condition in stress space renders a natural mathematical framework of plasticity theory from the view of differentiable manifold and its Lie group transformation. In the plasticity theory, Hong and Liu (2000) were the first to make an effort to study the internal symmetry group and the underlying spacetime structure for perfect elastoplasticity. The Lie group theory provides a universal tool for tackling a considerable number of differential equations when other means fail. Indeed, the group analyses may augment intuition in understanding and in using the symmetry for the formulation of physical models, and often disclose some possible approaches to solving complex problems.

Group theory as a mathematical tool to study the symmetry has an abundance of applications from various fields. Numerous problems in engineering sciences possess certain symmetry properties. If we can recognize them, a mathematical treatment adjusted to the symmetry properties may lead to a considerable simplification.

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With this in mind, the internal symmetries approach of the elastoplastic models equipped with the von-Mises yield criterion have been developed by Hong and Liu (1999a, 2000), Liu (2001, 2003a, 2004a,b,c), Liu and Hong (2001), and Mukherjee and Liu (2003). Then, Liu (2004d) and Liu and Chang (2004, 2005) have extended these studies to the Drucker–Prager model, quadratic yielding model and convex plastic model. Liu (2005, 2006) has explored the action of the Poincaré group for the kinematic hardening models. These authors explored the internal symmetry groups of the constitutive models of perfect elastoplasticity with or without considering large deformation, visco-elasticity, isotropic work-hardening elastoplasticity, mixed-hardening elastoplasticity, the Drucker–Prager plasticity, the models with yield functions quadratic or convex, as well as the general kinematic-hardening models, to ensure that the consistency condition is exactly satisfied at each time step once the computational schemes can take these symmetries into account. Along this line, some researchers are further developed the so-called exact integration schemes or the exponential-based integration schemes for more complex material models (Artioli et al., 2006; Cheviakov et al., 2013; Kossa and Szabo, 2009, 2010; Liu and Li, 2005; Rezaiee-Pajand and Nasirai, 2008; Rezaiee-Pajand et al., 2010; Rezaiee-Pajand and Sharifian, 2012; Szabo and Kossa, 2012; Wallin and Ristinmaa, 2008). Besides its use in the numerical computation of material plasticity model, the Lie-symmetry is also used as a predictive method to construct many material models under the guidance of a thermodynamic frame of relaxation (Ganghoffer et al., 2010; Magнет et al., 2005, 2007, 2008).

Lie groups have played a decisive role in our understanding of the geometry of differential equations. The concept of Lie groups, within their wider terminology and machinery of differential geometry, is very helpful in devising superior numerical methods to discretize the ordinary differential equations (ODEs) to retain the invariant property. By sharing the geometric structure and invariance with the original ODEs, the new methods are thought to be more accurate, more stable and more effective than the conventional numerical methods. In this paper we will develop a more simple Lie-group algorithm for the computation of plasticity from the Hamiltonian framework. The classical Hamiltonian mechanics is endowed with an even-dimensional phase space. In practice, there are many mechanical systems whose phase spaces are not canonical. That is, the phase manifold does not admit a cotangent bundle structure on it, but still has a Poisson bracket equipped with the properties of skew-symmetry, bilinearity, the Leibniz identity and the Jacobi identity. The most famous example is the Euler equation which is the governing equation of the motion of rigid body.

In addition to the work of Liu (2007) for a finite strain perfectly plastic equation, there is rare study about the generalized Hamiltonian structure and the Lie–Poisson bracket formulation of the constitutive models of plasticity. In order to develop the Lie–Poisson theory and use this framework to develop a highly efficient algorithm to automatically preserve the yield-surface, let us briefly sketch the following preliminaries for the Lie–Poisson formulation.

2. The Lie–Poisson formulation

Suppose that $P$ is a manifold. If there is a bracket $\{ \cdot, \cdot \}$ defined on the function space $C(P)$, which possesses the following properties:

\begin{align*}
\text{Skew - symmetry:} & \quad \{ F, G \} = -\{ G, F \}, \\
\text{Bilinearity:} & \quad \{ \lambda F + \mu G, H \} = \lambda \{ F, H \} + \mu \{ G, H \}, \lambda, \mu \in \mathbb{R}, \\
\text{Jacobi identity:} & \quad \{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} = 0, \\
\text{Leibniz identity:} & \quad \{ FG, H \} = F \{ G, H \} + \{ F, H \} G,
\end{align*}

then $\{ P, \{ \cdot, \cdot \} \}$ is a Poisson manifold (Marsden and Ratiu, 1994). If an observable function $F: P \to \mathbb{R}$ of a dynamical system can be governed by a generalized Hamiltonian function $H$ through

$$
\dot{F} = \{ F, H \},
$$

then $\{ P, \{ \cdot, \cdot \}, H \}$ is called a generalized Hamiltonian system. Throughout this paper a superimposed dot denotes the time derivative.

Suppose that $H: P \to \mathbb{R}$ is a smooth function on $P$. The generalized Hamiltonian vector field $X_H$ associated with $H$ is a unique smooth vector field on $P$, which for every smooth function $F: P \to \mathbb{R}$ satisfies

$$
X_H(F) = \{ F, H \}.
$$

Instead of the non-degeneracy of the classical Poisson bracket, the bracket defined on the non-canonical Poisson manifold is permitted to be degenerate. See Appendix A for a further explanation of the Lie–Poisson bracket.

The Lie–Poisson system is naturally formulated in the dual space $G^*$ of a Lie algebra $G$. The solutions of the system are coadjoint orbits of a certain Lie group, constrained on the non-linear submanifolds of $G^*$, known as the symplectic foliations. In the past several decades, the applications to dissipative systems that fit the Lie–Poisson formalism are numerous, for example, Bloch et al. (1996) and Pelino and Pasini (2001). Also, for its important applications in the real physical systems there were some numerical integrators developed to preserve the Lie–Poisson structure; see, for example, Austin et al. (1993), Channell and Svecal (1991), Engø and Faltinsen (2001), Ge and Marsden (1988), Li and Qin (1995) and McLachlan (1993).

3. Model specification

From now on we study the plastic equation for the perfectly plastic material model, which is expressed in the deviatoric stress space by Hong and Liu (1997)

$$
\begin{align*}
\dot{e} &= \epsilon^e + \epsilon^\theta, \\
\dot{\sigma} &= 2Ge^e, \\
\dot{\lambda} &= 2\tau_{\sigma}e^\theta, \\
|\dot{s}| &\leq \sqrt{2}\tau_{\sigma}, \\
\dot{\lambda} &\geq 0, \\
|\dot{s}| &\leq \sqrt{2}\tau_{\sigma}\dot{\lambda},
\end{align*}
$$

where the two material constants, namely the shear modulus $G$ and the shear yield strength $\tau_{\sigma}$, are determined experimentally and both are assumed to be positive. The bold-faced symbols $\dot{e}$, $\dot{\sigma}$, $\dot{\epsilon}^e$, $\dot{\epsilon}^\theta$ and $\dot{s}$ stand for the deviatoric parts of strain rate, elastic strain rate, plastic shear strain. The isotropic hardening model and the anisotropic elastic–plastic model will be described in Sections 7, 8 and 10.

In this paper we analyze the constitutive model of plasticity from a different point of view and attempt to achieve a deeper understanding of its underlying structures of Lie algebra properties and the Lie group symmetries. To this end we need an appropriate setting to make the representation clearer yet simpler, and, therefore, we formulate the constitutive model directly in a vector form by introducing the stress vector:
We use \( \mathbf{k} \) to denote the dimension of stress space and strain space. Here, \( \mathbf{Q} \) and \( \mathbf{q} \) are one pair of dual vectors in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \); \( \mathbf{Q} = (Q_1, \ldots, Q_n)^T \) denotes the generalized stress vector and \( \mathbf{q} = (q_1, \ldots, q_n)^T \) denotes the generalized strain rate vector. Throughout this paper a superscript \( T \) denotes the transpose, and
\[
|\mathbf{Q}| = \sqrt{\mathbf{Q}^T \mathbf{Q}} = \sqrt{\mathbf{q}^T \mathbf{q}}
\]
denotes the Euclidean norm of \( \mathbf{Q} \). The above constitutive model is re-postulated from the celebrated Prandtl–Reuss equation formulated by Prandtl (1924) and Reuss (1930), which is well known as the simplest three-dimensional constitutive law for describing a class of linearly elastic-perfectly plastic materials. After developing a suitable algorithm to preserve the Lie–Poisson structure, we will extend the present Lie-group algorithm to other models in Sections 7, 8 and 10.

