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A new method for solving hypersingular integral equations of the first kind

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ABSTRACT

A simple and efficient method for solving hypersingular integral equations of the first kind in reproducing kernel spaces is developed. In order to eliminate the singularity of the equation, a transform is used. By improving the traditional reproducing kernel method, which requires the image space of the operator to be W_2^1 and the operator to be bounded, the exact solutions and the approximate solutions of hypersingular integral equations of the first kind are obtained. The advantage of this numerical method lies in the fact that, on one hand, the approximate solution is continuous, and on the other hand, the approximate solution converges uniformly and rapidly to the exact solution. The validity of the method is illustrated with two examples.

1. Introduction

This aim of this work is to develop a numerical algorithm for the hypersingular integral equations of the first kind of the form

$$a(x)\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}\mathrm{d}t+\int_{-1}^{1}L(t,x)\varphi(t)\mathrm{d}t=f(x),\quad -1\le x\le 1,\tag{1.1}$$

where the unknown function $\varphi(x)$ has square-root zeros at the end-points, that is, $\varphi(x)=\sqrt{1-x^2}g(x)$ with $g(x)$ smooth, $a(x)$ is bounded and belongs to $L^2[-1,1]$, $L(t,x)$ is a regular square-integrable function of t and x, and $f(x)$ is smooth. The first integral is understood in the sense of Hadamard finite part, that is,

$$\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}\mathrm{d}t=\lim_{\varepsilon\to 0+}\left[\int_{-1}^{x-\varepsilon}\frac{\varphi(t)}{(t-x)^2}\mathrm{d}t+\int_{x+\varepsilon}^{1}\frac{\varphi(t)}{(t-x)^2}\mathrm{d}t-\frac{\varphi(x+\varepsilon)+\varphi(x-\varepsilon)}{\varepsilon}\right],\quad -1\le x\le 1.\tag{1.2}$$

Eq. (1.1) arises frequently in a variety of mixed boundary value problems in mathematical physics such as water wave scattering and radiation problems involving thin submerged plates [1–4] and fracture mechanics [5]. Usually, Eq. (1.1) is solved approximately by an expansion–collocation method [6].

In this work, we will present a new simple and effective method for the reproducing kernel space.

To solve Eq. (1.1), we first slide over the hypersingular integral term of Eq. (1.1). Note that in the sense of Cauchy principal value,

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\mathrm{d}t=-\pi x$$

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holds. Therefore,

$$\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}\mathrm{d}t = \int_{-1}^{1}\sqrt{1-t^2}\frac{g(t)}{(t-x)^2}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\sqrt{1-t^2}\frac{g(t)}{t-x}\mathrm{d}t$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-1}^{1}\sqrt{1-t^2}\frac{g(t)-g(x)}{t-x}\mathrm{d}t + g(x)\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\mathrm{d}t\right]$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-1}^{1}\sqrt{1-t^2}\frac{g(t)-g(x)}{t-x}\mathrm{d}t - \pi x g(x)\right]$$
$$= \int_{-1}^{1}\sqrt{1-t^2}\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2}\mathrm{d}t - \pi\left(xg'(x)+g(x)\right).$$

Then Eq. (1.1) can be converted into

$$a(x)\int_{-1}^{1}\sqrt{1-t^2}\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2}\mathrm{d}t + \int_{-1}^{1}\sqrt{1-t^2}L(t,x)g(t)\mathrm{d}t = f(x)+\pi a(x)\left(xg'(x)+g(x)\right)$$
$$= f_1(x), \quad -1 \le x \le 1. \tag{1.3}$$

In Eq. (1.3), $\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2}$ is regarded as $\frac{g''(x)}{2}$ as $t = x$. So $\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2} \in C([-1,1]\times[-1,1])$. This means that the singularity of Eq. (1.1) has been removed. When computing integrals, the Gauss–Chebyshev quadrature rule of the second kind is an appropriate choice.

