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Solution Procedures for Three-Dimensional Eddy Current Problems

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The problem under consideration is that of the scattering of time periodic electromagnetic fields by metallic obstacles. A common approximation here is that in which the metal is assumed to have infinite conductivity. The resulting problem, called the perfect conductor problem, involves solving Maxwell's equations in the region exterior to the obstacle with the tangential component of the electric field zero on the obstacle surface. In the interface problem different sets of Maxwell equations must be solved in the obstacle and outside while the tangential components of both electric and magnetic fields are continuous across the obstacle surface. Solution procedures for this problem are given. There is an exact integral equation procedure for the interface problem and an asymptotic procedure for large conductivity. Both are based on a new integral equation procedure for the perfect conductor problem. The asymptotic procedure gives an approximate solution by solving a sequence of problems analogous to the one for perfect conductors.

1. INTRODUCTION

This paper continues a study, begun in [4], of scattering of time harmonic electromagnetic fields by metallic obstacles, the *eddy current* problem. The work in [4] was restricted to a very special class of two-dimensional problems. Here we treat the full three-dimensional situation.

Although the technical details of the present paper are far more complicated than those of [4] the general outline is exactly the same. Two ideas are developed. The first is a boundary integral procedure for the eddy

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current problem. The second is an asymptotic procedure which applies for large conductivity and reflects the *skin effect* in metals. The key to both methods, just as in [4], is the introduction of a new integral equation procedure for the boundary value problem corresponding to perfect conductors.

Let Ω' be a bounded region in R^3 and $\Omega = (\bar{\Omega}')^c$. Ω is to represent air and Ω' a metallic conductor. We suppose there are incident electric and magnetic fields, $\mathbf{E}^0$ and $\mathbf{H}^0$, satisfying Maxwell's equations in air. The total fields $\mathbf{E}$, $\mathbf{H}$ satisfy the same Maxwell equations as $\mathbf{E}^0$ and $\mathbf{H}^0$ in Ω but a different set in Ω'. Across the interface $S = \partial\Omega = \partial\Omega'$ the tangential components of both $\mathbf{E}$ and $\mathbf{H}$ must be continuous.

We assume that $\mathbf{E}^0$ and $\mathbf{H}^0$ are time periodic with a single frequency and require that $\mathbf{E}$ and $\mathbf{H}$ should have the same property. We make the standard assumptions that conduction (displacement) currents can be neglected in air (metal). Then, with appropriate scaling, the eddy current problem is (see [11])

$$\begin{aligned} &\operatorname{curl} \mathbf{E} = \mathbf{H}, && \operatorname{curl} \mathbf{H} = \alpha^2 \mathbf{E} && \text{in } \Omega \\ &\operatorname{curl} \mathbf{E} = \mathbf{H}, && \operatorname{curl} \mathbf{H} = i\beta^2 \mathbf{E} && \text{in } \Omega' \qquad (\mathrm{P}_{\alpha\beta}) \\ &\mathbf{E}_\mathrm{T}^+ = \mathbf{E}_\mathrm{T}^-, && \mathbf{H}_\mathrm{T}^+ = \mathbf{H}_\mathrm{T}^- && \text{on } S. \end{aligned}$$

Here α and $\beta > 0$ are dimensionless parameters. The subscript T denotes tangential component and the superscripts plus and minus denote limits from Ω and Ω'. The fields $\mathbf{E}$ and $\mathbf{H}$ must be such that $\mathbf{E} - \mathbf{E}^0$ and $\mathbf{H} - \mathbf{H}^0$ represent scattered fields.

At higher frequencies the constant β is usually large and this leads to the *perfect conductor* approximation. Formally this means solving only the Ω equation and requiring that $\mathbf{E}_\mathrm{T} = \mathbf{0}$ on S. If we let $\mathbf{E}$ and $\mathbf{H}$ denote the scattered fields, we then obtain the problem

$$\begin{aligned} &\operatorname{curl} \mathbf{E} = \mathbf{H}, && \operatorname{curl} \mathbf{H} = \alpha^2 \mathbf{E} && \text{in } \Omega \\ & && \mathbf{E}_\mathrm{T} = -\mathbf{E}_\mathrm{T}^0 && \text{on } S. \end{aligned} \qquad (\mathrm{P}_{\alpha\infty})$$

THEOREM 1.1. *There exists at most one solution of* $(\mathrm{P}_{\alpha\beta})$ *for any* $\alpha > 0$ *and* $0 < \beta \leqslant \infty$.

The proof of uniqueness for $\mathrm{P}_{\alpha\infty}$ can be found in [9] and the proof for $(\mathrm{P}_{\alpha\beta})$ is a minor variation which we omit.

A first integral equation procedure for $(\mathrm{P}_{\alpha\beta})$ was given in [1]. There have been a number of subsequent procedures for $(\mathrm{P}_{\alpha\infty})$ (see [6, 9, 12]). All of these lead to integral equations of the second kind. We also give an integral equation procedure for $(\mathrm{P}_{\alpha\infty})$ but ours leads to a type of first-kind equation.

Our motivation for this new method, as in [4], is that it gives a simple procedure for the calculation of the quantity $\mathbf{H}_T$ on S. This enables us to formulate an integral equation procedure for $(P_{\alpha\beta})$ and to give our asymptotic scheme.

Our method is similar to that in [6] but contains important differences. In particular our analysis is in terms of Sobolev spaces. This facilitates the formulation of Galerkin schemes, a subject which is explored in [7].

Let us describe the asymptotic procedure. Let τ be the distance from S measured into Ω' along the normals to S. Then we obtain two different asymptotic expansions:

$$\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix} \sim \begin{matrix}\mathbf{E}^0\\ \mathbf{H}^0\end{matrix} + \sum_{n=0}^{\infty} \begin{matrix}\mathbf{E}_n\\ \mathbf{H}_n\end{matrix} \beta^{-n} \qquad \text{in } \Omega \qquad (A_\Omega)$$

$$\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix} \sim e^{-\sqrt{-i}\,\beta\tau} \sum_{n=0}^{\infty} \begin{matrix}\mathbf{E}_n\\ \mathbf{H}_n\end{matrix} \beta^{-n} \qquad \text{in } \Omega'. \qquad (A_{\Omega'})$$

In these formulas the $\mathbf{E}_n$ and $\mathbf{H}_n$ are independent of β. The exponential in $(A_{\Omega'})$ represents the skin effect.

The various coefficients can be computed recursively. $\mathbf{E}_0$, $\mathbf{H}_0$ in (A_Ω) is simply the perfect conductor approximation, that is, the solution of $(P_{\alpha\infty})$. One calculates the successive $\mathbf{E}_n$ and $\mathbf{H}_n$ in (A_Ω) by solving a sequence of problems of the same form as $(P_{\alpha\infty})$ but with boundary values determined from earlier coefficients. The $\mathbf{E}_n$ and $\mathbf{H}_n$ in $(A_{\Omega'})$ are obtained by solving ordinary differential equations in the variable τ.

The asymptotic procedure, when it is valid, gives a great reduction in complexity of solution since it involves solving only the boundary value problems of the form $(P_{\alpha\infty})$.

A theoretical numerical analysis of a Galerkin method for $(P_{\alpha\infty})$ is given in [7]. Reference [4] contains the results of numerical experiments in the two-dimensional case. These exhibited quite high accuracy in the integral equation methods and the validity of the asymptotic approximation over a very wide range of β values.

The plan of the paper is as follows. In Section 2 we describe our integral equation methods for $(P_{\alpha\infty})$ and $(P_{\alpha\beta})$. ([7] has a variational formulation which we propose to implement numerically). In Section 3 we first consider the special case of a half-space and illustrate the method formally. This suggests the theorems which should be true in the general case and we give a precise statement of them.

The equations which appear in our integral equation methods involve pseudo-differential operators on S. We give quite precise existence and regularity theorems for these in Section 3. The regularity results play a central role in the theoretical analysis of the variational procedures. The proofs of these theorems are technically complicated and are presented in Sections 4 and 5.

In Section 6 we describe our asymptotic procedure. For ease of presentation we will present it for the half-space case.

2. The Integral Equation Methods

Let us begin by stating the conditions under which we operate. In order to avoid technical assumptions about smoothness we will assume throughout that S is a regular analytic surface. There will be another condition which is based on the following well-known result.

THEOREM 2.1. *There exists a sequence* $\{\alpha_k\}$, $k = 1, 2,\ldots$, *such that if* $\alpha \neq \alpha_k$ *then* $\operatorname{curl} \mathbf{E} = \mathbf{H}$, $\operatorname{curl} \mathbf{H} = \alpha^2 \mathbf{E}$ *in* Ω', $\mathbf{E}_{\mathrm{T}} \equiv \mathbf{0}$ *on* S *implies* $\mathbf{E} \equiv \mathbf{H} \equiv \mathbf{0}$ *in* Ω'.

Throughout the paper we require

$$\alpha \neq \alpha_k, \qquad k = 1, 2,\ldots. \tag{2.1}$$

Our methods, like others, are based on the *Stratton–Chu* formulas from [11]. To describe these we need some notation. We will let $\mathbf{n}$ denote the exterior normal to S. Given any vector field $\mathbf{v}$ defined on S we have

$$\mathbf{v} = \mathbf{v}_{\mathrm{T}} + v_N \mathbf{n}, \qquad \mathbf{v}_{\mathrm{T}} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) \tag{2.2}$$

where $\mathbf{v}_{\mathrm{T}}$, which lies in the tangent plane, is the *tangential component* of $\mathbf{v}$.

We introduce the idea of a simple layer. We set

$$\varphi_\gamma(r) = r^{-1} e^{i\gamma r}. \tag{2.3}$$

$\varphi_\gamma(|x-y|)$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, is a fundamental solution of the Helmholtz equation, $\Delta w = -\gamma^2 w$ and when γ is real and positive it satisfies the Sommerfeld radiation condition. We define the *simple layer* V_γ for density ψ for the surface S by

$$V_\gamma(\psi) = c \int_S \psi(y)\, \varphi_\gamma(|x-y|)\, dS_y, \qquad c = (4\pi)^{-1}. \tag{2.4}$$

For a vector field $\mathbf{v}$ on S we define $V_\gamma(\mathbf{v})$ by (2.4) with $\mathbf{v}$ replacing ψ.

We collect in the following lemma some of the well-known results about V_γ.

LEMMA 2.1. *For any complex γ, $0 \leqslant \arg \gamma \leqslant \pi/2$ and any continuous ψ on S:*

(i) $V_\gamma(\psi)$ *is continuous in* R^3,

(ii) $\Delta V_\gamma(\psi) = -\gamma^2 V_\gamma(\overline{\psi})$ *in* $\Omega \cup \Omega'$,

(iii) $V_\gamma(\psi)(x) = O(|x|^{-1} e^{i\gamma|x|})$ *as* $|x| \to \infty$,

$$\left(\frac{\partial V_\gamma(\psi)}{\partial n}(x)\right)^{\pm} = \mp \frac{1}{2} \psi(x) + \int_S K_\gamma(x, y) \psi(y) \, dS_y \qquad \text{on } S,$$

(iv)[1]

$$K_\gamma(x, y) = O(|x - y|^{-1}) \qquad \text{as} \quad y \to x.$$

For vector densities $\mathbf{v}$, $V_\gamma(\mathbf{v})$ satisfies (i)–(iii) of Lemma 4.1. As we show in Section 4, (iv) yields the following additional result.