Using the on–off switching criteria, we can synthesize and convert the flow model (16)–(21) to a two-phase system:
\[
\frac{\mathbf{q}}{\mathbf{Q}} = \frac{k_0 \mathbf{q}}{\mathbf{Q}_0} 
\]
where \( \mathbf{q}_0 \) is subjected to
\[
\mathbf{q}_0 = 0 \quad \text{if} \quad q_0 |\mathbf{Q}| < Q_0 \quad \text{or} \quad \mathbf{q}^T \mathbf{Q}^T \mathbf{q} < 0. 
\]
In Eq. (22) the thermodynamic irreversible process demands \( \mathbf{q}_0 > 0 \); in contrast, the process with \( \mathbf{q}_0 < 0 \) is not allowed.

Inserting \( q_0 = 0 \) into Eq. (22) results in a very simple elastic equation \( \mathbf{Q} = \mathbf{k} \mathbf{q} \). Therefore, we do not discuss it furthermore; conversely, the plastic equation is more subtle with a rich internal property, and in this paper we are attempting to disclose its internal symmetry to search a different representation of the plastic equation from that of Hong and Liu (2000). They have found the Lorentz-group symmetry of the above flow model. The following two representations, respectively in the \( \mathbf{Q} \)-space, and in the \( \mathbf{X} \)-space, have been illustrated by Hong and Liu (2000).

### 3.1. Nonlinear representation in the \( \mathbf{Q} \)-space

Using Eq. (23) to eliminate \( \mathbf{q}_0 \) from Eq. (22) results in a non-linear system of \( n \) equations:
\[
\frac{\mathbf{q}}{\mathbf{Q}} = \frac{k_0 \mathbf{q}}{\mathbf{Q}_0} - \mathbf{q} + k_0 \mathbf{q} 
\]
which is a non-linear representation in the \( n \)-dimensional space of \( \mathbf{Q} \).

### 3.2. Linear representation in the \( \mathbf{X} \)-space

In the \( (\mathbf{Q}, \mathbf{q}_0) \)-space, the system of \( n + 1 \) equations, Eqs. (22) and (23), is non-linear. However, we may arrange the solution process simpler by making the problem linear in an augmented space.

Upon considering the integrating factor
\[
\lambda^0 := \exp \left( \frac{k_0 \mathbf{q}_0}{\mathbf{Q}_0} \right)
\]
Eq. (22) can be rearranged to
\[
\frac{d}{dt}(X^0 \mathbf{Q}) = k_0 \lambda^0 \mathbf{q} 
\]
Organizing Eqs. (27), (26) and (23) together, we have a linear differential equations system for the augmented stress:
\[
\mathbf{X} = A \mathbf{X}
\]
with
\[
A := \left[ \begin{array}{cc} \mathbf{I}_n & 0_{n \times n} \\ 0_{n \times 1} & -1 \end{array} \right]
\]
where
\[
\mathbf{X} = \lambda^0 \left[ \begin{array}{c} \mathbf{q} \\ \mathbf{Q}_0 \end{array} \right]
\]
is the augmented stress vector, satisfying
\[
||\mathbf{Q}|| = \mathbf{Q}_0 \iff \mathbf{X}^T \mathbf{g} \mathbf{X} = 0
\]
with
\[
\mathbf{g} = \left[ \begin{array}{cccc} 1 & \ldots & 0_{n \times 1} \\ 0_{1 \times n} & -1 \end{array} \right]
\]
being the metric tensor of the Minkowski space \( M^{n+1} \). \( \mathbf{I}_n \) is the identity tensor of order \( n \).

The linearity in Eq. (28) does not necessarily hold for other more complex elastoplastic models. Besides the elastic-perfectly plastic von-Mises model treated here and the von-Mises model with a linear kinematic hardening as treated by Hong and Liu (1999b), usually, the resultant differential equations system in the augmented stress space is quasilinear (Liu, 2003a), of which the coefficient matrix \( A \) also depends on \( \mathbf{X} \).

### 4. The Lie algebra of plastic equation

In order to derive a Lie algebra structure of the plastic Eq. (25), we write it with the following componential form:
\[ \dot{Q}_i = k_e \left[ \delta_{ij} - \frac{Q_j Q_i}{Q^2} \right] q_j, \]  

where \( \delta_{ij} \) is the Kronecker delta symbol. In this paper the Einstein sumation convention is adopted for the repeated indices. This equation can be viewed as an affine non-linear system with \( \dot{q}_i \) as input and \( \dot{Q}_i \) as output, which means that the above equation is linear in \( \dot{q}_i \) but non-linear in \( \dot{Q}_i \).

For the non-linear dynamical system:
\[ \frac{dx_i(t)}{dt} = \eta^i(x^1, \ldots, x^n, t), \quad 1 \leq \mu \leq n, \]  

if the general solution \( x(t) = (x^1(t), \ldots, x^n(t))^T \) can be expressed as a function of \( m \) particular solutions \( x^1(t), \ldots, x^n(t) \) and \( n \) integration constants \( c_1, \ldots, c_n \):
\[ x(t) = F(x^1, \ldots, x^n, c_1, \ldots, c_n), \]  

then Eq. (34) is said to admit a superposition principle; see, e.g., Hong and Liu (1997). Lie has proved that Eq. (34) permits a superposition principle iff it can be written as
\[ \frac{dx_i}{dt} = \sum_{\lambda=1}^{s} Z_{i}(t) \xi_{\lambda}(x), \]  

and its vector fields
\[ Y_{i} = \xi_{i}(x) \frac{\partial}{\partial x^{i}}, \quad i = 1, \ldots, s, \]  

constitute a finite-dimensional Lie algebra, whose dimension \( r \) satisfies \( s \leq r \leq mn \). The theorem stated below is proved in Appendix B.

Theorem 1. The plastic Eq. (25) admits a superposition principle.

The Lie algebra consists of \( \mathfrak{g} \) in Eq. (B1) is indeed the algebra of so\((n, 1)\) of the \( n + 1 \)-dimensional proper orthochronous Lorentz group \( SO_0(n, 1) \), in an extension to the large deformation model as observed by Hong and Liu (1999a). The vector fields generated from a spin term may appear. Hong and Liu (1997) have derived the superposition formula for Eq. (25), which was based on the linear differential Eq. (28). Here, we are proved the superposition principle by using the Lie theory.

5. The Lie–Poisson bracket formulation of plastic equation

Upon defining the Poisson tensor \( J \) to be
\[ J = qQ^T - Qq^T, \]  

Eq. (25) can be written as
\[ \dot{Q} = J\dot{V}H, \]  

where
\[ H = \frac{k_e |\dot{Q}|^2}{2Q_0^2} \]  

is a generalized Hamiltonian function. The theorem stated below is a main result of this paper and is very important, whose proof is given in Appendix C.