In this work, by solving Eq. (1.3), we will give the exact solution of Eq. (1.1), denoted by a series, in the reproducing kernel space. After truncating the series, the approximate solution is obtained. The approximate solution converges uniformly and quickly to the exact solution of Eq. (1.1) on the interval $[-1, 1]$. It is worth pointing out that, unlike other numerical method solutions, our numerical approximate solution is continuous and converges uniformly to the exact solution of Eq. (1.1). The two experiments at the end show the efficiency of our method.

2. Preliminaries

Definition 2.1. $W[-1,1] = \{u(x) | u''(x)$ is an absolutely continuous real-valued function

on $[-1,1]$ and $u'''[x] \in L^2[-1,1]\}$. (2.1)

The inner product and the norm of $W[-1,1]$ are defined as follows:

$$(u,v)_{W[-1,1]} = \sum_{i=0}^{2} u^{(i)}(-1)v^{(i)}(-1) + \int_{-1}^{1} u'''(x)v'''(x)\mathrm{d}x, \quad \forall u,\ v \in W[-1,1]. \tag{2.2}$$

Theorem 2.1. *The reproducing kernel of space W is*

$$R(x,y) = \begin{cases} R_1(x,y), & y \le x \\ R_2(x,y), & y > x \end{cases}$$

where

$$R_1(x,y) = \frac{1}{120}\left(276+195y+40y^2+y^5+5x\left(39+56y+18y^2-y^4\right)+10x^2(1+y)^2(4+y)\right).$$
$$R_2(x,y) = \frac{1}{120}\left(276+195x+40x^2+x^5+5y\left(39+56x+18x^2-x^4\right)+10y^2(1+x)^2(4+x)\right).$$

See the Appendix for the proof of Theorem 2.1, above.

3. Several lemmas

Let us discuss how to solve $\varphi(x)$ from Eq. (1.1) in the next two sections.
Put

$U[-1,1] = \{u(x) | u(x)$ is real and absolutely continuous on the interval $[-1,1]\}$.

Define operator A as follows:

$$(Ag)(x) = a(x)\int_{-1}^{1}\sqrt{1-t^2}\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2}\mathrm{d}t + \int_{-1}^{1}\sqrt{1-t^2}L(t,x)g(t)\mathrm{d}t, \quad -1 \le x < 1. \tag{3.1}$$

We can easily obtain the following four lemmas.

Lemma 3.1. *Suppose that* $u(x) \in C[-1,1]$, *and for a fixed* $x_i \in [-1,1]$, $u(x) \in C^1[-1,x_i], u(x) \in C^1[x_i,1]$; *then* $u(x) \in U[-1,1]$.

Lemma 3.2. *Let* $k(t,x) \in C([-1,1]\times[-1,1])$; *then* $\int_{-1}^{1} k(t,x)\mathrm{d}t \in C[-1,1]$.

Lemma 3.3. *Let* $h(t) \in L^2[-1,1]$, $k(t,x) \in C([-1,1]\times[-1,1])$; *then* $\int_{-1}^{1} h(t)k(t,x)\mathrm{d}t \in C[-1,1]$.

Lemma 3.4. *Let* $k(t,x) \in L^2([-1,1]\times[-1,1])$, $h(t) \in C[-1,1]$; *then* $\int_{-1}^{1} k(t,x)h(t)\mathrm{d}t \in L^2[-1,1]$.

Lemma 3.5. *Operator* A *maps* $W[-1,1]$ *into* $L^2[-1,1]$.

Proof. Taking any $g(x) \in W[-1,1]$, using $\frac{-g'(x)(t-x)+g(t)-g(x)}{(t-x)^2} \in C([-1,1]\times[-1,1])$ and Lemma 3.2, it follows that the first integral in A belongs to $C[-1,1]$. Besides, from $L(t,x) \in L^2([-1,1]\times[-1,1])$ and Lemma 3.4, one obtains that the second integral in A belongs to $L^2[-1,1]$. Hence, A maps $W[-1,1]$ into $L^2[-1,1]$. □

Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[-1,1]$. Put

$$\psi_i(x) = [A_t R(x,t)](x_i). \tag{3.2}$$

Theorem 3.1. $\psi_i(x) \in W[-1,1]$.