LEMMA 2.2. *For any complex γ, $0 \leqslant \arg \gamma \leqslant \pi/2$ and any continuous* $\mathbf{v}$ *on* S,

$$(n \times \operatorname{curl} V_\alpha(\mathbf{v})(x))^{\pm} = \pm \tfrac{1}{2} v(x) + \tfrac{1}{2} \int_S \mathbf{K}_\gamma(x, y) \, \mathbf{v}(y) \, dS_y$$

where the matrix function K_γ *satisfies* $K_\gamma(x, y) = O(|x - y|^{-1})$ *as* $y \to x$.

The Stratton–Chu formulas state the following. If $\operatorname{Curl} \mathbf{E} = \mathbf{H}$ and $\operatorname{curl} \mathbf{H} = i\beta^2 \mathbf{E}$ in Ω' then

$$\left.\begin{aligned} \mathbf{E} &= V_{\sqrt{i}\,\beta}(\mathbf{n} \times \mathbf{H}) - \operatorname{curl} V_{\sqrt{i}\,\beta}(\mathbf{n} \times \mathbf{E}) + \operatorname{grad} V_{\sqrt{i}\,\beta}(\mathbf{n} \cdot \mathbf{E}) \\ \mathbf{H} &= \operatorname{curl} V_{\sqrt{i}\,\beta}(\mathbf{n} \times \mathbf{H}) - \operatorname{curl} \operatorname{curl} V_{\sqrt{i}\,\beta}(\mathbf{n} \times \mathbf{E}) \end{aligned}\right\} \quad \text{in } \Omega'. \tag{2.5}$$

Similarly, if $\operatorname{curl} \mathbf{E} = \mathbf{H}$, $\operatorname{curl} \mathbf{H} = \alpha^2 \mathbf{E}$ in Ω and $\mathbf{E}$ and $\mathbf{H}$ represent scattered fields then

$$\left.\begin{aligned} \mathbf{E} &= V_\alpha(\mathbf{n} \times \mathbf{H}) - \operatorname{curl} V_\alpha(\mathbf{n} \times \mathbf{E}) + \operatorname{grad} V_\alpha(\mathbf{n} \cdot \mathbf{E}) \\ \mathbf{H} &= \operatorname{curl} V_\alpha(\mathbf{n} \times \mathbf{H}) - \operatorname{curl} \operatorname{curl} V_\alpha(\mathbf{n} \times \mathbf{E}) \end{aligned}\right\} \quad \text{in } \Omega. \tag{2.6}$$

If $\mathbf{n} \times \mathbf{H}$, $\mathbf{n} \times \mathbf{E}$ and $\mathbf{n} \cdot \mathbf{E}$ were all known then (2.6) would yield a solution of $(\mathrm{P}_{\alpha\infty})$ but this is too much information; we know only $\mathbf{n} \times \mathbf{E}$. The standard treatment of $(\mathrm{P}_{\alpha\infty})$ starts from (2.6) but sets $\mathbf{n} \times \mathbf{H}$ and $\mathbf{n} \cdot \mathbf{E}$ equal to zero and replaces $-\mathbf{n} \times \mathbf{E}$ by an unknown tangential field $\mathbf{L}$:

$$\mathbf{E} = \operatorname{curl} V_\alpha(\mathbf{L}), \qquad \mathbf{H} = \operatorname{curl} \operatorname{curl} V_\alpha(\mathbf{L}). \tag{2.7}$$

[1] Here again the plus and minus denote limits from Ω and Ω'.

Imposition of the boundary condition then yields an integral equation of the second kind for $\mathbf{L}$ in the tangent space to S.

The method (2.7) is analogous to solving the Dirichlet problem for the scalar Helmholtz equation with a double layer. It has the drawback, for our purposes, that having found $\mathbf{L}$ it is hard to determine $\mathbf{H}_{\mathrm{T}}$ (or equivalently $\mathbf{n} \times \mathbf{H}$) on S. It is not too difficult to see that calculating $\mathbf{n} \times \mathbf{H}$ on S involves finding a second normal derivative of $V_\alpha(\mathbf{L})$.

Our method for $(\mathrm{P}_{\alpha\infty})$ is analogous to solving the scalar problems with a simple layer (see [5]). (In [4] our method reduces to exactly that of [5].) We again use (2.6) but this time we set $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ and replace $\mathbf{n} \times \mathbf{H}$ and $\mathbf{n} \cdot \mathbf{E}$ by unknowns $\mathbf{J}$ and M. Thus we take

$$\mathbf{E} = V_\alpha(\mathbf{J}) + \operatorname{grad} V_\alpha(M), \qquad \mathbf{H} = \operatorname{curl} V_\alpha(\mathbf{J}). \tag{2.8}$$

If we can determine $\mathbf{J}$ then in this case we can use Lemma 2.2 to determine $\mathbf{n} \times \mathbf{H}$, hence $\mathbf{H}_T$ on S.

We need equations to determine $\mathbf{J}$ and M. The first of these comes from the boundary condition. For any field ψ defined in a region containing S we will have $\operatorname{grad} \psi = (\operatorname{grad} \psi)_T + (\operatorname{grad} \psi)_N \mathbf{n}$ and there is a surface differential operator $\operatorname{grad}_{\mathrm{T}}$ such that $\operatorname{grad}_{\mathrm{T}} \psi = (\operatorname{grad} \psi)_{\mathrm{T}}$ on S. We set

$$A_\alpha^{(1)}(\mathbf{J}, M) \equiv A^{11}(\mathbf{J}) + A^{12}(M) = V_\alpha(\mathbf{J})_{\mathrm{T}} + \operatorname{grad}_{\mathrm{T}} V_\alpha(M) \tag{2.9}$$

and then the boundary condition in $(\mathrm{P}_{\alpha\infty})$ becomes

$$A_\alpha^{(1)}(\mathbf{J}, M) = -\mathbf{E}_{\mathrm{T}}^0. \tag{2.10}$$

Another equation is required. To see what it is we note that the equations for $(\mathrm{P}_{\alpha\infty})$ require that $\operatorname{div} \mathbf{E} \equiv \operatorname{div} \mathbf{H} \equiv 0$ in Ω. For (2.7) this is automatic but for (2.8) only $\operatorname{div} \mathbf{H}$ is automatically zero; hence we must somehow guarantee that $\operatorname{div} \mathbf{E} = 0$. We assert that it suffices to make $\operatorname{div} \mathbf{E} \equiv 0$ on S. For it follows from (2.8) and (2.3) that $\Delta \mathbf{E} = -\alpha^2 \mathbf{E}$; hence $\Delta \operatorname{div} \mathbf{E} = -\alpha^2 \operatorname{div} \mathbf{E}$ in Ω. Moreover $\operatorname{div} \mathbf{E}$ satisfies the radiation condition. Hence, by uniqueness for the scalar exterior Dirichlet problem $\operatorname{div} \mathbf{E} \equiv 0$ on S implies $\operatorname{div} \mathbf{E} = 0$ in Ω.

Our other condition, then, is $\operatorname{div} \mathbf{E} \equiv 0$ on S. We can state this condition in a little more useful fashion. Note first that $\operatorname{div} \operatorname{grad} V_\alpha(M) = \Delta V_\alpha(M) = -\alpha^2 V_\alpha(M)$. We can also simplify $\operatorname{div} V_\alpha(\mathbf{J})$ on S. First we note that for any field $\mathbf{v}$ defined in a neighbourhood of S we can define the surface divergence $\operatorname{div}_{\mathrm{T}}$ by $\operatorname{div} \mathbf{v} = \operatorname{div}_{\mathrm{T}} \mathbf{v} + \mathbf{n} \partial v / \partial n$. (This operator, like $\operatorname{grad}_{\mathrm{T}}$, is discussed in Section 4.) We have:

LEMMA 2.3. *For any differentiable tangential field* $\mathbf{v}$,

$$\operatorname{div} V_\gamma(\mathbf{v}) = V_\gamma(\operatorname{div}_{\mathrm{T}} \mathbf{v}) \qquad \textit{on } S.$$

Proof.

$$\operatorname{div} V_\gamma(\mathbf{v})(x) = c \int_S \mathbf{v}(y) \cdot \operatorname{grad}_x \varphi_\gamma(|x-y|)\, ds_y$$

$$= -c \int_S \mathbf{v}(y) \cdot \operatorname{grad}_y \varphi_\gamma(|\mathbf{x}-\mathbf{y}|)\, ds_y$$

$$= -c \int_S \operatorname{div}_{\mathrm{T}}(\mathbf{v}(y)\, \varphi_\gamma(|x-y|))\, ds_y$$

$$+ c \int_S \operatorname{div}_{\mathrm{T}} \mathbf{v}(y)\, \varphi_\gamma(|x-y|)\, ds_y$$

$$= V_\gamma(\operatorname{div}_{\mathrm{T}} v)(x)$$

since $\int_S \operatorname{div}_{\mathrm{T}} \mathbf{w}(y)\, ds_y = 0$ for any $\mathbf{w}$.

If we define

$$A_\alpha^{(2)}(\mathbf{J}, M) \equiv A^{21}(\mathbf{J}) + A^{22}(M) = -V_\alpha(\operatorname{div}_{\mathrm{T}} \mathbf{J}) + \alpha^2 V_\alpha(M) \tag{2.11}$$

then the condition $\operatorname{div} \mathbf{E} \equiv 0$ on S becomes

$$A_\alpha^{(2)}(\mathbf{J}, M) = 0. \tag{2.12}$$

We set $A_\alpha(\mathbf{J}, M) = (A^{(1)}(\mathbf{J}, M), A^{(2)}(\mathbf{J}, M))$ and combine (2.10) and (2.12) into

$$A_\alpha(\mathbf{J}, M) = (-\mathbf{E}_{\mathrm{T}}^0, 0). \tag{$\mathrm{E}_{\alpha\infty}$}$$

Our procedure for $(\mathrm{P}_{\alpha\beta})$ proceeds as follows. We let $\mathbf{E}$ and $\mathbf{H}$ this time denote the total fields. We again use (2.8) in Ω and we use its analog in Ω'. Thus we put

$$\begin{aligned} &\mathbf{E} = \mathbf{E}^0 + V_\alpha(\mathbf{J}) + \operatorname{grad} V_\alpha(M), \qquad \mathbf{H} = \mathbf{H}^0 + \operatorname{curl} V_\alpha(\mathbf{J}) \qquad \text{in } \Omega \\ &\mathbf{E} = V_{\sqrt{i}\,\beta}(\mathbf{j}) + \operatorname{grad} V_{\sqrt{i}\,\beta}(m), \qquad \mathbf{H} = \operatorname{curl} V_{\sqrt{i}\,\beta}(\mathbf{j}) \qquad \text{in } \Omega'. \end{aligned} \tag{2.13}$$