Table 1

<table>
<thead>
<tr>
<th>Equation (s)</th>
<th>Space</th>
<th>Metric</th>
<th>Dim.</th>
<th>Yielding</th>
<th>Linearity</th>
<th>Lie-algebra</th>
<th>Lie-group</th>
</tr>
</thead>
<tbody>
<tr>
<td>(25)</td>
<td>( L^1 )</td>
<td>( I_0 )</td>
<td>( n )</td>
<td>Yield surface</td>
<td>Non-linear</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(28)</td>
<td>( M^{n+1} )</td>
<td>( g )</td>
<td>( n+1 )</td>
<td>Cone</td>
<td>Linear</td>
<td>( SO(n, 1) )</td>
<td>( SO(n, 1) )</td>
</tr>
<tr>
<td>(39)</td>
<td>( \mathfrak{g} )</td>
<td>( J )</td>
<td>( n )</td>
<td>Coadjoint orbit</td>
<td>Non-linear</td>
<td>( \mathfrak{M}^i ), ( i = 1, \ldots, n )</td>
<td>( SO(n) )</td>
</tr>
</tbody>
</table>

Theorem 2. The plastic Eq. (39) is a Lie–Poisson bracket system. The solutions of Eq. (39) are the coadjoint orbits being acted by the Lie group \( SO(n) \), constrained on the yield manifold of \( \mathfrak{g} \), known as a symplectic foliation with the generalized Hamiltonian function \( H \) being constant on it.

Given an initial point \( Q(t_0) \) on the yield manifold, a solution to the plastic Eq. (39) stays on the same coadjoint orbit \( \mathfrak{C}_Q(t) \) for all time until unloading happens. Along the coadjoint orbit the generalized Hamiltonian function \( H \) defined by Eq. (40) is a constant.

Up to here we have investigated the plastic behavior from several theoretical aspects, explored three types of representations of the perfectly plastic equation. The three representations are compared in Table 1 by displaying the underlying space, metric (structure) tensor, dimension, yielding, linearity, Lie-algebra and Lie-group. Each has its philosophy as being a different aspect of the same material model of perfect plasticity.

In the next three sections we will explore the above theoretical results directly in terms of the non-linear ODEs in Eq. (39), and more importantly we can develop the corresponding numerical algorithms based on the Lie-group \( SO(n) \) to the perfectly plastic model and its extension to other models.

6. Numerical integration based on the symmetry group \( SO(n) \)

6.1. The Lie-group \( SO(n) \)

Eq. (25) can be expressed as
\[ \dot{Q} = \frac{Q}{Q_0} \frac{Q}{Q_0} k_d q, \]  

which is a special case of the Jordan dynamics developed by Liu (2000a):
\[ \dot{x} = [y, z, u] := y \cdot zu - u \cdot zy. \]  

The triplet \( y, z \) and \( u \) are functions of \( x \) and \( t \).

We have proven that the plastic equation admits a Lie-algebra \( so(n) \) in the previous section. Indeed, from Eq. (41) we can directly write
\[ n = \mathfrak{f} \otimes n - n \otimes \mathfrak{f} n, \]  

where
\[ n := \frac{Q}{Q_0}, \quad f = \frac{k_e}{Q_0} q, \]  

and \( \mathfrak{f} \otimes n \) denotes the dyadic operation of \( \mathfrak{f} \) and \( n \), i.e., \( \mathfrak{f} \otimes n = (n \cdot z) \mathfrak{f} \). Because the coefficient matrix in Eq. (43) is skew-symmetric, the Lie-group generated from the above dynamical system is \( SO(n) \).

6.2. The derivation of Lie-group \( SO(n) \)

For the purpose of a numerical computation we can approximate the specified strain path by a rectilinear strain path with a constant value of \( q \) in a small time interval, and all the variables are assumed to be known at the previous time step. So we need a numerical process to update the variables from this time step to the next time step. In order to develop a numerical scheme from
Eq. (43), we suppose that the coefficient matrix is constant and we let

\[ \mathbf{a} = \mathbf{f}, \quad \mathbf{b} = \mathbf{n} \quad (45) \]

be two constant vectors, which can be obtained by taking the values of \( \mathbf{f} \) and \( \mathbf{n} \) at a suitable mid-point of \( t \in [t_0 = 0, t] \), where \( t \leq t_0 + \Delta t \). \( \Delta t \) is a small time stepsize and the value of \( \mathbf{n} \) at \( t = t_0 = 0 \) denoted by \( \mathbf{n}_0 \) is supposed to be already known from the previous time step. Thus we have

\[ \mathbf{n} = \mathbf{b} \cdot \mathbf{n}_0 - \mathbf{a} \cdot \mathbf{n}_0. \quad (46) \]

Let

\[ z = \mathbf{a} \cdot \mathbf{n}, \quad w = \mathbf{b} \cdot \mathbf{n} \quad (47) \]

and Eq. (46) becomes

\[ \mathbf{n} = wa - zb. \quad (48) \]

At the same time we can derive the following ODEs for \( z \) and \( w \):

\[ \frac{d}{dt} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -\mathbf{a} \cdot \mathbf{b} \quad ||\mathbf{a}||^2 \\ -\mathbf{b}^2 \quad \mathbf{a} \cdot \mathbf{b} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}. \quad (49) \]

Through some operations we can derive the following transition matrix for \( z \) and \( w \):

\[ \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) - \frac{\mathbf{a} \cdot \mathbf{b}^2}{||\mathbf{a}||^2} \sin(\omega t) - \frac{\mathbf{a} \cdot \mathbf{b}^2}{||\mathbf{a}||^2} \sin(\omega t) \\ -\frac{\mathbf{b}^2}{||\mathbf{b}||^2} \sin(\omega t) + \frac{\mathbf{a} \cdot \mathbf{b}^2}{||\mathbf{a}||^2} \sin(\omega t) \end{pmatrix} \begin{pmatrix} z_0 \\ w_0 \end{pmatrix}. \quad (50) \]

where

\[ \omega = \sqrt{||\mathbf{a}||^2 ||\mathbf{b}||^2 - (\mathbf{a} \cdot \mathbf{b})^2} \]

and \( z_0 \) and \( w_0 \) are the initial values of \( z \) and \( w \) at an initial time \( t = t_0 = 0 \). Inserting Eq. (50) into Eq. (48), using \( z_0 = \mathbf{a} \cdot \mathbf{n}_0 \), \( w_0 = \mathbf{b} \cdot \mathbf{n}_0 \), and integrating the resultant equation we can derive

\[ \mathbf{n} = \begin{pmatrix} \mathbf{a} & \mathbf{b}^T \\ \mathbf{b} & \mathbf{a}^T \end{pmatrix} \begin{pmatrix} \mathbf{a} \cdot \mathbf{n}_0 \\ \mathbf{b} \cdot \mathbf{n}_0 \end{pmatrix}, \quad (52) \]

where \( \mathbf{n}_0 \) is the initial value of \( \mathbf{n} \) at an initial time \( t = t_0 = 0 \), and

\[ \begin{array}{c}
\mathbf{a}_{11} = \frac{1}{\omega} \sin(\omega t) + \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2} \cos(\omega t) - 1, \\
\mathbf{a}_{12} = \frac{||\mathbf{a}||^2}{\omega^2} [1 - \cos(\omega t)], \\
\mathbf{a}_{21} = \frac{||\mathbf{b}||^2}{\omega^2} [\cos(\omega t) - 1], \\
\mathbf{a}_{22} = \frac{1}{\omega} \sin(\omega t) + \frac{\mathbf{b} \cdot \mathbf{a}}{||\mathbf{b}||^2} [1 - \cos(\omega t)].
\end{array} \quad (53) \]

Let \( \mathbf{R}(t) \) be the coefficient matrix in Eq. (52), it is easy to check that

\[ \mathbf{R}(t) = \mathbf{R}(t), \quad (54) \]

which means that \( \mathbf{R}(t) \) is a single-parameter Lie-group element of \( SO(n) \).