Proof. (a) From

$$\psi_i(x) = [A_t R(x,t)](x_i) = a(x_i)\int_{-1}^{1}\sqrt{1-t^2}\,\frac{-\frac{\partial}{\partial t}R(x,x_i)(t-x_i)+R(x,t)-R(x,x_i)}{(t-x_i)^2}\mathrm{d}t + \int_{-1}^{1}\sqrt{1-t^2}L(t,x_i)R(x,t)\mathrm{d}t,$$

it follows that

$$\begin{aligned}\psi_i''(x) &= a(x_i)\int_{-1}^{1}\sqrt{1-t^2}\,\frac{-\frac{\partial^3}{\partial x^2\partial t}R(x,x_i)(t-x_i)+\frac{\partial^2}{\partial x^2}R(x,t)-\frac{\partial^2}{\partial x^2}R(x,x_i)}{(t-x_i)^2}\mathrm{d}t \\ &\quad + \int_{-1}^{1}\sqrt{1-t^2}L(t,x_i)\frac{\partial^2}{\partial x^2}R(x,t)\mathrm{d}t \\ &= \mathrm{I}+\mathrm{II},\\ \psi_i'''(x) &= a(x_i)\int_{-1}^{1}\sqrt{1-t^2}\,\frac{-\frac{\partial^4}{\partial x^3\partial t}R(x,x_i)(t-x_i)+\frac{\partial^3}{\partial x^3}R(x,t)-\frac{\partial^3}{\partial x^3}R(x,x_i)}{(t-x_i)^2}\mathrm{d}t \\ &\quad + \int_{-1}^{1}\sqrt{1-t^2}L(t,x_i)\frac{\partial^3}{\partial x^3}R(x,t)\mathrm{d}t \\ &= \mathrm{III}+\mathrm{IV}.\end{aligned}$$

Since $L(t,x_i) \in L^2[-1,1]$, $\frac{\partial^2}{\partial x^2}R(x,t)$, $\frac{\partial^3}{\partial x^3}R(x,t)$, $\frac{-\frac{\partial^3}{\partial x^2\partial t}R(x,x_i)(t-x_i)+\frac{\partial^2}{\partial x^2}R(x,t)-\frac{\partial^2}{\partial x^2}R(x,x_i)}{(t-x_i)^2} \in C([-1,1]\times[-1,1])$ and Lemma 3.3, we obtain I, II, IV $\in C[-1,1]$. Because $\frac{-\frac{\partial^4}{\partial x^3\partial t}R(x,x_i)(t-x_i)+\frac{\partial^3}{\partial x^3}R(x,t)-\frac{\partial^3}{\partial x^3}R(x,x_i)}{(t-x_i)^2}$ belongs to $C([-1,x_i]\times[-1,1])$ and $C([x_i,1]\times[-1,1])$, it follows that III $\in C[-1,x_i]$ and $C[x_i,1]$. Therefore, $\psi_i''(x) \in C[-1,1]$, $\psi_i'''(x) \in C[-1,x_i]$ and $C[x_i,1]$. It follows that $\psi_i''(x) \in U[-1,1]$ and $\psi_i'''(x) \in L^2[-1,1]$ from Lemma 3.1. Thus, according to the definition of $W[-1,1]$, $\psi_i(x) \in W[-1,1]$. □

4. Main results

Theorem 4.1. $\{\psi_i\}_{i=1}^{\infty}$ *defined by* (3.2) *is complete in* $W[-1,1]$.