We must insure that $\operatorname{div} \mathbf{E} = 0$ in Ω and Ω'; hence we obtain $A_\alpha^2(\mathbf{J}, M) = A^2_{\sqrt{i}\,\beta}(\mathbf{j}, m) = 0$. We must have $\mathbf{E}_{\mathrm{T}}^+ = E_{\mathrm{T}}^-$ on S; hence $A_\alpha^{(1)}(\mathbf{J}, M) = A^{(1)}_{\sqrt{i}\,\beta}(\mathbf{j}, m) - \mathbf{E}_{\mathrm{T}}^0$ on S. We must also have $\mathbf{H}_{\mathrm{T}}^+$ on S and for this we use Lemma 2.2. Define

$$K_\gamma(\mathbf{v})(x) = \int_S \mathbf{K}_\gamma(x, y)\, \mathbf{v}(y)\, ds_y. \tag{2.14}$$

Then by Lemma 2.2 we will have $(\mathbf{n} \times \mathbf{H})^+ = (\mathbf{n} \times \mathbf{H})^-$; hence $\mathrm{H}_\mathrm{T}^+ = \mathrm{H}_\mathrm{T}^-$ on S if

$$\mathbf{J} + K_\alpha(\mathbf{J}) = -\mathbf{j} + K_{\sqrt{i}\,\beta}(\mathbf{j}) - 2\mathbf{n} \times \mathbf{H}^0 \qquad \text{on } S. \tag{2.15}$$

Let us again combine the equations for $(\mathrm{P}_{\alpha\beta})$ by writing

$$\begin{aligned} C_{\alpha\beta}(\mathbf{J}, M, \mathbf{j}, m) = (&A_\alpha^{(1)}(\mathbf{J}, M) - A_{\sqrt{i}\,\beta}^{(1)}(\mathbf{j}, m), A_\alpha^{(2)}(J, M), \\ &\mathbf{J} + K_\alpha(\mathbf{J}) + \mathbf{j} - K_{\sqrt{i}\,\beta}(\mathbf{j}), A_{\sqrt{i}\,\beta}^{(2)}(\mathbf{j}, m)). \end{aligned} \tag{2.16}$$

Then the equations are

$$C_{\alpha\beta}(\mathbf{J}, M, \mathbf{j}, m) = (-\mathbf{E}_\mathrm{T}^0, 0, -2\mathbf{n} \times \mathbf{H}^0, 0). \tag{$\mathrm{E}_{\alpha\beta}$}$$

It is not difficult to verify that if we have $\operatorname{div} \mathbf{E} = 0$ in (2.8) then (2.8) gives a solution of Maxwell's equations with a similar result for (2.13). Thus we have the following result:

THEOREM 2.2. (i) *If* $(\mathbf{J}, M)$ *is a solution of* $(\mathrm{E}_{\alpha\infty})$ *with* $\mathbf{J}$ *differentiable and* M *continuous then* (2.8) *yields a solution of* $(P_{\alpha\infty})$.

(ii) *If* $(\mathbf{J}, M, \mathbf{j}, m)$ *is a solution of* $(\mathrm{E}_{\alpha\beta})$ *with* $\mathbf{J}, \mathbf{j}$ *differentiable and* M, m *continuous then* (2.13) *yields a solution of* $(\mathrm{P}_{\alpha\beta})$.

Remarks. 1. The method for $(\mathrm{P}_{\alpha\beta})$ is capable of handling the case of dielectric obstacles as well. Here one simply replaces $i\beta^2$ in Ω' by β^2 and the method proceeds in exactly the same way if $\sqrt{i}\,\beta$ is replaced by β in all the formulas.

2. The method for $(\mathrm{P}_{\alpha\infty})$ can clearly be used for the interior boundary problem.

3. The method can be modified to handle the boundary problem, either interior or exterior, in which $\mathbf{H}_\mathrm{T}$ is specified on S. If one seeks a solution of $\operatorname{curl} \mathbf{E} = \mathbf{H}$, $\operatorname{curl} \mathbf{H} = i\beta^2 \mathbf{E}$ in Ω', for instance, with $\mathbf{n} \times \mathbf{H} = \mathscr{H}$ on S then one can take

$$\mathbf{E} = V_{\sqrt{i}\,\beta}(\mathbf{j}) + \operatorname{grad} V_{\sqrt{i}\,\beta}(m), \qquad \mathbf{H} = \operatorname{curl} V_{\sqrt{i}\,\beta}(\mathbf{j})$$

with $-\mathbf{j} + K_{\sqrt{i}\,\beta}\mathbf{j} = 2\mathscr{H}$, $A_{\sqrt{i}\,\beta}^{(2)}(\mathbf{j}, m) = 0$ on S.

In later sections we will establish that Eqs. $(\mathrm{E}_{\alpha\infty})$ and $(\mathrm{E}_{\alpha\beta})$ have solutions which will be as smooth as desired if $\mathbf{E}^0$ and $\mathbf{H}^0$ are sufficiently smooth. In [7] we give a variational formulation for Eqs. $(\mathrm{E}_{\alpha\infty})$ and discuss approximate solutions with finite elements. This analysis is related to that in [2] and [3].

3. The Half-Space Case

In this section we give a formal treatment of the case in which Ω is the half-space $x_3 > 0$, $x = (x_1, x_2, x_3)$. All our formulas are greatly simplified. We have $\mathbf{n}(x) = \mathbf{e}_3$ and the first simplification is that K_γ in Lemma 2.1(v) is identically zero. It is also true that the matrix $\mathbf{K}_\gamma$ in Lemma 2.2 is zero as we will verify shortly.

We have here $M(y) = M(y_1, y_2)$ and

$$V_\gamma(M)(x) = \varphi_\gamma * M \qquad \text{on } S, \tag{3.1}$$

where the star denotes convolution. Similarly

$$\mathbf{J}(y) = J^1(y_1, y_2)\,\mathbf{e}_1 + J^2(y_1, y_2)\mathbf{e}_2$$

and

$$V_\gamma(\mathbf{J}) \equiv V_\gamma(\mathbf{J})_{\mathrm{T}} = \varphi_\gamma * J^1\,\mathbf{e}_1 + \varphi_\gamma * J^2\mathbf{e}_2 \qquad \text{on } S. \tag{3.2}$$

We also have

$$\operatorname{grad}_{\mathrm{T}} V_\gamma(M) = \frac{\partial}{\partial x_1}(\varphi_\gamma * M)\,\mathbf{e}_1 + \frac{\partial}{\partial x_2}(\varphi_\gamma * M)\,\mathbf{e}_2 \tag{3.3}$$

$$\operatorname{div}\mathbf{J}(y) = J^1_{y_1}(y_1, y_2) + J^2_{y_2}(y_1, y_2), \qquad V_\gamma(\operatorname{div}\mathbf{J}) = \varphi_\gamma * \operatorname{div}\mathbf{J}.$$

Finally, we have, for $\mathbf{v} = v^1(y_1, y_2)\,\mathbf{e}_1 + v^2(y_1, y_2)\mathbf{e}_2$,

$$\begin{aligned}\mathbf{n} \times \operatorname{curl} V_\gamma(\mathbf{v}) &= \mathbf{e}_3 \times \Big\{-\frac{\partial}{\partial x_3} V_\gamma(v^2)\,\mathbf{e}_1 + \frac{\partial}{\partial x_3} V_\gamma(v^1)\,\mathbf{e}_2 \\ &\quad + \left(\frac{\partial}{\partial x_1} V_\gamma(v^2) - \frac{\partial}{\partial x_2} V_\gamma(v^1)\right) e_3\Big\} \\ &= -\frac{\partial}{\partial x_3} V_\gamma(v^1)\,\mathbf{e}_1 - \frac{\partial}{\partial x_3} V_\gamma(v^2)\,\mathbf{e}_2.\end{aligned}$$

Thus, from Lemma 2.1(iv) with $K_\gamma = 0$ we conclude

$$(\mathbf{n} \times \operatorname{curl} V_\gamma(\mathbf{v}))^{\pm} = \pm\tfrac{1}{2}\mathbf{v}. \tag{3.4}$$

With the simplifications Eqs. $(\mathrm{E}_{\alpha\infty})$ become

$$\begin{aligned}\varphi_\alpha * \mathbf{J} + \frac{\partial}{\partial x_1}\varphi_\alpha * M e_1 + \frac{\partial}{\partial x_2}\varphi_\alpha * M e_2 &= 4\pi\mathscr{E} \\ -\varphi_\alpha * \operatorname{div}\mathbf{J} + \alpha^2\varphi_\alpha * M &= 0,\end{aligned} \tag{3.5}$$

where $\mathscr{E}(x_1, x_2) = -E_T^0(x_1, x_2, 0) = -E_1^0(x_1, x_2, 0)\,\mathbf{e}_1 - E_2^0(x_1, x_2, 0)\,\mathbf{e}_2$. These equations are easily solved with Fourier transforms. For any $f(\mathbf{v})$ which are functions of x_1, x_2 we write

$$f^{\wedge}(\xi)(\mathbf{v}^{\wedge}(\xi)), \qquad \xi = (\xi_1, \xi_2)$$

for the transforms. We have $(f * g)^{\wedge} = 2\pi f^{\wedge} g^{\wedge}$. We transform equations (3.5) and obtain

$$\varphi_\alpha^{\wedge}(\mathbf{J}^{\wedge} + i\xi M^{\wedge}) = 2\mathscr{E}^{\wedge}, \qquad \varphi_\alpha^{\wedge}(-i\xi \cdot J^{\wedge} + \alpha^2 M^{\wedge}) = 0. \tag{3.5'}$$

From these we find

$$\begin{aligned} M^{\wedge} &= (\varphi_\alpha^{\wedge}(\alpha^2 - |\xi|^2))^{-1}\, 2i\xi \cdot \mathscr{E}^{\wedge} \\ J^{\wedge} &= -i\xi M^{\wedge} + 2(\varphi_\alpha^{\wedge})^{-1}\, \mathscr{E}^{\wedge}. \end{aligned} \tag{3.6}$$

We can identify the solution from (3.6) by using the following result.

LEMMA 3.1. $\varphi_\gamma^{\wedge} = (|\xi|^2 - \gamma^2)^{-1/2}$.

Proof. Suppose $V_\gamma(f) = g$ on $x_3 = 0$. Then we have $f^{\wedge}\varphi_\gamma^{\wedge} = 2g^{\wedge}$. On the other hand consider the function $v(x) = V_\gamma(f)(x)$ for $x_3 > 0$. We have $\Delta v = -\gamma^2 v$ with $v(x_1, x_2, 0) = g(x_1, x_2)$. We can solve this problem another way. We assert that

$$v(x) = -2\frac{\partial}{\partial x_3} V_\gamma(g)(x).$$

This follows immediately from Lemma 2.1 specialized to the present case and uniqueness. We have, then,

$$-2\frac{\partial}{\partial x_3} V_\gamma(f) = 4\frac{\partial^2}{\partial x_3^2} V_\gamma(g) = -4(\gamma^2 + \Delta_2)\, V_\gamma(g) \qquad \text{in } x_3 > 0$$

where $\Delta_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. If we let $x_3 \downarrow 0$ and use Lemma 2.1 on the left side we obtain

$$f = -\pi^{-1}(\Delta_2 + \gamma^2)\, \gamma_\gamma * g$$

or taking transforms, $f^{\wedge} = 2(|\xi|^2 - \gamma^2)\, \varphi_\gamma^{\wedge}\, g^{\wedge}$. Comparing the two expressions for $f^{\wedge}$ yields the conclusion.