6.3. Yield surface preserving scheme

Now we need a numerical process to update the stress from this time step \( t_0 \) to the next time step \( t_1 \). As being the simplest scheme we can take \( t = t_0 = 0 \), and thus

\[ \mathbf{a} = \frac{k_{\mathbf{b}}}{Q_0}, \quad \mathbf{b} = \mathbf{n}_0. \]

If we use the implicit scheme we can take \( t = t_0 + \Delta t \) with \( 0 \leq \theta \leq 1 \) and other variables are also defined at the mid-point, which however requires an iteration to solve the updated stress \( \mathbf{Q} \). Then inserting the above \( \mathbf{a} \) and \( \mathbf{b} \) into Eqs. (51)–(53) and through some elementary operations, we can derive the following Lie-group \( SO(n) \) scheme to solve Eq. (39) in the plastic state:

\[ \begin{align*}
\mathbf{Q} = & \frac{k_{\mathbf{b}}}{Q_0} \mathbf{Q}_0, \\
c_k = & \frac{k_{\mathbf{b}}}{Q_0} \mathbf{Q}_k, \\
\omega_k = & \sqrt{\mathbf{b}_0^2 - c_k^2}, \\
\mathbf{Q}_{k+1} = & \left[ \cos(\omega_k t) \right] \mathbf{Q}_k + \mathbf{a} \cdot \mathbf{n} \cdot \mathbf{b} \cdot \mathbf{n}. \quad (55) \]
\end{align*} \]

Next we give a proof that the above scheme preserves the yield surface. By taking the square norms of both the sides of the last equation in Eq. (55) we have

\[ ||\mathbf{Q}_{k+1}||^2 = \left[ \cos(\omega_k t) - \frac{c_k \sin(\omega_k t)}{\omega_k} \right] ||\mathbf{Q}_k||^2 + 2 \left[ \cos(\omega_k t) - \frac{c_k \sin(\omega_k t)}{\omega_k} \right] k_{\mathbf{b}} \frac{\sin(\omega_k t)}{\omega_k} \mathbf{Q}_k \cdot \mathbf{Q}_k + \frac{k_{\mathbf{b}}^2 \sin^2(\omega_k t)}{\omega_k^2} ||\mathbf{Q}_0||^2. \quad (56) \]

Inserting

\[ ||\mathbf{Q}_k||^2 = \frac{b_0^2 Q_0^2}{k_k}, \quad \mathbf{Q}_k \cdot \mathbf{Q}_k = \frac{c_k^2 Q_0^2}{k_k} \]

into Eq. (56) leads to

\[ ||\mathbf{Q}_{k+1}||^2 = \left[ \cos^2(\omega_k t) - 2 \cos(\omega_k t) \sin(\omega_k t) \frac{c_k}{\omega_k} + \frac{c_k^2 \sin^2(\omega_k t)}{\omega_k^2} \right] ||\mathbf{Q}_0||^2 + 2 \left[ \cos^2(\omega_k t) - \frac{c_k \sin(\omega_k t)}{\omega_k} \right] \frac{c_k \sin(\omega_k t) Q_0^2}{\omega_k} + \frac{k_{\mathbf{b}}^2 \sin^2(\omega_k t) Q_0^2}{\omega_k^2}. \quad (57) \]

Because of \( ||\mathbf{Q}_0||^2 = Q_0^2 \), we can obtain

\[ ||\mathbf{Q}_{k+1}||^2 = \left[ \cos^2(\omega_k t) - 2 \cos(\omega_k t) \sin(\omega_k t) \frac{c_k}{\omega_k} + \frac{c_k^2 \sin^2(\omega_k t)}{\omega_k^2} \right] \frac{Q_0^2}{2} + 2 \left[ \cos^2(\omega_k t) - \frac{c_k \sin(\omega_k t)}{\omega_k} \right] \frac{c_k \sin(\omega_k t) Q_0^2}{\omega_k} + \frac{k_{\mathbf{b}}^2 \sin^2(\omega_k t) Q_0^2}{\omega_k^2} \]

\[ = \left[ \cos^2(\omega_k t) + \frac{k_{\mathbf{b}}^2}{Q_0^2} \sin^2(\omega_k t) \right] Q_0^2 = Q_0^2. \quad (58) \]

where we have used \( b_0^2 - c_k^2 = \omega_k^2 \). This ends the proof.

7. Yield surface preserving scheme for isotropic hardening material model

Instead of the constant value of \( Q_0 \) in the above perfectly plastic model, we can take the isotropic hardening effect into account, where \( Q_0(Q_0) \) is now a function of \( \theta_0 \). Then we can derive the same equation as that in Eq. (43), but we need to supplement another equation to compute \( \theta_0 \) and \( Q_0(Q_0) \):

\[ \frac{Q_0}{Q_0 + k} (59) \]
Then we can derive the following SO(2) scheme to solve the isotropic hardening problem in the plastic state:

\[ b_k = \frac{k_{01} q_k}{Q_0(q_0^2)}, \]
\[ c_k = \frac{k_{01} q_k}{Q_0(q_0^2)}, \]
\[ \omega_k = \sqrt{\frac{b_k^2}{c_k} - \frac{c_k}{b_k^2}}, \]
\[ n_{k+1} = \left[ \cos(\omega_k \Delta t) - \frac{c_k \sin(\omega_k \Delta t)}{\omega_k} \right] n_k + \frac{k_{01} \sin(\omega_k \Delta t)}{Q_0(q_0^2) \omega_k} q_k. \]
\[ q_{k+1} = q_0 + \frac{k_{01} n_{k+1}}{Q_0(q_0^2)} q_k, \]
\[ Q_{k+1} = Q_0(q_0^{k+1}) n_{k+1}. \] (60)

8. Yield surface preserving scheme for anisotropic elastic-perfectly plastic material model

Instead of Eq. (17) we can consider an anisotropic elastic relation between \( Q \) and \( \mathbf{q} \) by

\[ \dot{Q} = K\mathbf{q}, \] (61)

where \( K \) is an elastic modulus tensor.

We can derive

\[ n = \left[ \frac{KQ}{Q \cdot KQ} n \cdot KQ \right] = \left[ KQ \otimes KQ - KQ \otimes KQ \otimes KQ \right] n. \] (62)

By inserting

\[ a = K\mathbf{q}, \quad b = \frac{KQ_0}{Q_0^2}, \]

into Eqs. (51)–(53) and through some elementary operations, we can derive the SO(n) scheme to preserve the yield surface for the anisotropic elastic-perfectly plastic material model:

\[ a_k = K\mathbf{q}_k, \]
\[ b_k = \frac{KQ_0}{Q_0^2} k\mathbf{q}_k, \]
\[ c_k = a_k \cdot b_k, \]
\[ \omega_k = \sqrt{\frac{\left| a_k \right|^2 \left| b_k \right|^2 - c_k^2}, \]
\[ a_{11} = \frac{1}{\omega_k} \sin(\omega_k \Delta t) + \frac{c_k}{\omega_k} \cos(\omega_k \Delta t) - 1, \]
\[ a_{12} = \frac{b_k}{\omega_k} \left[ 1 - \cos(\omega_k \Delta t) \right], \]
\[ a_{22} = \frac{1}{\omega_k} \sin(\omega_k \Delta t) + \frac{c_k}{\omega_k} \cos(\omega_k \Delta t) - 1, \]
\[ a_{22} = \frac{1}{\omega_k} \sin(\omega_k \Delta t) + \frac{c_k}{\omega_k} \left[ 1 - \cos(\omega_k \Delta t) \right], \]
\[ Q_{k+1} = Q_0 + a_{22} a_k \cdot Q_k + a_{22} b_k \cdot Q_k a_k - \left[ a_{11} a_k \cdot Q_k + a_{22} b_k \cdot Q_k a_k \right], \] (63)

9. Numerical examples

9.1. Perfect plasticity

In order to assess the performance of the numerical method in Section 6.3, we compute three numerical examples.
For the perfectly plastic model under a rectilinear strain path with a constant strain rate $\dot{\gamma} = c$, from Eq. (25) we can find the fixed point in the stress space by

$$\begin{align*}
\dot{Q} &= -\frac{k_c}{Q_0^2} Q^T c + k_c c = 0, \quad (70)
\end{align*}$$

from which we can solve $\dot{Q}$ to be the fixed point (Hong and Liu, 1998) by

$$Q = Q_0 \frac{c}{\|c\|}. \quad (71)$$

A more detailed behavior analysis was given by Hong and Liu (1998). In Fig. 3(a) and (b) we plot the portraits of $(Q_1, Q_1)$ and $(Q_2, Q_2)$ for the above example, where the six fixed points are marked sequentially by the Arabic numbers from 1 to 6.