Proof. From Theorem 3.1, $\psi_i(x) \in W[-1,1]$. Let $u \in W[-1,1]$ such that $(u,\psi_i)=0$. From

$$0 = (u,\psi_i) = (u(x), A_t R(x,t)(x_i)) = [A_t(u(x),R(x,t))](x_i) = (A_t u(t))(x_i) \tag{4.1}$$

and the density of $\{x_i\}_{i=1}^{\infty}$ being in $[-1,1]$, we obtain $Au=0$. Note that $Au=0$ has a unique solution. We obtain $u=0$. □

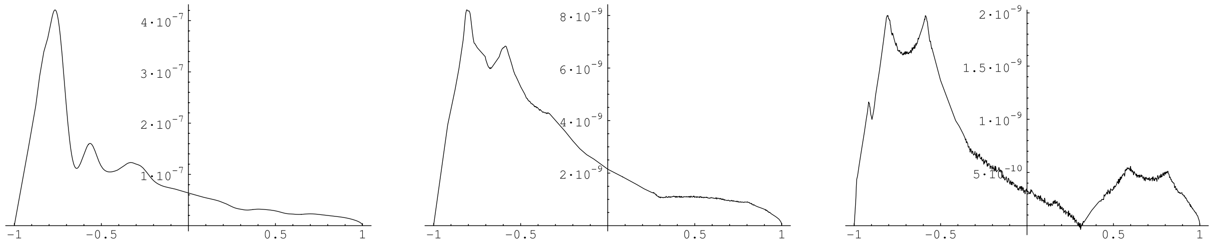

Fig. 1. The figure of the absolute error $\varphi_n - \varphi$ for Example 5.1 for $n = 5, 15, 25$.

Using the Gram–Schmidt process, we orthonormalize the sequence $\{\psi_i\}_{i=1}^{\infty}$ and obtain the orthonormal system $\{\overline{\psi}_i\}_{i=1}^{\infty}$, that is,

$$\overline{\psi}_i = \sum_{k=1}^{i} \beta_{ik}\psi_k, \quad \beta_{ii} > 0,\ i = 1, 2, \ldots.$$

$\{\overline{\psi}_i\}_{i=1}^{\infty}$ is an orthonormal basis of $W[-1, 1]$.

Theorem 4.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of $[-1, 1]$; then:*

(a) *The exact solution of Eq. (1.1) is $\varphi(x) = \sqrt{1-x^2}\sum_{i=1}^{\infty} \tilde{f}_i \bar{\psi}_i$, where $f_1(x) = f(x) + \pi a(x)\left(xg'(x) + g(x)\right)$, $\tilde{f}_i = \sum_{k=1}^{i} \beta_{ik} f_1(x_k)$, $i = 1, 2, \ldots$.*

(b) *$\varphi_n(x) = \sqrt{1-x^2}\sum_{i=1}^{n} \tilde{f}_i \bar{r}_i$ converges uniformly to the exact solution $\varphi(x)$ of Eq. (1.1).*

Proof. (a) Let $\varphi(x)$, $g(x)$ be the exact solutions of Eq. (1.1) and (1.3) respectively. We have

$$(g, \psi_k) = (g, [A_t R(x, t)](x_k)) = A_t(g, R(x, t))(x_k) = Ag(x_k) = f_1(x_k).$$

So,

$$(g, \bar{\psi}_i) = \sum_{k=1}^{i} \beta_{ik}(g, \psi_k) = \sum_{k=1}^{i} \beta_{ik} f_1(x_k) = \tilde{f}_i.$$

Hence, $\varphi(x) = \sqrt{1-x^2}g(x) = \sqrt{1-x^2}\sum_{i=1}^{\infty}(g, \bar{r}_i)\bar{\psi}_i = \sqrt{1-x^2}\sum_{i=1}^{\infty}\tilde{f}_i\bar{\psi}_i$.