From Lemma 3.1 we have

$$(\varphi_\alpha^{\wedge}(\alpha^2 - |\xi|^2))^{-1} = -(|\xi|^2 - \alpha^2)^{-1/2} = -\varphi_\alpha^{\wedge}.$$

Thus from (3.6), we deduce that

$$M = -\pi^{-1}\varphi_\alpha * \operatorname{div} \mathscr{E} = -4V_\alpha(\operatorname{div} \mathscr{E}). \tag{3.7}$$

We have also

$$\begin{aligned}(\hat{\varphi_\alpha})^{-1} &= (|\xi|^2 - \alpha^2)^{1/2} \\ &= (|\xi|^2 - \alpha^2)(|\xi|^2 - \alpha^2)^{-1/2} \\ &= (|\xi|^2 - \alpha^2)\,\hat{\varphi_\alpha}.\end{aligned}$$

Hence $(3.6)_2$ and (3.7) yield

$$\begin{aligned}\mathbf{J} &= \operatorname{grad}_T(4V_\alpha(\operatorname{div} \mathscr{E})) - 2(\Delta_2 + \alpha^2)(\pi)^{-1}\,\varphi_\alpha * \mathscr{E} \\ &= 4 \operatorname{grad}_T V_\alpha(\operatorname{div} \mathscr{E}) - 2(\Delta_2 + \alpha^2)\,V_\alpha(\mathscr{E}).\end{aligned} \tag{3.8}$$

The results (3.7) and (3.8) are formal but they suggest the appropriate theorems for $(E_{\alpha\infty})$. Let $H^r(R^2)$ denote the Sobolev space of order r for R^2, that is, the completion of $C_0^\infty(R^2)$ under the norm

$$|u|_r^2 = \int_{R^2} (1 + |\xi|^2)^r\, |\hat{u}(\xi)|^2\, d\xi, \tag{3.8$'$}$$

let $\mathbf{H}^r(R^2)$ be the space of a vector function with components in $H^r(R^2)$, and let $H^{r,s} = \mathbf{H}^r(R^2) \times H^s(R^2)$. Lemma 3.1 implies that the map $\psi \to V_\gamma(\psi)$ is a pseudo-differential operator, with symbol $(|\xi|^2 - \gamma^2)^{-1/2}$, and hence of order minus one in the sense of [7]. It follows that $\psi \to V_\gamma(\psi)(\mathbf{v} \to V_\gamma(\mathbf{v}))$ maps $H^r(R^2)(\mathbf{H}^r(R^2))$ into $H^{r+1}(R^2)(\mathbf{H}^{r+1}(R^2))$. Moreover div_T and grad_T are operators of order one and hence take $\mathbf{H}^r(R^2)$ and $H^r(R^2)$ into $H^{r-1}(R^2)$ and $\mathbf{H}^{r-1}(R^2)$, respectively.

The results of the preceding paragraph show that, in the half-space case, $A_\alpha^{(1)}$, defined by (2.9), takes $H^{r-1,r}$ into $\mathbf{H}^r(R^2)$ while $A_\alpha^{(2)}$, defined by (2.11), takes $H^{r-1,r}$ into $H^{r-1}(R^2)$. Hence A_α maps $H^{r-1,r}$ into $H^{r,r-1}$. Equations (3.7) and (3.8) give a formula for the inverse of A_α. These formulas suggest the correct result for Eqs. $(E_{\alpha\infty})$ in the general case. Let $H^r(S)$, $\mathbf{H}^r(S)$ and $H^{r,s}$ be as above but for the surface S. See [10] for the appropriate definitions. Then we will establish the following result.

THEOREM 3.1.[2] *Suppose* $\mathbf{F} \in \mathbf{H}^r(S)$ *for some real r. Then the equations*

$$A_\alpha(\mathbf{J}, M) = (\mathbf{F}, 0) \tag{3.9}$$

have a unique solution $(\mathbf{J}, M) \in H^{r-1,r}$.

[2] Recall we are assuming $\alpha \neq \alpha_k$.

This result in conjunction with Theorem 2.2 yields an existence theorem for $(P_{\alpha\infty})$.

COROLLARY 3.1. *If* $\mathbf{E}^0 \in H^k(\mathbb{R}^3)$ *for* $k > \frac{5}{2}$ *then there exists a solution of* $(P_{\alpha\infty})$.

Proof. If $\mathbf{E}^0 \in H^k(\mathbb{R}^3)$, $k \geqslant 1$, then the trace theorem implies that $\mathbf{E}^0_T \in H^{k-1/2}(S)$. By Theorem 3.1, $(E_{\alpha\infty})$ has a solution $(\mathbf{J}, M) \in H^{k-3/2, k-1/2}$. But for $k - \frac{3}{2} > 2$, $\mathbf{J}$ and M will then be differentiable hence we have a solution of $(P_{\alpha\infty})$ by Theorem 2.2.

We turn now to $(P_{\alpha\beta})$. For the half-space case Eqs. $(E_{\alpha\beta})$ become

$$\begin{gathered}
\varphi_\alpha * \mathbf{J} - \varphi_{\sqrt{i}\,\beta} * \mathbf{j} + \frac{\partial}{\partial x_1}(\varphi_\alpha * M - \varphi_{\sqrt{i}\,\beta} * m)\,\mathbf{e}_1 \\
+ \frac{\partial}{\partial x_2}(\varphi_\alpha * M - \varphi_{\sqrt{i}\,\beta} * m)\,\mathbf{e}_2 = 4\pi\mathscr{E} \\
-\varphi_\alpha * \operatorname{div}\mathbf{J} + \alpha^2\varphi_\alpha * M = 0, \qquad -\varphi_{\sqrt{i}\,\beta} * \operatorname{div}\mathbf{j} + \beta^2\varphi_{\sqrt{i}\,\beta} * m = 0 \\
\mathbf{J} + \mathbf{j} = \mathscr{H},
\end{gathered} \tag{3.10}$$

where $\mathscr{H}(x^1, x^2) = -2\mathbf{e}_3 \times \mathbf{H}^0(x_1, x_2, 0) = 2H^0_2(x_1, x_2, 0)\,\mathbf{e}_1 - 2H^0_1(x_1, x_2, 0)\mathbf{e}_2$. If we transform we obtain

$$\begin{gathered}
\hat{\varphi}_\alpha(\mathbf{J}^\wedge + i\xi M^\wedge) - \hat{\varphi}_{\sqrt{i}\,\beta}(\mathbf{j}^\wedge + i\xi m^\wedge) = 2\mathscr{E}^\wedge \\
-i\xi \cdot \mathbf{J}^\wedge + \alpha^2 M^\wedge = -i\xi \cdot \mathbf{j}^\wedge + i\beta^2 m^\wedge = 0, \qquad \mathbf{J}^\wedge + \mathbf{j}^\wedge = \mathscr{H}^\wedge.
\end{gathered} \tag{3.11}$$

We solve (3.11). We observe first that if we take the inner product of $(3.11)_1$ with $i\xi$, use $(3.11)_{2,3}$, and substitute from Lemma (3.1) we obtain $(\hat{\varphi}_\alpha)^{-1} M^\wedge - (\hat{\varphi}_{\sqrt{i}\,\beta})^{-1} m^\wedge = -2i\xi \cdot \mathscr{E}^\wedge$. Next we take the product of $(3.11)_4$ with $i\xi$ and use $(3.11)_{2,3}$ to obtain $\alpha^2 M^\wedge + \beta^2 m^\wedge = i\xi \cdot \mathscr{H}^\wedge$. We can solve these two equations for $M^\wedge$ and $m^\wedge$. Put

$$D^\wedge = (\alpha^2(\hat{\varphi}_{\sqrt{i}\,\beta})^{-1} + \beta^2(\hat{\varphi}_\alpha)^{-1})^{-1}. \tag{3.12}$$

Then

$$\begin{gathered}
M^\wedge = 2\beta^2 D^\wedge i\xi \cdot \mathscr{E}^\wedge + (\hat{\varphi}_{\sqrt{i}\,\beta})^{-1} D^\wedge i\xi \cdot \mathscr{H}^\wedge \\
m^\wedge = 2\alpha^2 D^\wedge i\xi \cdot \mathscr{E}^\wedge + (\hat{\varphi}_\alpha)^{-1} D^\wedge i\xi \cdot \mathscr{H}^\wedge.
\end{gathered} \tag{3.13}$$

Once $M^\wedge$ and $m^\wedge$ are determined one can solve (3.11) for $\mathbf{J}^\wedge$ and $\mathbf{j}^\wedge$. We will have

$$\begin{gathered}
J^\wedge = (\hat{\varphi}_\alpha + \hat{\varphi}_{\sqrt{i}\,\beta})^{-1}\{2\mathscr{E}^\wedge + \hat{\varphi}_{\sqrt{i}\,\beta}\mathscr{H}^\wedge - \hat{\varphi}_\alpha i\xi M^\wedge + \hat{\varphi}_{\sqrt{i}\,\beta} i\xi m^\wedge\} \\
\mathbf{j}^\wedge = -J^\wedge + \mathscr{H}^\wedge.
\end{gathered} \tag{3.14}$$

We can no longer determine the solution of $(E_{\alpha\beta})$ explicitly but we can still use (3.13) and (3.14) to determine the regularity. We observe that Lemma 3.1 implies that the following estimates hold for large $|\xi|$:

$$\hat{\varphi_\alpha}(\xi) = O(|\xi|^{-1}), \qquad \hat{\varphi_{\sqrt{i}\,\beta}}(\xi) = O(|\xi|^{-1}), \qquad \hat{D}(\xi) = O(|\xi|^{-1}). \quad (3.15)$$

We will make the same regularity assumption on $\mathscr{E}$ as in $(P_{\alpha\infty})$, that is, $\mathscr{E} \in \mathbf{H}^r(R^2)$. Now recall that $\mathscr{E} = -\mathbf{E}_T$ while $\mathscr{H} = 2H_2^0\mathbf{e}_1 - 2H_1^0\mathbf{e}_2$. Since we have $H^0 = \operatorname{curl} \mathbf{E}^0$ we can anticipate then that $\mathscr{H} \in \mathbf{H}^{r-1}(R^2)$.

Our goal is to have the same regularity that we had for $(P_{\alpha\infty})$, that is, $\mathbf{J}, j \in \mathbf{H}^{r-1}(R^2)$, $M, m \in H^r(R^2)$. The formulas (3.13) and (3.14) show, however, that we will not get this for arbitrary $\mathscr{E} \in \mathbf{H}^r(R^2)$ and $\mathscr{H} \in \mathbf{H}^{r-1}(R^2)$. For we will then have $\operatorname{div}_T \mathscr{E} \in H^{r-1}(R^2)$ and $\operatorname{div}_T \mathscr{H} \in H^{r-2}(R^2)$. Then by (3.15), the first terms on the right in (3.13) will be in $H^r(R^2)$ but the second terms will be in $H^{r-2}(R^2)$. In order to get the regularity we want we require of $\mathscr{H}$ that it lie in $\mathbf{H}^r$. This also turns out to be so in the general case. We will prove the following.