**Example 3.** Fig. 4 displays an example for the model subjected to an input of a cyclic triangular path in two dimensions as shown in Fig. 4(a). The first cycle consists of three pieces 1, 2, 3, the second cycle consisting of pieces 4, 5, 6 repeats in the strain space the locus of the first cycle of pieces 1, 2 and 3, and so forth. The material constants used are $k_c = 200,000$ MPa and $Q_0 = 400$ MPa. Only the responses of the first two cycles are displayed because after that the responses were found to be almost repeated and stabilized. The results include the stress path in Fig. 4(b), and the hysteresis loops in Fig. 4(c) and (d). The response graph of the stress...
path in Fig. 4(b) as can be seen is very different from the input graph of the strain path in Fig. 4(a). One of two main features is that the strain path is closed, but the corresponding stress response has an open path. It is obvious that the Lie-group $\text{SO}(n)$ scheme gave very accurate responses and supplied a completely faithful result of the consistency condition as shown in Fig. 4(e). In Fig. 3(c) and (d) we plot the portraits of $(Q_1, \dot{Q}_1)$ and $(Q_2, \dot{Q}_2)$ for this example, where the three fixed points are marked sequentially by the Arabic numbers 1, 2 and 3. We can observe that both in Figs. 2(e) and 4(e) there appears a plateau of constant error of the consistency condition, where the plastic equation reaches to a fixed point and the stresses are kept unchanged. From this point we can also observe that the Lie-group $\text{SO}(n)$ scheme can preserve the fixed point behavior very well.

9.2. Isotropic hardening

In order to assess the performance of the numerical method in Section 7, we employ a numerical example with $k_e = 200,000$ MPa and

$$Q_0(q_0) = 250 - 50 \exp(-5q_0).$$

First we consider a strain control case with the strain control components $q_1$ and $q_2$ shown in Fig. 5(a). Fig. 5 illustrates the response to an input of a cyclic two-triangular path in two-dimensional strain space $(q_1, q_2)$. The results include the stress path in Fig. 5(b), the hysteresis loops in Fig. 5(c) and (d), as well as the error of consistency condition calculated by the above numerical method in Fig. 5(e). The $\text{SO}(n)$ group-preserving scheme gives the error of the consistency condition in the order of $10^{-12}$. 

![Fig. 2. The responses to an input of a cyclic two-triangular strain path in (a), and (b) displaying its corresponding stress path, (c) the cyclic axial stress-axial strain curve, (d) the cyclic shear stress–shear strain curve, and (e) error in satisfying the consistency condition for the numerical scheme.](image-url)
Then, Fig. 6 displays an example for the isotropic hardening model with \( k_e = 200,000 \text{ MPa} \) and 
\[ Q_0(q_0) = 450 - 50 \exp (-5q_0), \]
which is subjected to an input of a cyclic triangular path in two dimensions as shown in Fig. 6(a). The results include the stress path in Fig. 6(b) as can be seen is very different from the input graph of the strain path in Fig. 6(a). It is obvious that the Lie-group \( \text{SO}(n) \) scheme can achieve a completely faithful result of the consistency condition as shown in Fig. 6(e).

9.3. Anisotropic elastic-perfect plastic material

In order to assess the performance of the numerical method in Section 8, we employ a numerical example with 
\[ Q_0 = 200 \text{ MPa} \]
\[ K = \begin{bmatrix} 200,000 & 50,000 \\ 50,000 & 100,000 \end{bmatrix} \text{ MPa}. \]

We consider a strain control case with the strain control components \( q_1 \) and \( q_2 \) as shown in Fig. 7(a). The stress path is shown in Fig. 7(b), the hysteresis loops are shown in Fig. 7(c) and (d), and the error of consistency condition is shown in Fig. 7(e). The group-preserving scheme of \( \text{SO}(n) \) gives the error of the consistency condition in the order of \( 10^{-12} \).

10. Anisotropic elastic-perfectly plastic model with a specified dissipation rate

In the course of metal forming for the anisotropic elastic-plastic material, the work hardening usually leads to a cost consumption by increasing the applied loading in order to produce a desired large strain deformation. To this end we propose the following model endowed with an anisotropy and without work hardening and deforming under a specified dissipation rate:

\[ \dot{q} = q^e + q^p, \quad (73) \]
\[ Q = Kq^e, \quad (74) \]
\[ Q_0(q_0) = K_q^p, \quad (75) \]
\[ \|Q\| = Q_0. \quad (76) \]
\[ q_0 = p > 0. \quad (77) \]

Here, \( K \) is an anisotropic modulus, \( Q_0 \) is a constant yield stress, and \( q_0 = p \) is a specified constant or time-dependent dissipation rate, whose amount depends on the mechanical power we can supply. From the above equations we can derive the ODEs for \( Q \) and \( q_0 \):

\[ \dot{Q} = \left[ I - \frac{KQQ}{Q \cdot KQ} \right] Kq, \quad (78) \]
\[ \dot{q}_0 = \frac{Q_0Q \cdot Kq}{Q \cdot KQ}. \quad (79) \]

In order to satisfy Eq. (77) we can consider the controlled strain rate to be

\[ \dot{q} = \frac{pQ \cdot KQ}{Q^3_0} K^{-1}Q, \quad (80) \]

which is one kind of the state-feedback control law for the input strain rate. Inserting it into Eq. (79) and using Eq. (76) we can derive

\[ \dot{q}_0 = \frac{pQ_0 \cdot Q \cdot KQ}{Q^3_0 \cdot KQ} = p > 0. \quad (81) \]

Inserting Eq. (80) into Eq. (78) we have
\[ Q = \frac{pQ \cdot KQ}{Q_0^3} \left[ I_n - \frac{KQQ^T}{Q^2} \right] Q, \]  
which can rearranged to

\[ \dot{Q} = \frac{pQ \cdot KQ}{Q_0^3} \left[ \frac{KQ}{Q_2} \cdot Q \cdot Q - \frac{pKQ}{Q_0^3} \right] Q. \]  

Thus the numerical scheme in Section 8 can be applied to solve the above equation by taking

\[ a_k = Q_k, \quad b_k = \frac{pKQ_k}{Q_0^3}. \]

We can prove that the eigenvector \( Q \) of the modulus matrix \( K \)

\[ KQ = \lambda Q, \]  
is a fixed point of the dynamical system (82). Inserting the above equation into Eq. (82) we can derive

\[ \dot{Q} = \frac{pQ \cdot KQ}{Q_0^3} \left[ I_n - \frac{KQQ^T}{Q^2} \right] Q = \frac{pQ \cdot KQ}{Q_0^3} \left[ Q - \frac{Q}{\lambda} \right] Q = 0. \]  

Thus when the orbit of \( Q \) tends to the eigenvector, we have \( \dot{Q} = 0 \), which means that the eigenvector is a critical point of the dynamical system (82).

In order to display the above controlled responses we fix \( p = 0.0005 \) and calculate the numerical solution in a time interval of \( 0 \leq t \leq 200 \) s with the initial conditions given by \( Q_1 = Q_0 \) and \( Q_2 = 0 \), where \( Q_0 = 500 \) MPa and the matrix \( K \) is also given by Eq. (72). Fig. 8 illustrates the responses. The stress path is shown in Fig. 8(a), of which we can observe that the orbit tends to the fixed point. The controlled strain path and the time histories of \( q_1 \) and \( q_2 \) are shown, respectively, in Fig. 8(b) and (c). The error of consistency condition calculated by the above numerical method is shown in Fig. 8(d). The SO(n) group-preserving scheme leads to the error of the consistency condition in the order of \( 10^{-11} \).