(b) Write $g_n(x) = \sum_{i=1}^{n} \tilde{f}_i \bar{\psi}_i$; then $\varphi_n(x) = \sqrt{1-x^2}g_n(x)$. Since

$$\|R(x, y)\|_W^2 = (R(x, y), R(x, y)) = R(x, x)$$

is a polynomial of x, we obtain that for any $-1 \le x \le 1$,

$$\begin{aligned} |\varphi_n(x) - \varphi(x)| &= \sqrt{1-x^2}|g_n(x) - g(x)| = \sqrt{1-x^2}|(g_n(y) - g(y), R(x, y))| \\ &\le \|g_n - g\| \cdot \|R(x, y)\|_W \le M\|g_n - g\|. \end{aligned}$$

So, from $\|g_n - g\|_W \to 0$, as $n \to \infty$, the conclusion follows. □

5. Numerical Examples

In this section, we will demonstrate the effectiveness of the proposed method by considering two concrete examples of Eq. (1.1). Denote by $\varphi(x)$ and $\varphi_n(x)$ the exact solution and the approximate solution of the examples considered, respectively. And in the computation, the nine-point Gauss–Chebyshev quadrature rule of the second kind is used.

Example 5.1. Considering the following hypersingular integral equation:

$$\int_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\mathrm{d}t + \int_{-1}^{1}(t+x)\varphi(t)\mathrm{d}t = \frac{\pi}{2}(1-6x^2) + \frac{\pi}{8}x, \quad -1 \le x \le 1. \tag{5.1}$$

$\varphi(x) = \sqrt{1-x^2}x^2$ is the exact solution of Eq. (5.1). Take $\{x_i\}_{i=1}^{m} = \{-1+\frac{2i}{n}\}_{i=0}^{n} \cup \{-1+\frac{2i}{n}+\frac{1}{n}\}_{i=0}^{n-1} \cup \{-1+\frac{i}{n}+\frac{1}{2n}\}_{i=0}^{2n-1} \cup \{-1+0.004i\}_{i=1}^{20} \cup \{1-0.004i\}_{i=20}^{1}$. The absolute errors $\varphi_n - \varphi$ for $n = 5, 15, 25$ are given in Fig. 1.

Example 5.2. Another hypersingular integral equation is given by

$$\int_{-1}^{1} \frac{\varphi(t)}{(t-x)^2}\mathrm{d}t + \int_{-1}^{1} tx\varphi(t)\mathrm{d}t = -8\pi x^3 + \frac{17}{8}\pi x - \pi, \quad -1 \le x \le 1. \tag{5.2}$$

whose exact solution is $\varphi(x) = \sqrt{1-x^2}(1+2x^3)$. $\{x_i\}_{i=1}^{m}$ is chosen the same as in Example 5.1. The absolute errors $\varphi_n - \varphi$ for $n = 5, 15, 25$ are given in Fig. 2.

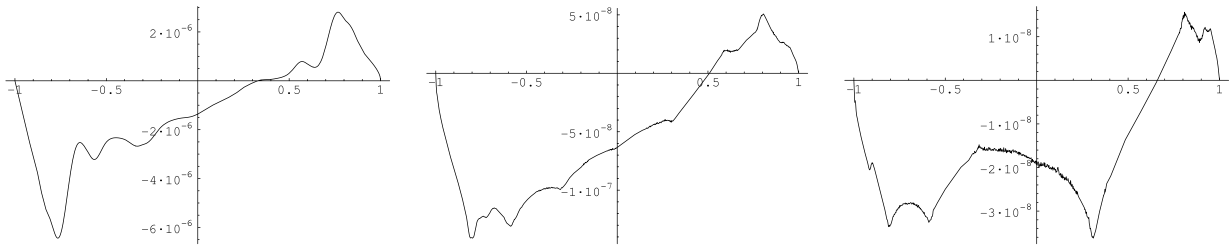

Fig. 2. The figure of the absolute error $\varphi_n - \varphi$ for Example 5.2 for $n = 5, 15, 25$.

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Appendix

In the following, the proof of Theorem 2.1 will be given.