THEOREM 3.2. *Suppose* $\mathscr{E} \in \mathbf{H}^r(s)$ *and* $\mathscr{H} \in \mathbf{H}^{r+1}(s)$ *for some real* r. *Then the equations*

$$C_{\alpha\beta}(\mathbf{J}, M, \mathbf{j}, m) = (\mathscr{E}, 0, \mathscr{H}, 0)$$

have a unique solution with $(\mathbf{J}, M)$ *and* $(\mathbf{j}, m)$ *in* $H^{r-1,r}$.

COROLLARY 3.2. *If* $\mathbf{E}^0 \in H^k(\mathbb{R}^3)$ *for* $k > \frac{11}{2}$ *then there exists a solution of* $(P_{\alpha\beta})$.

Proof. If $\mathbf{E}^0 \in H^k(\mathbb{R}^3)$ then it follows that $\mathbf{H} = \operatorname{curl} \mathbf{E}$ is in $H^{k-1}(\mathbb{R}^3)$. We have, accordingly,

$$\mathbf{E}_T \in H^{k-1/2}(S), \qquad n \times \mathbf{H} \in H^{k-3/2}(S).$$

These are the forcing terms in Eqs. $(E_{\alpha\beta})$ and by Theorem 3.2 we conclude that those equations have a solution with $\mathbf{J}, \mathbf{j} \in H^{k-7/2}(S)$ and $M, m \in H^{k-5/2}(S)$. But then if $k - \frac{5}{2} > 2$ it follows that $\mathbf{J}, \mathbf{j}, M, m$ are all differentiable and we can use Theorem 2.2 again to obtain the existence of a solution of $(P_{\alpha\beta})$.

4. PRELIMINARY RESULTS

In this section we develop the necessary machinery to prove the theorems of the preceding sections. As a first step we discuss some geometric ideas. We introduce coordinate systems for S. These consist of a finite number of coordinate patches $S_1, \ldots, S_N$ covering S. For each patch there is a region $\Gamma_k \subset R^2$ and a map X_k such that $x = \mathbf{X}_k(u)$, $u = (u_1, u_2) \in R^2$, covers S_k. The mappings are compatible on overlapping regions. To say that S is a regular analytic surface means that the individual maps from Γ_k to Γ_e on overlaps are analytic and that $\mathbf{X}_{k,u_1}$ and $\mathbf{X}_{k,u_2}$ are linearly independent.

We use the $\mathbf{X}_k$ to generate local coordinate systems in R^3. We set

$$\mathbf{e}_1(u) = X_{u_1}, \qquad \mathbf{e}_2(u) = X_{u_2}, \qquad \mathbf{e}_3(u) = \mathbf{e}_1(u) \times \mathbf{e}_2(u) \tag{4.1}$$

where we have suppressed the subscript k. Then the equations

$$x = \mathbf{X}(u) + u_3 \mathbf{e}_3(u), \qquad u \in \Gamma, \quad |u_3| < \delta \tag{4.2}$$

will define a coordinate system for a region $U_k \subset R^3$ with $u_3 = 0$ corresponding to S_k. We will assume that $u_3 > 0$ corresponds to Ω.

One must use these coordinate systems to define the various quantities in Section 2. It simplifies our formulas and calculations if the coordinate systems are orthonormal, that is,

$$\mathbf{e}_i(u) \cdot \mathbf{e}_j(u) = \delta_{ij} \tag{4.3}$$

and we will assume that this is so. (Such choices can always be made.)

Given any vector field $\mathbf{v}$ in one of the U_k it can be represented as

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 \equiv \mathbf{v}_{\mathrm{T}} + v_3 \mathbf{e}_3. \tag{4.4}$$

One readily checks that $\mathbf{v}_{\mathrm{T}} = \mathbf{e}_3 \times (\mathbf{v} \times \mathbf{e}_3)$. Since the coordinate system is orthonormal if we are given a scalar field χ or a vector field $\mathbf{v}$ in u^k we can calculate $\operatorname{grad} \chi$, $\operatorname{div} \mathbf{v}$, and $\operatorname{curl} \mathbf{v}$ by the usual formulas and we can define the surface operators $\operatorname{grad}_{\mathrm{T}}$ and $\operatorname{div}_{\mathrm{T}}$ by

$$\operatorname{grad}_{\mathrm{T}} \chi = (\operatorname{grad} \chi)_{\mathrm{T}} = \chi_{u_1} \mathbf{e}_1 + \chi_{u_2} \mathbf{e}_2, \qquad \operatorname{div}_{\mathrm{T}} \mathbf{v} = v_{1,u_1} + v_{2,u_2}. \tag{4.5}$$

We turn now to the integral operators V_γ. Here we use the ideas of [10]. We introduce a partition of unity $\sum \xi_k \equiv 1$ subordinate to the S_k. Then we define $V_\gamma(\psi)$ for a scalar field ψ on S by

$$V_\gamma(\psi)(x) = \sum_k c \int_{\Gamma_k} \psi(\mathbf{X}_k(u))\, \xi_k(u)\, \phi_\gamma(|x - X_k(u)|)\, du. \tag{4.6}$$ [3]

[3] The orthonormality of the coordinate system implies that the surface element is unity.

For $x \in S$, (4.6) gives

$$V_\gamma(\psi) = \sum_j \sum_k c\xi_j \int_{\Gamma_k} \psi(X_k(u))\, \xi_k(X_k(u))\, \phi_\gamma(|X_j - X_k(u)|)\, du. \tag{4.7}$$

Analogous expressions hold for $V_\gamma(\mathbf{v})$ when $\mathbf{v}$ is a vector field on S.

Formula (4.7) is the basis for the idea of pseudo-differential operators on S. If $\psi \in C_0^\infty(S_k)$ for some patch S_k then $V_\gamma(\psi)$ will be in $C^\infty(S_k)$. The idea is to extend that definition to ψ's which need not be C^∞ but lie in some Sobolev space on S. It is clear from (4.7) that one need concentrate only on the quantities $\chi V_\gamma(\psi)$ where χ and ψ have support in the same patch S_k.

Let $\chi, \psi \in C_0^\infty(S_k)$. Then we have

$$\chi V_\gamma(\psi) = \chi(X(U))\, c \int_{\Gamma_k} \psi(X(u))\, \phi_\gamma(|X(U) - X(u)|)\, du$$

$$= \int_{R^2} \tilde{\psi}(u)\, K_\gamma(U, u - U)\, du, \tag{4.8}$$

where

$$K_\gamma(U, u - U) = c\chi(X(U))\, \phi_\gamma(|X(U) - X(u)|). \tag{4.9}$$

In the terminology of [10], K_γ is called the *kernel* of the pseudo-differential operator V_γ. Let us introduce the Fourier transform $\psi\hat{}$ of $\tilde{\psi}$,

$$\psi\hat{}\,(\xi) = (2\pi)^{-1/2} \int_{R^2} \tilde{\psi}(u)\, e^{i\xi \cdot u}\, du.$$

Then (4.8) may be rewritten in the form

$$\chi V_\gamma(\psi) = \int_{R^2} \psi\hat{}\,(\xi)\, a_\gamma(U, \xi)\, e^{i\xi \cdot x}\, d\xi \tag{4.10}$$

where

$$a_\gamma(U, \xi) = (2\pi)^{-1/2}\, \chi(X(U)) \int_{R^2} e^{-i\xi \cdot \eta} K(U, \eta)\, d\eta. \tag{4.11}$$

$a_\gamma(U, \xi)$ is called the *symbol* of V_γ. Note that

$$a_\gamma(U, \xi) = \chi(X(U))\, K_\gamma(U, \cdot)\hat{}\,(\xi).$$

Suppose that $K_\gamma(U, \eta)$ has an asymptotic expansion of the form

$$K_\gamma(U, \eta) \sim \sum_{n=r}^{\infty} K_\gamma^n(U, \eta) \tag{4.12}$$

where K^n_γ is homogeneous of degree n in η. It will follow then that the (distributional) Fourier transform of K_γ, that is, a_γ, has an expansion of the form

$$a_\gamma(U,\xi) \sim \sum_{n=r} a^n_\gamma(U,\xi) \tag{4.13}$$

where a^n_γ is homogeneous of degree $-n-2$ in ξ. In the terminology of [10] one says that if (4.12) holds then V_γ is an operator of *order* r and that $a^r_\gamma(U,\xi)$ is its *top order* symbol. The operator V_γ is called *elliptic* if $a^r_\gamma(U,\xi) \neq 0$ for $\xi \neq 0$.

We now summarize some results from [10] on pseudo-differential operators on S. Let $H^t(S)$ denote the Sobolev space of order t on S. (These are defined by introducing a partition of unity on S and using (3.8′) locally.)

LEMMA 4.1. *Suppose A is a pseudo-differential operator of order r on S. Then*

(i) *A is a continuous map from $H^t(S)$ into $H^{t-r}(S)$ for any t.*

(ii) *If A is elliptic the map A: $H^t(S) \to H^{t-r}(S)$ is Fredholm.*

(iii) *If A is elliptic then $\psi \in H^t(S)$ and $A\psi \in H^s(S)$ implies $\psi \in H^{s+r}(S)$ and there is a constant $C_{t,s}$ such that $\|\psi\|_{s+r} \leqslant C_{t,s}\|A\psi\|_s$.*

The above results extend to mappings from tangential fields on S into tangential fields. A pseudo-differential operator $\mathbf{v} \to V_\gamma(\mathbf{v})$ is elliptic if the determinant of its top order symbol is non-zero for $\xi \neq 0$ and Lemma 4.1 holds for such operators.