11. Conclusions

In this paper we have investigated the plastic equations from a theoretical aspect of the generalized Hamiltonian formalism, which highlights the stress yielding behavior as a coadjoint orbit on the symplectic foliation in the dual Lie algebra space. The yield function plays the role of a generalized Hamiltonian function in the Lie–Poisson bracket system. The physical meaning of this
formulation relies on the fact that the yielding behavior of materials is realized by the coadjoint orbit on the yield surface to preserve the generalized Hamiltonian (yield) function invariant.

The non-linear problems of plasticity were usually treated by workers in plasticity with various numerical schemes, which often encounter the tremendous difficulties of plastic non-linearity and yielding inconsistency. The passage directly from the flow models to a numerical scheme, if no care is taken of, may alter or destroy the underlying structure of the model, resulting in an unstable, inefficient, and inaccurate computation.

A numerical scheme based on the representation in the Minkowskian X-space has been shown to be efficient, since the representation in the X-space is linear, and moreover, easy to retain the Lie group symmetry $SO(n, 1)$ of the model, thus facilitating the fulfillment of the consistency condition. The extensions of X-space representation in the canonical Minkowski space together with the Lorentz group symmetry to other plasticity models have also been conducted in several papers. However, we need to stress that the Lorentz-group $SO(n, 1)$ used in the above schemes is non-compact, which may enlarge the computational error due to the numerical instability induced by a large time stepsize used in the algorithm. So in this paper we have developed the rotation group $SO(n)$ algorithm based on the theoretical development of the Lie-Poisson formulation. In contrast, the Lie-group $SO(n)$ is compact, which is more stable than the Lorentz-group $SO(n, 1)$ algorithm, allowing a large-step computation, and the computational error will be not enlarged. One numerical example was used to demonstrate that the $SO(n)$ scheme is accurate than the exponential-based scheme derived from the Lorentz-group in the augmented stress space. To test the four proposed models it can be seen that the Lie-group $SO(n)$ algorithm is essentially and automatically preserved the yield-surface. This study provided not only a deeper understanding of the mathematical structure of the plastic equation, but also a sound foundation of the numerical solution for non-linear problems of plasticity.

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Appendix A. The Jacobi identity

In this appendix we describe the Jacobi identity in the Lie–Poisson bracket, which is used to prove the Lie–Poisson bracket formulation of the plastic equation.

Suppose that \( C : P \to \mathbb{R} \) is a non-constant smooth function on \( P \).

If \( \{ C, F \} = 0 \) for all smooth function \( F : P \to \mathbb{R} \), then \( C \) is a Casimir function on \( P \).

When \( P \) is a finite-dimensional manifold with dimension \( n \), the local coordinates of \( P \) can be assigned as \( x = (x_1, \ldots, x_n) \), and the Poisson bracket on \( P \) can be written as

\[
\{ F, G \} := J_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j},
\]

where \( J_{ij}(x) \) is a Poisson tensor.

Given an \( n \times n \) matrix function \( J(x) \) with components \( J_{ij}(x) \) defined on the open set \( P \subset \mathbb{R}^n \), the necessary and sufficient conditions of \( J(x) \) to be a Poisson tensor are

\[
J_{ij} = -J_{ji}, \quad i, j = 1, 2, \ldots, n
\]

and

\[
J_{ij}J_{jk} + J_{ik}J_{jk} + J_{kj}J_{ik} = 0, \quad i, j, k = 1, 2, \ldots, n.
\]

where \( J_{ik} \) denotes \( \partial J_{ik} / \partial x_k \).

For all smooth function \( H : P \to \mathbb{R} \) defined on \( P \), the bundle mapping \( B : TP \to TP \) is denoted by \( B(dH(x)) = X_{dH(x)} \). TP and TP are, respectively, the tangent and cotangent bundles on the Poisson manifold \( P \). The rank of the Poisson bracket at a point \( x \in P \) is defined as the rank of the linear mapping \( B_{x_0} : T_x P \to T_x P \). A point \( x \) on the Poisson manifold \( P \) is called a regular point, if the ranks for all points in the neighborhood of \( x \in P \) are the same; otherwise, \( x \) is a singular point. The rank of \( B \) at \( x \in P \) and the rank of Poisson tensor \( J(x) \) at point \( x \) are the same. Because of the skew-symmetry of \( J(x) \), the rank is always even.

Suppose that the rank of the Poisson tensor \( J(x) \) at a regular point \( x_0 \) is \( n - m, m > 0 \), then there exist \( m \) functionally independent Casimir functions defined in the neighborhood of the point \( x_0 \).

Especially, when \( J(x) \) is a linear function of \( x \), the bracket (A1) is called a Lie–Poisson bracket, and Eq. (5) is a Lie–Poisson bracket system, written as

\[
x = J(x) VH(x).
\]
where the gradient operator $\nabla$ denotes the derivative with respect to $x$. We usually write such a $J_{ij}(x)$ to be

$$J_{ij} = C_{kij} x^k; \quad (A5)$$

where $C_{kij} = /C_{0} C_{kji}$ and at the same time the Jacobi identity (A3) takes the form:

$$C_{ijk}^{'} C_{k^i} + C_{k}^{'} C_{k^i} C_{k^j} C_{k^j} = 0. \quad (A6)$$

It is known that for this case the underlying space can be given a Lie algebra structure with the structure constants $C_{kij}$ in a suitable basis (Marsden and Ratiu, 1994).

**Appendix B. The proof of Theorem 1**

In this appendix we prove Theorem 1 in Section 4. The $n$ vector fields of Eq. (33) are

$$g_j = \delta_j e_i - \frac{Q_j}{Q_0} e_i, \quad 1 \leq j \leq n, \quad (B1)$$

where $e_i$, $i = 1, \ldots, n$ are unit bases. The Lie bracket of $g_j$ and $g_j$ is

$$[g_j, g_j] = \frac{\partial g_j}{\partial Q_j} g_j - \frac{\partial g_j}{\partial Q_j} g_j. \quad (B2)$$

From Eq. (B1) it follows that

$$\frac{\partial g_j}{\partial Q_j} = \frac{Q_j}{Q_0} - \frac{Q_j}{Q_0} g_j. \quad (B3)$$

where $g_j$ is the $i$th component of $g_j$. By using the above equation we can prove that

$$[g_j, g_j] = \frac{Q_j}{Q_0} g_j - \frac{Q_j}{Q_0} g_j. \quad (B4)$$

Inserting Eq. (B1) for $g_j$ the right-hand side can be further reduced to

$$\frac{Q_j}{Q_0} g_j - \frac{Q_j}{Q_0} g_j = \frac{Q_j}{Q_0} (\delta_{j2} \delta_{j3} - \delta_{j3} \delta_{j2}) e_i. \quad (B5)$$

Fig. 7. For anisotropic elastic-perfect plasticity the responses to an input of a cyclic two-triangular strain path in (a), and (b) displaying its corresponding stress path, (c) the cyclic axial stress-axial strain curve, (d) the cyclic shear stress–shear strain curve, and (e) error in satisfying the consistency condition for the numerical scheme.
Thus, from Eqs. (B7) and (B8) it follows that which is zero if any two indices of \(B\) are equal, +1 if \(\{i_1, i_2, \ldots, i_k\}\) is an odd permutation, and -1 if \(\{i_1, i_2, \ldots, i_k\}\) is an even permutation. Recalling that

\[
e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} = \delta_{i_2} \delta_{i_3} - \delta_{i_3} \delta_{i_2},
\]

(B6)

from Eqs. (B4) and (B5) we have

\[
[g_i, g_j] = -\frac{1}{Q_0} e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} Q_j e_i.
\]

(B7)

This motivates us to consider the vector fields

\[
w_{i_1 i_2 \ldots i_n} := e_{i_1 i_2 \ldots i_k} Q_i e_i.
\]

(B8)

There are total \(n(n-1)/2\) linearly independent vector fields of \(w\). Thus, from Eqs. (B7) and (B8) it follows that

\[
[g_i, g_j] = -\frac{1}{Q_0} e_{i_1 i_2 \ldots i_k} w_{i_1 i_2 \ldots i_k}.
\]