Proof. Using the formula for integration by parts three times, one has

$$\int_{-1}^{1} u'''v'''\mathrm{d}x = u''v^{(3)}\mid_{-1}^{1} - u'v^{(4)}\mid_{-1}^{1} + uv^{(5)}\mid_{-1}^{1} - \int_{-1}^{1} uv^{(6)}\mathrm{d}x, \tag{A.1}$$

and further

$$\begin{aligned}(u, v) &= \sum_{i=0}^{2} u^{(i)}(-1)v^{(i)}(-1) + \int_{-1}^{1} u'''v'''\mathrm{d}x \\ &= u(-1)[v(-1) - v^{(5)}(-1)] + u(1)v^{(5)}(1) + u'(-1)[v'(-1) + v^{(4)}(-1)] \\ &\quad - u'(1)v^{(4)}(1) + u''(-1)[v''(-1) - v^{(3)}(-1)] + u''(1)v^{(3)}(1) - \int_{-1}^{1} uv^{(6)}\mathrm{d}x,\end{aligned}$$

and thus

$$\begin{aligned}(u(y), R(x, y)) = {} & u(-1)\left[R(x, -1) - \frac{\partial^5}{\partial y^5}R(x, -1)\right] + u(1)\frac{\partial^5}{\partial y^5}R(x, 1) + u'(-1)\left[\frac{\partial}{\partial y}R(x, -1) + \frac{\partial^4}{\partial y^4}R(x, -1)\right] \\ & - u'(1)\frac{\partial^4}{\partial y^4}R(x, 1) + u''(-1)\left[\frac{\partial^2}{\partial y^2}R(x, -1) - \frac{\partial^3}{\partial y^3}R(x, -1)\right] + u''(1)\frac{\partial^3}{\partial y^3}R(x, 1) \\ & - \int_{-1}^{1} u(y)\frac{\partial^6}{\partial y^6}R(x, y)\mathrm{d}y.\end{aligned}$$

In order to obtain $(u(y), R(x, y)) = u(x)$, it is enough to require the following equalities to hold:

$$-\frac{\partial^6}{\partial y^6}R(x, y) = \delta(y - x) \tag{A.2}$$

$$R(x, -1) - \frac{\partial^5}{\partial y^5}R(x, -1) = 0, \qquad \frac{\partial^5}{\partial y^5}R(x, 1) = 0 \tag{A.3}$$

$$\frac{\partial}{\partial y}R(x, -1) + \frac{\partial^4}{\partial y^4}R(x, -1) = 0, \qquad \frac{\partial^4}{\partial y^4}R(x, 1) = 0 \tag{A.4}$$

$$\frac{\partial^2}{\partial y^2}R(x, -1) - \frac{\partial^3}{\partial y^3}R(x, -1) = 0, \qquad \frac{\partial^3}{\partial y^3}R(x, 1) = 0. \tag{A.5}$$

From (A.2), we have $\frac{\partial^6}{\partial y^6}R(x, y) = 0$ as $y \neq x$. Its characteristic equation is $\lambda^6 = 0$. Hence, $\lambda = 0$ (sixfold); we obtain

$$R(x, y) = \begin{cases} c_0(x) + c_1(x)y + c_2(x)y^2 + c_3(x)y^3 + c_4(x)y^4 + c_5(x)y^5 & y \le x, \\ d_0(x) + d_1(x)y + d_2(x)y^2 + d_3(x)y^3 + d_4(x)y^4 + d_5(x)y^5 & y > x. \end{cases} \tag{A.6}$$

Integrating both sides of (A.2) from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and letting $\varepsilon \to 0$, one gets

$$\frac{\partial^5}{\partial y^5} R(x, x-0) - \frac{\partial^5}{\partial y^5} R(x, x+0) = 1. \tag{A.7}$$

Meanwhile, integrating indefinitely on both sides of (A.2) with respect to y, we can obtain in turn that for a fixed x, as a function of y, $\frac{\partial^i}{\partial y^i} R(x, y)$, $i = 4, 3, 2, 1, 0$, are all continuous, that is

$$\frac{\partial^i}{\partial y^i} R(x, x+0) = \frac{\partial^i}{\partial y^i} R(x, x-0), \quad i = 0, 1, 2, 3, 4. \tag{A.8}$$

Substituting the results c_i, d_i obtained from (A.3)–(A.5), (A.7), (A.8) into (A.6), $R(x, y)$ is obtained. □

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