We apply the preceding ideas to the operators V_γ. We first obtain the expansion (4.12). Consider the first the expression $|\mathbf{X}(U)-\mathbf{X}(u)|$. Recall that we are assuming S is analytic. It follows that the functions X are analytic and that

$$|\mathbf{X}(U)-\mathbf{X}(u)| = \sum_{v=1}^{\infty} M_v(U, u-U)$$

where M_v is homogeneous of degree v in $u-U$. Moreover it is easy to check that the orthonormality of the coordinate system gives $M_1(U, u-U) = |u-U|$. Next we see from (2.3) that $\phi_\gamma(r) = r^{-1}\sum_{j=0}^{\infty}\beta^j_\gamma r^j$. Thus we find

$$\phi_\gamma(|X(U)-X(u)|) = |u-U|^{-1} + \sum_{v=0}^{\infty} k^v_\gamma(U, u-U) \tag{4.14}$$

with k^v_γ homogeneous of degree v. Substitution of (4.14) yields (4.12) with $r=-1$ and

$$K^{-1}_\gamma(U,\eta) = c\chi(X(U))\,|\eta|^{-1}. \tag{4.15}$$

If we use Lemma 3.1 we see that

$$a_\gamma^{-1}(U, \xi) = c\chi(X(U))\,|\xi|^{-1}. \tag{4.16}$$

In our later computations a central role will be played by the special operator V_i. We observe that

$$\phi_\gamma(r) = \phi_i(r) + (i\gamma + 1) + \Phi_\gamma(r), \qquad \Phi_\gamma(r) = \sum_{k=1}^{\infty} \delta_k r^k \tag{4.17}$$

and we write accordingly

$$V_\gamma(\psi) = V_i(\psi) + \Gamma_\gamma(\psi) + W_\gamma(\psi), \tag{4.18}$$

where

$$\Gamma_\gamma(\psi) = (i\gamma + 1)\, c \int_S \psi\, ds, \quad W_\gamma(\psi)(x) = c \int_S \psi(y)\, \Phi_\gamma(|x-y|)\, ds_y. \tag{4.19}$$

Since $\Phi_\gamma(|x-y|)$ starts with terms of order $|x-y|$ the same type of argument as above shows that W_γ is an operator of order minus three. The operator Γ_γ takes $H^r(S)$ into $H^t(S)$ for any t. We summarize:

LEMMA 4.2. *$V_\gamma = V_i + \tilde{W}_\gamma$ where $\tilde{W}_\gamma$ is a continuous map for $H^t(S)$ into $H^{t+3}(S)$.*

Now we obtain a result for V_i.

LEMMA 4.3. *The map $\psi \to V_i(\psi)$ is bijective from $H^r(S)$ into $H^{r+1}(S)$ for any* real r.

Proof. From (4.15) and (4.16) we see that V_i is an elliptic pseudo-differential operator of order minus one. Hence it maps $H^r(S)$ into $H^{r+1}(S)$ for any r and is a Fredholm operator. We assert that as a map from $H^{-1/2}(S)$ to $H^{1/2}(S)$, V_i is self-adjoint. Indeed the dual of $H^{-1/2}(S)$ is $H^{1/2}(S)$ and for smooth function ψ and χ one has

$$\int_S \psi(x)\, V_i(\chi)(x)\, ds_x = \int_S \chi(x)\, V_i(\psi)(x)\, ds_x$$

because ϕ_i depends only on $|x-y|$. Since V_i is Fredholm we can thus conclude that V_i is bijective from $H^{-1/2}(S)$ to $H^{1/2}(S)$ if we can show that $V_i(\psi) = 0$ implies $\psi \equiv 0$. But once we know the result for $r = -\frac{1}{2}$ the regularity result (iii) of Lemma 4.1 shows that V_i is bijective from $H^r(S)$ to $H^{r+1}(S)$ for any real r.

Suppose, then, that $V_i(\psi) = 0$ for $\psi \in H^{-1/2}(S)$. We conclude by

Lemma 4.1(iii) that $\psi \in H^r(S)$ for any real r and hence ψ is continuous. Put $v(x) = c \int_S \psi(y)\, \phi_i(|x-y|)\, ds_y$. By Lemma 2.1, v is continuous in R^3 and satisfies $\Delta v - v = 0$ in Ω and Ω'; moreover $v = O(e^{-|x|}/|x|)$ as $|x| \to \infty$ and $v \equiv 0$ on S. Easy Green's theorem arguments imply that $v \equiv 0$ in Ω and Ω' and Lemma 2.1(iv) then implies $\psi \equiv 0$.

We can establish results analogous to those above for the map $\mathbf{v} \to V_\gamma(\mathbf{v})_{\mathrm{T}}$.

LEMMA 4.4. *$(V_\gamma)_{\mathrm{T}} = (V_i)_{\mathrm{T}} + \mathscr{W}_\gamma$ where:*

(i) *$(V_i)_{\mathrm{T}}$ is bijective $\mathbf{H}^r(S)$ into $\mathbf{H}^{r+1}(S)$,*

(ii) *$\mathscr{W}_\gamma$ maps $\mathbf{H}^r(S)$ continuously into $\mathbf{H}^{r+3}(S)$.*

Proof. We recall that $(V_\gamma(\mathbf{v}))_{\mathrm{T}} = n \times (V_\gamma(\mathbf{v}) \times \mathbf{n})$. We substitute the series (4.17) into V_γ to obtain $V_\gamma(\mathbf{v}) = V_i(\mathbf{v}) + \Gamma_\gamma(\mathbf{v}) + W_\gamma(\mathbf{v})$ in analogy to (4.18). Just as before, the tangential component of $W_\gamma(\mathbf{v})$ will produce an operator $\mathscr{W}_\gamma$ of order minus three. Similarly

$$\mathbf{n}(x) \times (\Gamma_\gamma(\mathbf{v}) \times \mathbf{n}(x)) = (i\gamma + 1)\, c\mathbf{n}(x) \times \left\{ \int_S \mathbf{v}(y) \times \mathbf{n}(y)\, ds \right.$$
$$\left. + \int_S \mathbf{v}(y) \times (\mathbf{n}(x) - \mathbf{n}(y))\, ds_y \right\}. \tag{4.20}$$

But $\mathbf{n}(X(U)) - \mathbf{n}(\mathbf{X}(u)) = O(|U-u|)$ so the second term on the right of (4.20) will be an operator of order minus three. This verifies (ii). In order to verify (i) we first write, as above,

$$V_i(\mathbf{v})_{\mathrm{T}}\,(x) = \mathbf{n}(x) \times V_i(\mathbf{v} \times \mathbf{n}) + \mathbf{n}(x) \times c \int_S \mathbf{v}(y)$$
$$\times (n(x) - \mathbf{n}(y))\, \phi_\gamma(|x-y|)\, ds_y$$
$$\equiv \mathscr{V}(\mathbf{v}) + \mathscr{W}(\mathbf{v}) \tag{4.21}$$

where $\mathscr{W}$ is an operator of order minus two. One verifies, just as with $V_i(\psi)$, that $\mathscr{V}$ is an elliptic operator of order minus one, hence takes $\mathbf{H}^r(S)$ into $\mathbf{H}^{r+1}(S)$ and is Fredholm. We assert that, once again, $\mathscr{V}$ is self-adjoint from $\mathbf{H}^{-1/2}(S)$ to $\mathbf{H}^{1/2}(S)$, the calculation being as before.

We want to show that $\mathscr{V}$ is bijective from $\mathbf{H}^r(S)$ to $\mathbf{H}^{r+1}(S)$ for any real r. From the preceding paragraph and our earlier argument it suffices to show that $\mathscr{V}(\mathbf{v}) \equiv 0$ implies $\mathbf{v} \equiv 0$. Suppose $\mathscr{V}(\mathbf{v}) \equiv 0$ and set $\mathbf{w}(x) = c \int_S (\mathbf{v}(y) \times \mathbf{n}(y)\ \ \phi_i(|x-y|)\, ds_y$. Then $\Delta \mathbf{w} - \mathbf{w} = 0$ in Ω and Ω' from Lemma 2.1. We do not, however, have $\mathbf{w} \equiv 0$ on S but only $n \times \mathbf{w} = 0$ on S, which is equivalent to $\mathbf{w}_{\mathrm{T}} \equiv 0$ so we need further argument.

We apply Green's theorem to Ω' and obtain the following:

$$\int_{\Omega'} |\mathbf{w}|^2\, dx = \int_{\Omega'} \mathbf{w} \cdot \Delta \mathbf{w}\, dx$$
$$= -\int_S \mathbf{w} \cdot \left(\frac{\partial \mathbf{w}}{\partial n}\right)^- ds - \int_{\Omega'} \operatorname{tr}(\nabla \mathbf{w} (\nabla \mathbf{w})^{\mathrm{T}})\, dx. \tag{4.22}$$

If we do the same calculation for Ω (with a limiting argument) we obtain

$$\int_{\Omega} |w|^2\, dx = \int_{\Omega} \mathbf{w} \cdot \Delta \mathbf{w}\, dx$$
$$= \int_S \mathbf{w} \cdot \left(\frac{\partial \mathbf{w}}{\partial n}\right)^+ ds - \int_{\Omega} \operatorname{tr}(\nabla \mathbf{w} (\nabla \mathbf{w})^{\mathrm{T}})\, dx. \tag{4.23}$$

We add (4.22) and (4.23) and note that, by Lemma 2.1, $(\partial \mathbf{w}/\partial n)^- - (\partial \mathbf{w}/\partial n)^+ = \mathbf{v} \times \mathbf{n}$, which is tangential to S. Hence we have, on S,

$$\mathbf{w} \cdot \left(\left(\frac{\partial \mathbf{w}}{\partial n}\right)^+ - \left(\frac{\partial \mathbf{w}}{\partial n}\right)^+\right) = 0.$$

From this we conclude $\mathbf{w} \equiv 0$ in Ω and Ω'. But then $\mathbf{v} \times \mathbf{n} \equiv 0$, on S, which is equivalent to $\mathbf{v} \equiv 0$.

We show now that $(V_i)_{\mathrm{T}}$ is bijective. Consider the equation $V_i(\mathbf{v})_{\mathrm{T}} = \mathbf{f}$ for $\mathbf{f} \in \mathbf{H}^{r+1}(S)$, $r \geqslant -\frac{1}{2}$. From (4.21) and our result about $\mathscr{V}$ this is equivalent to

$$\mathbf{v} + \mathscr{V}^{-1}\mathscr{W}\mathbf{v} = \mathscr{V}^{-1}\mathbf{f}. \tag{4.24}$$

Now $\mathscr{V}^{-1}\mathscr{W}$ takes $\mathbf{H}^r(S)$ into $\mathbf{H}^{r+1}(S)$ since $\mathscr{W}$ is of order minus two. Since $\mathbf{H}^{r+1}(S)$ is compact in $H^r(S)$ we conclude that (4.24) is a Riesz–Schauder system. If $\mathbf{v}$ is a solution of the corresponding homogeneous system then an argument analogous to that for $\mathscr{V}$ shows that $\mathbf{v} \equiv 0$. Hence (4.24) has a unique solution and the proof of Lemma 4.4 is complete.

In the next section we will need one more property of the operator $(V_\gamma)_{\mathrm{T}}$ and we give this property now.

LEMMA 4.5. $\operatorname{div}_{\mathrm{T}} V_\gamma(\mathbf{v})_{\mathrm{T}} = V_\gamma(\operatorname{div}_{\mathrm{T}} \mathbf{v}) + \mathscr{E}_\gamma(\mathbf{v})$ *where* $\mathscr{E}_\gamma$ *is a continuous map from* $\mathbf{H}^r(S)$ *into* $H^{r+1}(S)$.

Proof. Our first observation is that, just as in (4.21), we have

$$V_\gamma(\mathbf{v})_{\mathrm{T}} = n \times (V_\gamma(\mathbf{v}) \times n) = n \times V_\gamma(\mathbf{v} \times \mathbf{n}) + \mathscr{W}_\gamma(\mathbf{v}), \tag{4.25}$$

where $\mathscr{W}_\gamma$ is of order minus two. Since div_T is an operator of order one it suffices to verify the result for the first term on the right of (4.25).