(B9)

From Eq. (B8) it follows that

\[
\frac{\partial w_{i_1 i_2 \ldots i_k}}{\partial Q_j} = e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k},
\]

(B10)

where \(w_{i_1 i_2 \ldots i_k}\) is the \(i\)th component of \(w_{i_1 i_2 \ldots i_k}\). Hence, by Eqs. (B3), (B10), (B1) and (B8) through some calculations we can find

\[
w_{i_1 i_2 \ldots i_k} = e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} g_i.
\]

(B11)

\[
w_{i_1 i_2 \ldots i_k} w_{i_1 i_2 \ldots i_k} = e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} w_{i_1 i_2 \ldots i_k} - e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k} e_{i_1 i_2 \ldots i_k}.
\]

(B12)

Therefore, the \(n\) vector fields of Eq. (33) and the spin term \(w\) in Eq. (B8) constitute a finite-dimensional Lie algebra. This ends the proof. \(\Box\)

Appendix C. The proof of Theorem 2

In this appendix we prove Theorem 2 in Section 5. We can prove that \(J\) defined by Eq. (38) satisfies Eqs. (A2) and (A3). The first condition of skew-symmetry is obvious. Let us write

\[
J_{ij} = \delta_{i} Q_j - Q_i\delta_j.
\]

\[
J_{ij} = \delta_{i} \delta_j - \delta_i \delta_j.
\]

By using them we can prove that

\[
J_{i} dJ_{k} + J_{j} dJ_{k} + J_{k} dJ_{i} = [\delta_{i} Q_j - Q_i \delta_j] [\delta_{j} Q_k - \delta_j Q_k] + [\delta_{j} Q_k - Q_j \delta_k] [\delta_{k} Q_i - \delta_k Q_i] - Q_i Q_i, Q_j - Q_i Q_j - Q_i Q_i, Q_k - Q_i Q_k - Q_i Q_k - Q_i Q_k - Q_i Q_k - Q_i Q_k - Q_i Q_k - Q_i Q_k.
\]

\[
+ Q_i Q_i, Q_k + Q_i Q_k, Q_i + Q_i Q_i, Q_i + Q_i Q_i, Q_k = 0.
\]
Thus, \( \mathbf{J} \) satisfies Eqs. (A2) and (A3). Moreover, because \( \mathbf{J} \) is a linear function of \( \mathbf{Q} \), the bracket (A1) with the above \( \mathbf{J} \) is a Lie–Poisson bracket. Consequently, the plastic Eq. (39) is a Lie–Poisson bracket system (A4).

As that presented in Eq. (A5), from Eq. (38) we can identify the structure constants to be

\[
C_{ij}^1 = 
\begin{bmatrix}
0 & -\dot{q}_2 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\dot{q}_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \dot{q}_1 & 0 & \ldots & 0 \\
-\dot{q}_n & 0 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\end{bmatrix}, \\
C_{ij}^2 = 
\begin{bmatrix}
0 & \dot{q}_2 & \dot{q}_3 & \ldots & \dot{q}_n \\
\dot{q}_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\dot{q}_1 & 0 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
0 & 0 & \ldots & 0 \\
\end{bmatrix}, \\
C_{ij}^3 = 
\begin{bmatrix}
0 & -\dot{q}_n & 0 & \ldots & 0 \\
\dot{q}_1 & -\dot{q}_2 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \dot{q}_1 & 0 & \ldots & 0 \\
-\dot{q}_n & 0 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\end{bmatrix}.
\]

Suppose that \( \mathbf{Q} = \mathbf{Q}_k \mathbf{e}_k \) and that \( \{ \mathbf{e}_k, k = 1, \ldots, n \} \) forms a basis of the dual Lie algebra \( \mathfrak{g}^* \). The above structure constants can be used to construct a Lie algebra denoted by \( \mathfrak{g} \):

\[
[f, g] = C_{ij}^k f_i g_k, \quad (C4)
\]

where \( \{ f_k, k = 1, \ldots, n \} \) forms a basis of the Lie algebra \( \mathfrak{g} \) and \( \{ \mathbf{e}_k \} \) is the commutator; see, e.g., Varadarajan, 1984.

Next, we consider the adjoint representation of the Lie algebra \( \mathfrak{g} \). For each \( f \in \mathfrak{g} \) the operator \( \text{ad} f \) that maps \( g \in \mathfrak{g} \) into \( [f, g] \) is a linear transformation of \( \mathfrak{g} \) onto itself, i.e.,

\[
(\text{ad} f) g = [f, g].
\]

As supposed \( \{ f_k, k = 1, \ldots, n \} \) are bases for the Lie algebra \( \mathfrak{g} \), we have

\[
(\text{ad} f) f_k = C_{ij}^k f_i, \quad (C6)
\]

Therefore the matrix associated with the transformation \( \text{ad} f \),

\[
M_{ik} = C_{ij}^k. \quad (C7)
\]

Corresponding to the structure constants given in Eqs. (C1)–(C3), the following \( M \) are available:

\[
M_1 = C_{ij}^1 = 
\begin{bmatrix}
0 & -\dot{q}_2 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
0 & \dot{q}_1 & 0 & \ldots & 0 \\
0 & 0 & \dot{q}_1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \dot{q}_1 \\
\end{bmatrix}, \quad (C8)
\]

\[
M_2 = C_{ij}^2 = 
\begin{bmatrix}
\dot{q}_2 & 0 & 0 & \ldots & 0 \\
-\dot{q}_1 & 0 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dot{q}_1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & \dot{q}_2 \\
\end{bmatrix}.
\]

In order to prove that the above \( \{ M_k, k = 1, \ldots, n \} \) is indeed a matrix basis for the Lie algebra \( \mathfrak{g} \) and satisfies Eq. (C4), let us rewrite them to be

\[
M_1 = \dot{q}_1 \mathbf{I}_n + \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix} \quad \text{\textup{ith row}}, \quad (C11)
\]

\[
M_2 = \dot{q}_1 \mathbf{I}_n + \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix} \quad \text{\textup{jth row}}, \quad (C12)
\]

From them we obtain

\[
[M, M_j] = M M_j - M_j M = 
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0& 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\]

\[
[M_1, M_j] = M_1 M_j - M_j M_1 = 
\begin{bmatrix}
\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \ldots & \dot{q}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}.
\]

\[
[M_2, M_j] = M_2 M_j - M_j M_2 = 
\begin{bmatrix}
\dot{q}_1 & 0 & 0 & \ldots & 0 \\
-\dot{q}_1 & 0 & -\dot{q}_3 & \ldots & -\dot{q}_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \dot{q}_2 \\
\end{bmatrix}.
\]

In above, \( (\dot{q}_1 \dot{q}_2, \ldots, \dot{q}_1 \dot{q}_n) \) locates at the \textup{ith row} and \( (-\dot{q}_1 \dot{q}_1, \ldots, -\dot{q}_1 \dot{q}_n) \) locates at the \textup{jth row}. On the other hand, from Eqs. (C1)–(C3) we find that the structure constants \( C_{ij}^k \) are all zeros except for \( k = i \) or \( k = j \), of which we have \( C_{ij}^0 = -\dot{q}_i \) and \( C_{ij}^i = \dot{q}_i \). Thus, we can obtain

\[
C_{ij}^0 M_k = C_{ij}^k M_0 + C_{ij}^j M_k = -\dot{q}_i M_0 + \dot{q}_i M_j. \quad (C14)
\]

Inserting Eqs. (C11) and (C12) for \( M_0 \) and \( M_1 \) into the above equation we can obtain the right-hand side of Eq. (C13), that is,

\[
[M_1, M_j] = C_{ij}^0 M_k. \quad (C15)
\]

This ends the proof of Eq. (C4). Consequently, \( \{ M_k, k = 1, \ldots, n \} \) forms a matrix basis for the Lie algebra \( \mathfrak{g} \).