In local coordinates we have

$$V_\gamma(\mathbf{v}\times\mathbf{n}) = \sum_{i=1}^{3} (V_\gamma(\mathbf{v}\times\mathbf{n})\cdot\mathbf{e}_i(U))\,\mathbf{e}_i(U)$$

and

$$\begin{aligned}\mathbf{n}\times V_\gamma(\mathbf{v}\times\mathbf{n}) &= e_3(U)\times(V_\gamma(\mathbf{v}\times\mathbf{n}))\\ &= -(V_\gamma(\mathbf{v}\times\mathbf{n})\cdot\mathbf{e}_2(U))\,\mathbf{e}_1(U) + (V_\gamma(\mathbf{v}\times\mathbf{n})\cdot\mathbf{e}_1(U))\,\mathbf{e}_2(U).\end{aligned} \tag{4.26}$$

We have also $\mathbf{v}(u)\times\mathbf{n}(u) = -v^2(u)\,\mathrm{e}_1(u) + v^1(u)\,e_2(u)$. Now we calculate as follows:

$$\begin{aligned}&\operatorname{div}_T(\mathbf{n}\times V_\gamma(\mathbf{v}\times\mathbf{n}))\\ &\quad= -\frac{\partial}{\partial U_1}(V_\gamma(\mathbf{v}\times\mathbf{n})\cdot\mathbf{e}_2(U)) + \frac{\partial}{\partial U_2}(V_\gamma(\mathbf{v}\times\mathbf{n})\cdot\mathbf{e}_1(U))\\ &\quad= c\int_S \Big\{-(-v^2(u)\,\mathbf{e}_1(u) + v^1(u)\,\mathbf{e}_2(u))\cdot\mathbf{e}_2(U)\frac{\partial}{\partial U_1}\phi_\gamma(|X(U)-X(u)|)\\ &\qquad+ (-v^2(u)\,\mathbf{e}_1(u) + v^1(u)\,\mathbf{e}_2(u))\cdot\mathbf{e}_1(U)\frac{\partial}{\partial U_2}\phi_\gamma(X(U)-X(u))\Big\}\,du.\end{aligned}$$

Then, as in earlier calculations, we can approximate $|X(U)-X(u)|$ by $|U-u|$ so that $\partial/\partial U_1$ and $\partial/\partial U_2$ can be replaced by $-\partial/\partial u_1$ and $-\partial/\partial u_2$, respectively. Continuing the approximation we can replace $\mathbf{e}_1(U)$ and $\mathbf{e}_2(U)$ by $\mathbf{e}_1(u)$ and $e_2(\mathbf{u})$ and hence replace $(-v^2(u)\,\mathbf{e}_1(u) + v^1(u)\,\mathbf{e}_2(u))\cdot e_2(U)$ by $v^1(u)$ and $(-v^2(u)\,\mathbf{e}_1(u) + v^1(u)\,\mathbf{e}_2(u))\cdot\mathbf{e}_1(U)$ by $-v^2(u)$. An integration by parts yields the integral $c\int_\Sigma \operatorname{div}\mathbf{v}(y)\,\phi_\gamma(|x-y|)\,ds_y$ modulo terms which will correspond to $\mathscr{E}_\gamma$.

The next results of this section give a proof of Lemma 2.2. As in our earlier calculations we need only consider $V_\gamma(\mathbf{v})(x)$ for $\mathbf{v}$ having support in a patch S_k and x lying in U_k. We have

$$\operatorname{curl}_x(v(y)\,\phi_\gamma(|x-y|) = -\mathbf{v}(y)\times\operatorname{grad}_x\phi_\gamma(|x-y|). \tag{4.27}$$

In addition, we have

$$\begin{gathered}\operatorname{grad}_x\phi_\gamma(|x-y|) = \phi'_\gamma(|x-y|)\frac{x-y}{|x-y|}\\ \phi'_\gamma(r) = -\frac{1}{r^2} + O(1) \qquad \text{as} \quad r\to 0.\end{gathered} \tag{4.28}$$

Now we use local coordinates and approximate. We have $x - y = \mathbf{X}(U) + U_3 e_3(U) - \mathbf{X}(u)$. One can check that

$$\mathbf{X}(u) = \mathbf{X}(U) + \mathbf{e}_1(U)(u_1 - U_1) + \mathbf{e}_2(U)(U_2 - U_2) + \mathbf{a}(U, u - U)$$

where $\mathbf{a}$ is homogeneous of degree two in $u - U$. This yields, in view of the orthonormality of the coordinate system,

$$|x - y|^2 = |U - u|^2 + U_3^2 + 2U_3 \mathbf{e}_3(U) \cdot \mathbf{a} + b(U, u - U) \tag{4.29}$$

where b is homogeneous of degree three in $u - U$. Thus we have by (4.28)

$$\begin{aligned} \operatorname{grad}_x \phi_\gamma(|x - y|) = \{&\mathbf{e}_1(U)(u_1 - U_1) + \mathbf{e}_2(U)(u_2 - U_2) \\ &+ U_3 \mathbf{e}_3(U) + \cdots\}[|U - u|^2 + U_3^2]^{-3/2}. \end{aligned} \tag{4.30}$$

Since $\mathbf{v}$ is a tangential field we have $\mathbf{v}(u) = v^1(u)\,\mathbf{e}_1(u) + v^2(u)\,\mathbf{e}_2(u)$ and $\mathbf{v} \times \operatorname{grad}_x \phi_\gamma(|x - y|) = [|U - u|^2 + U_3^2]^{-3/2}$.

$$\begin{aligned} &[-v^1(u)\{\mathbf{e}_1(u) \times \mathbf{e}_1(U)(u_1 - U_1) + \mathbf{e}_1(u) \times \mathbf{e}_2(U)(u_2 - U_2) + U_3 \mathbf{e}_1(u) \times \mathbf{e}_3(U)\} \\ &-v^2(u)\{\mathbf{e}_2(u) \times \mathbf{e}_1(U)(u_1 - U_1) + e_2(u) \times e_2(U)(u_2 - U_2) + U_3 \mathbf{e}_2(u) \times \mathbf{e}_3(U)\} \\ &+ \cdots]. \end{aligned} \tag{4.31}$$

To calculate the operator in Lemma 2.2 we have to take the cross product of $\mathbf{n}$ with curl $V_\gamma(\mathbf{v})$. In local coordinates we have $\mathbf{n} = \mathbf{e}_3(U)$ and we note the following formulas:

$$\begin{aligned} \mathbf{e}_3(U) \times (\mathbf{e}_1(u) \times \mathbf{e}_1(U)) &= -\mathbf{e}_1(U) \cdot \mathbf{e}_3(U) \\ \mathbf{e}_3(U) \times (\mathbf{e}_1(u) \times \mathbf{e}_2(U)) &= -e_2(U)\,\mathbf{e}_1(u) \cdot \mathbf{e}_3(U) \\ \mathbf{e}_3(U) \times (\mathbf{e}_2(u) \times \mathbf{e}_1(U)) &= -e_1(U)\,\mathbf{e}_3(U) \cdot e_2(u) \\ \mathbf{e}_3(U) \times (\mathbf{e}_2(u) \times \mathbf{e}_2(U)) &= -\mathbf{e}_2(U)\,\mathbf{e}_3(U) \cdot \mathbf{e}_2(u) \\ \mathbf{e}_3(U) \times (\mathbf{e}_1(u) \times \mathbf{e}_3(U)) &= \mathbf{e}_1(u) - \mathbf{e}_3(U)\,\mathbf{e}_3(U) \cdot \mathbf{e}_1(u) \\ \mathbf{e}_3(U) \times (\mathbf{e}_2(u) \times \mathbf{e}_3(U)) &= \mathbf{e}_2(u) - \mathbf{e}_3(U) \cdot e_2(u). \end{aligned} \tag{4.32}$$

The first four quantities in (4.32) are zero to order $|u - U|$ while the last two are $\mathbf{e}_1(u)$ and $\mathbf{e}_2(u)$, respectively. Thus (4.31) yields

$$\mathbf{n} \times \operatorname{curl} V_\gamma(\mathbf{v}) = \int \left\{ [v^1(u)\,\mathbf{e}_1(u) + v^2(u)\,\mathbf{e}_2(u)] \times \frac{-U_3}{(|U - u|^2 + U_3^2)^{3/2}} + \cdots \right\} du.$$

If we take the limit as $U_3 \downarrow 0$, $U_3 \uparrow 0$ in the first term we obtain $\frac{1}{2}\mathbf{v}(u)$ and $-\frac{1}{2}\mathbf{v}(u)$, respectively. A careful calculation shows that the error terms are $O(|U-u|^{-1})$ and this is the statement in Lemma 2.2.

Our final result in this section is another fact that can be obtained from [10]. The surface operator $\Delta_T = \mathrm{div}_T\,\mathrm{grad}_T$ is a pseudo-differential operator of order 2 on S; hence so is $\Delta_T - \beta^2 I$. The latter is elliptic and self-adjoint as an operator on $H^{-1/2}(S)$. Hence we can use the same argument as for V_i to obtain the following result.

LEMMA 4.6. *$\Delta_T - i\beta^2 I$ is bijective from $H^r(S)$ into $H^{r-2}(S)$ for any real r.*

5. PROOFS OF THEOREMS

In this section we provide the proof of Theorems 3.1 and 3.2. We begin with the proof of Theorem 3.1. Let us recall what must be shown. From Section 2,

$$A_\alpha(\mathbf{J}, M) = (V_\alpha(\mathbf{J})_T + \mathrm{grad}_T\, V_\alpha(M), \qquad -V_\alpha(\mathrm{div}_T\,\mathbf{J}) + \alpha^2 V_\alpha(M)) \tag{5.1}$$

and we are seeking a solution of

$$A_\alpha(\mathbf{J}, M) = (\mathbf{F}, 0), \qquad \mathbf{F} = -\mathbf{E}_T^0. \tag{5.2}$$

We use perturbation theory as we did in the proof of Lemma 4.4; that is, we reduce (5.2) to a Riesz–Schauder system. We first use Lemmas 4.2 and 4.4 to write

$$A_\alpha(\mathbf{J}, M) = A_i(\mathbf{J}, M) + B_\alpha(J, M), \tag{5.3}$$

where

$$B_\alpha(\mathbf{J}, M) = (\mathscr{W}_\alpha(\mathbf{J}) + \mathrm{grad}_T\, \tilde{W}_\alpha(M), -\tilde{W}_\alpha(\mathrm{div}\,\mathbf{J}) + (\alpha^2+1)\, V_i(M) + \alpha^2 \tilde{W}_\alpha(M). \tag{5.4}$$

We establish first that A_i is invertible.