Let us consider the Lie group \( \mathbf{G} \) generated from the matrix \( \mathbf{M} \):

\[
\dot{\mathbf{G}}_i = \mathbf{G} \mathbf{G}_i(0) = \mathbf{I}_n, i \text{ not summed}. \quad (C16)
\]

Since the two matrices on the right-hand side of Eq. (C11) commute, we can solve the above \( \mathbf{G} \) as
\[ G = \begin{bmatrix}
0 & e^{\psi(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{\psi(0)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\psi(0)} & 0 & 0 & 0 & 0 \\
v_{0} & v_{0} & v_{0} & v_{0} & v_{0} & v_{0} & v_{0} & v_{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  
(C17)

where

\[ \nu_{\mu}(t) = -\int_{0}^{t} q_{i}(\zeta) \exp[q_{i}(\zeta) - q_{i}(0)] d\zeta. \]  
(C18)

The above \( G \) is a dilatational translation in the \( \mathbb{R}^n \) plane \( G_{\mu} \) constant denoted by \( DT_{\mu}(n-1) \). The right-action of \( DT_{\mu}(n-1) \) on \( \mathbb{R}^{n-1} \) is a dilation followed by a translation by the vector \( \nu_{i} \) and has the expression:

\[ Q^{T}_{\mu} \exp(\psi(0)) L_{\mu-1}, \nu_{i} = \exp(\psi(0)) Q_{\mu} + \nu_{i}^{T}. \]  
(C19)

for any \( Q_{\mu} = (Q_{1}, Q_{2}, ..., Q_{n-1}, Q_{n}) \) \( \in \mathbb{R}^{n-1} \), where \( \nu_{i} = (v_{0}, v_{1}, ..., v_{n-1}) \) \( \in \mathbb{R}^{n-1} \). Note that \( DT_{\mu}(n-1) \) embeds into \( G_{\mu}(n, \mathbb{R}) \) as that done in Eq. (C17); thus one can operate with \( DT_{\mu}(n-1) \) as one would with matrix Lie groups by using the embedding.

Corresponding to the Lie algebra \( \mathfrak{g} \) there exists a Lie group denoted by \( G \) which is composed of all \( DT_{\mu}(n-1) \), and the adjoint representation of the Lie group is denoted by \( Ad_{G} \), \( g \in G \).

\[ Ad_{G} : \mathfrak{g} \rightarrow G. \]  
(C20)

\( G^{*} \) is foliated by the coadjoint orbits:

\[ Q_{\mu} = (Ad_{G}^{*}, Q, g \in G) \subset G^{*}, \]  
(C21)

where the coadjoint action \( Ad_{G}^{*} \) is defined by

\[ \langle Ad_{G}^{*}(w), v \rangle = \langle w, Ad_{G}(v) \rangle, \quad w \in \mathfrak{g}^{*}, \quad v \in G. \]  
(C22)

Here \( < \cdot , \cdot > \) denotes a non-degenerate pairing between \( \mathfrak{g}^{*} \) and \( G \).

For matrices the adjoint action and coadjoint action are, respectively,

\[ Ad_{G}^{*} V = g^{T} V g, \]  
(C23)

\[ Ad_{G} V = V g. \]  
(C24)

Differentiating the pair in Eq. (C22) with respect to \( g \) then letting \( g \) equal to identity, we obtain

\[ \langle ad_{G}^{*} w, v \rangle = \langle w, ad_{G} v \rangle, \quad w \in \mathfrak{g}^{*}, \quad v \in G, \]  
(C25)

where \( u = (d/dt) g(t) |_{t=0} \) \( \equiv \) ad \( ^{\dagger} \) is the coadjoint representation of the Lie algebra \( \mathfrak{g} \). Then we have

\[ ad_{G}^{*} w = -J[w] f, \quad w \in \mathfrak{g}^{*}. \]  
(C26)

Therefore the matrix associated with the transformation \( ad_{G}^{*} \) is

\[ (M)_{ik}^{\dagger} = -C_{ik}. \]  
(C27)

Since \( C_{ik} \) is a skew-symmetric matrix for each \( i \), the corresponding coadjoint action is found to be an \( n \)-dimensional rotation group, denoted by \( SO(n) \). This ends the proof. \( \square \)

**Appendix D. The consistent tangent operators of Eqs. (69) and (55)**

(I) The tangent operator of Eq. (69).

First we note that \( \exp(\Delta A_{k}) \), \( a_{k} \) and \( b_{k} \) in Eqs. (67) and (68) can be written as

\[ \exp(\Delta A_{k}) = \begin{bmatrix}
I_{n} + \frac{a_{0} - 1}{2a_{0}^{2}} \Delta q_{k}^{T} & b_{k} a_{0}\Delta q_{k}^{T} \\
b_{k} a_{0}^{2} \Delta q_{k} & a_{0}
\end{bmatrix}, \]  
(D1)

\[ a_{k} = \cosh(k ||\Delta q_{k}||/Q_{0}), \quad b_{k} := \sinh(k ||\Delta q_{k}||/Q_{0}). \]  
(D2)

where

\[ \Delta q_{k} = q_{k+1} - q_{k}. \]  
(D3)

Correspondingly, Eq. (69) can be written as

\[ Q_{k+1} = \frac{Q_{0} Q_{k} + Q_{0} \frac{a_{0} - 1}{2a_{0}^{2}} \Delta q_{k}^{T} Q_{k} \Delta q_{k} + b_{k} a_{0}^{2} \Delta q_{k}^{T} \Delta q_{k}^{T}}{a_{0} Q_{0} + b_{k} a_{0}^{2} Q_{0}}. \]  
(D4)

The tangent operator is given by

\[ T_{k+1} = \frac{\partial Q_{k+1}}{\partial q_{k+1}}. \]  
(D5)

By using

\[ \frac{\partial Q_{k+1}}{\partial q_{k+1}} = I_{n}, \quad \frac{\partial}{\partial q_{k+1}} Q_{k+1} = \frac{\Delta q_{k+1}}{||\Delta q_{k+1}||}, \quad \frac{\partial a_{k}}{\partial q_{k+1}} = k_{c} a_{k}, \quad \frac{\partial b_{k}}{\partial q_{k+1}} = k_{e} b_{k}. \]  
(D6)

where

\[ \nu_{0} = a_{0} Q_{0} + b_{k} a_{0}^{2} Q_{0} \]  
(C28)

\[ v_{k} = Q_{0} Q_{k} + Q_{0} \frac{a_{0} - 1}{2a_{0}^{2}} \Delta q_{k}^{T} Q_{k} \Delta q_{k} + b_{k} a_{0}^{2} \Delta q_{k}^{T} \Delta q_{k}. \]  
(C29)

(II) The tangent operator of Eq. (55). By using Eq. (D3), Eq. (55) can be written as

\[ Q_{k+1} = \cos W_{k} C_{k} \sin W_{k} Q_{k} + \frac{k_{e} \sin W_{k}}{W_{k}} \Delta q_{k}. \]  
(D8)

where

\[ B_{k} = k_{c} ||\Delta q_{k}||/Q_{0}, \]  
(C30)

\[ C_{k} = k_{e} Q_{0} \cdot \Delta q_{k}, \]  
(C31)

\[ W_{k} = \sqrt{B_{k}^{2} - C_{k}^{2}}. \]  
(D9)

Then through some elementary operations we can derive the consistent tangent operator of Eq. (55):
\[ T_{k+1} = Q_k \left( K - C_k - W_k \right) \left( \frac{\partial W_k}{\partial W_{k+1}} \right)^T + \sin W_k \left( \frac{\partial W_k}{\partial W_{k+1}} \right)^T + \frac{1}{2} \left( \frac{\partial W_k}{\partial W_{k+1}} \right)^T \frac{\partial W_k}{\partial W_{k+1}} \frac{\partial W_k}{\partial W_{k+1}}^T \].

(D10)

where

\[ \frac{\partial C_k}{\partial W_{k+1}} = \frac{k}{Q_0} Q_k \frac{\partial W_k}{\partial W_{k+1}}. \]

(D11)

References