LEMMA 5.1. *For any real r, A_i is bijective from $H^{r-1,r}$ into $H^{r,r-1}$.*

Proof. We consider the equations

$$V_i(\mathbf{J})_T + \mathrm{grad}_T\, V_i(M) = \mathbf{F}, \qquad -V_i(\mathrm{div}_T\,\mathbf{J}) - V_i(M) = G. \tag{5.5}$$

Suppose we have a solution. Then we form div_T of the first equation and use

Lemma 4.5. If we substitute for $V_i(\operatorname{div}_T \mathbf{J})$ from the second equation we obtain[4]

$$-V_i(M) + \Delta_T V_i(M) = -\mathscr{E}_i(\mathbf{J}) + \operatorname{div}_T \mathbf{F} + G. \tag{5.6}$$

Now from Lemma 4.3, V_i is invertible and from Lemma 4.6, $\Delta_T - I$ is invertible. Hence (5.6) yields

$$M = \mathscr{M}(\mathbf{J}) + M^0 \tag{5.7}$$

$$\mathscr{M}(\mathbf{J}) = -V_i^{-1}(\Delta_T - I)^{-1}\,\mathscr{E}_i(\mathbf{J}), \qquad M^0 = V_i^{-1}(\Delta_T - I)^{-1}\,(\operatorname{div}_T \mathbf{F} + G). \tag{5.8}$$

Our next step is to use Lemma 4.4 to invert the first equation in (5.5). We obtain

$$\mathbf{J} = \mathscr{T}(M) + \mathbf{J}^0, \tag{5.9}$$

$$\mathscr{T}(M) = -((V_i)_T)^{-1}\operatorname{grad}_T V_i(M), \qquad \mathbf{J}^0 = ((V_i)_T)^{-1}\,(\mathbf{F}). \tag{5.10}$$

Now we eliminate M between (5.7) and (5.9) to obtain

$$\mathbf{J} = \mathscr{T}\mathscr{M}\mathbf{J} + \mathbf{J}^0 + \mathscr{T}M^0. \tag{5.11}$$

We assert that (5.11) is a Riesz–Schauder equation on $\mathbf{H}^{r-1}(S)$. To see this we study the regularity of $\mathscr{T}\mathscr{M}$. Since V_i^{-1}, $(\Delta_T - I)^{-1}$ and $\mathscr{E}_i$ are of orders one, minus two, and minus one, respectively, (5.8) shows that $\mathscr{M}$ takes $\mathbf{H}^{r-1}(S)$ into $H^{r+1}(S)$. On the other hand $(V_i)_T^{-1}$, grad_T, and V_i are of orders one, one, and minus one, respectively; hence, by (5.10), $\mathscr{T}$ takes $\mathbf{H}^{r+1}(S)$ into $\mathbf{H}^r(S)$. Thus $\mathscr{T}\mathscr{M}$ maps $\mathbf{H}^{r-1}(S)$ into $\mathbf{H}^r(S)$ and is, accordingly, compact. One checks the same way that if $\mathbf{F} \in \mathbf{H}^r(S)$ then $\mathbf{J}^0 + \mathscr{T}\mathbf{M}^0$ is in $\mathbf{H}^{r-1}(S)$.

It follows from the above that (5.11) will have a unique solution in $\mathbf{H}^{r-1}(S)$ if the corresponding homogeneous equation has only the zero solution. If this is the case we can define $M \in H^r(S)$ by (5.7). Then it is easy to reverse our steps to show that $\mathbf{J}, M$ satisfy (5.5).

Suppose then that $\mathbf{J}$ is a solution of the homogeneous equation (5.11). Form M from (5.7) with $M^0 = 0$. In analogy to (2.8) we define $\mathbf{E}$ and $\mathbf{H}$ by

$$\mathbf{E} = V_i(\mathbf{J}) + \operatorname{grad} V_i(M), \qquad \mathbf{H} = \operatorname{curl} V_i(J). \tag{5.12}$$

Then, just as in Section 2, we will have

$$\operatorname{curl}\mathbf{E} = \mathbf{H}, \qquad \operatorname{curl}\mathbf{H} = -\mathbf{E} \text{ in } \Omega, \qquad \mathbf{E}_T = 0 \text{ on } S. \tag{5.13}$$

Since $\mathbf{E}$ and $\mathbf{H}$ vanish exponentially for large $|x|$, an easy variation of the

[4] Recall that $\Delta_T \chi = \operatorname{div}_T \operatorname{grad}_T \chi$.

proof of Theorem 1.1 gives $\mathbf{E} \equiv \mathbf{H} \equiv \mathbf{0}$ in Ω. But as in an earlier proof we can also consider the expression (5.12) for $x \in \Omega'$ and find that the same equations are satisfied. And, again, one concludes easily that $\mathbf{E} \equiv \mathbf{H} \equiv \mathbf{0}$ in Ω'. We claim that these facts imply that $\mathbf{J} \equiv \mathbf{0}$ and $M \equiv 0$. From (5.12) and Lemma 2.1, one concludes first that $M = (\mathbf{E} \cdot \mathbf{n})^- - (\mathbf{E} \cdot \mathbf{n})^+ = 0$. Then from Lemma 2.2 we obtain $\mathbf{J} = (\mathbf{n} \times \mathbf{H})^+ - (\mathbf{n} \times \mathbf{H})^- \equiv \mathbf{0}$ and the proof of Lemma 5.1 is complete.

Once we have Lemma 5.1 we use (5.3) to write (5.2) as

$$(\mathbf{J}, M) = -A_i^{-1} B_\alpha(\mathbf{J}, M) + A_i^{-1}(\mathbf{F}, 0). \tag{5.14}$$

LEMMA 5.2. *$A_i^{-1} B_\alpha$ is a continuous map from $H^{r-1,r}(S)$ into $H^{r,r+1}(S)$.*

Proof. We have only to count the orders of the various operators. From Lemmas 4.2 and 4.4, $\mathscr{W}_\alpha$ and $\tilde{\mathscr{W}}_\alpha$ are both of order minus three, hence take $\mathbf{H}^{r-1}(S)$ and $H^r(S)$ into $\mathbf{H}^{r+2}(S)$ and $H^{r+3}(S)$, respectively. Grad_T is of order one and V_α is of order minus one; hence, by (5.4), B_α takes $H^{r-1,r}$ into $H^{r+2,r+1}$. Then, by Lemma 5.1, $A_i^{-1} B_\alpha$ takes $H^{r-1,r}(S)$ into $H^{r+1,r+2}(S)$. It follows that this operator is compact on $H^{r-1,r}(S)$. Moreover for $\mathbf{F} \in H^r$, $A_i^{-1}(\mathbf{F}, 0) \in H^{r-1,r}$; thus (5.14) is a Riesz–Schauder system on $H^{r-1,r}$.

We can argue that if $(\mathbf{J}, M)$ is a solution of the homogeneous equation (5.14) then $(\mathbf{J}, M) = (\mathbf{0}, 0)$ almost exactly as in the proof of Lemma 5.1. The only difference is that $\mathbf{E}, \mathbf{H}$ defined by (5.12) satisfy

$$\mathrm{curl}\, \mathbf{E} = \mathbf{H}, \qquad \mathrm{curl}\, \mathbf{H} = -\mathbf{E} \text{ in } \Omega \text{ and } \Omega', \qquad \mathbf{E}_\mathrm{T} = 0 \text{ on } S. \tag{5.15}$$

By Theorem 1.1, $\mathbf{E} \equiv \mathbf{H} \equiv \mathbf{0}$ in Ω and by our hypothesis 2.1, $\mathbf{E} \equiv \boldsymbol{H} \equiv \mathbf{0}$ in Ω'. Then the previous argument shows that $\mathbf{J} \equiv \mathbf{0}$, $M \equiv 0$. This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is very similar to the one just given but is a little complicated by the regularity requirements. We will outline the ideas, omitting a few technical details.

Let us again recall the equations we have to solve. They are

$$\begin{aligned} &V_\alpha(\mathbf{J})_\mathrm{T} + \mathrm{grad}_\mathrm{T}\, V_\alpha(M) - V_{\sqrt{i}\,\beta}(\mathbf{j})_\mathrm{T} - \mathrm{grad}_\mathrm{T}\, V_{\sqrt{i}\,\beta}(m) = \mathscr{E} \\ &-V_\alpha(\mathrm{div}_\mathrm{T}\, \mathbf{J}) + \alpha^2 V_\alpha(M) = 0 \\ &\mathbf{J} + K_\alpha(\mathbf{J}) + \mathbf{j} - K_{\sqrt{i}\,\beta}(\mathbf{j}) = \mathscr{H} \\ &-V_{\sqrt{i}\,\beta}(\mathrm{div}_\mathrm{T}\, \mathbf{j}) + i\beta^2 V_{\sqrt{i}\,\beta}(m) = 0. \end{aligned} \tag{5.16}$$

Once again we want to reduce (5.16) to a Riesz–Schauder system. To this end we need the following result.

LEMMA 5.3. *For any* real r:

(i) *The map* $\psi \to V_{\sqrt{i}\,\beta}\psi$ *is bijective from* $H^r(S)$ *into* $H^{r+1}(S)$.

(ii) *The map* $\mathbf{v} \to V_{\sqrt{i}\,\beta}(\mathbf{v})_{\mathrm{T}}$ *is bijective from* $\mathbf{H}^r(S)$ *into* $\mathbf{H}^{r+1}(S)$.

Proof. Condition (i) is a corollary of Lemmas 4.2 and 4.3. Indeed by those lemmas we can write the equation $V_{\sqrt{i}\,\beta}\psi = f$ as $\psi + V_i^{-1}\tilde{W}_{\sqrt{i}\,\beta}\psi = V^{-1}f$, a Riesz–Schauder system. An argument analogous to several we have given shows that $\psi + V_i^{-1}\tilde{W}_{\sqrt{i}\,\beta}\psi = 0$ implies $\psi = 0$. Condition (ii) follows from Lemma 4.4 in the same way.

Now let us begin on (5.16). We take $\mathrm{div}_{\mathrm{T}}$ of $(5.16)_1$ and use Lemma 4.5 and $(5.16)_{2,4}$ to obtain the equation

$$(\Delta_{\mathrm{T}} + \alpha^2)\, V_\alpha(M) - (\Delta_{\mathrm{T}} + i\beta^2)\, V_{\sqrt{i}\,\beta}(m) + \mathscr{E}_\alpha(\mathbf{J}) - \mathscr{E}_{\sqrt{i}\,\beta}(\mathbf{j}) = \mathrm{div}\,\mathscr{E}. \quad (5.17)$$

Now we use Lemma 4.2 and rewrite (5.17). We see from Lemma 4.2 that

$$V_\alpha(M) = V_{\sqrt{i}\,\beta}(M) + \tilde{\mathscr{V}}_{\alpha\beta}(M) \quad (5.18)$$

where $\tilde{\mathscr{V}}_{\alpha\beta}$ is of order minus three. Accordingly we can write (5.17) as

$$\begin{aligned}(\Delta_{\mathrm{T}} + i\beta^2)\, V_{\sqrt{i}\,\beta}(m) = {} & (\Delta_{\mathrm{T}} + i\beta^2)\, V_{\sqrt{i}\,\beta}(M) + (\Delta_{\mathrm{T}} + \alpha^2)\, \tilde{\mathscr{V}}_{\alpha\beta}(M) \\ & + (\alpha^2 - i\beta^2)\, V_{\sqrt{i}\,\beta}(M) + \mathscr{E}_\alpha(\mathbf{J}) - \mathscr{E}_{\sqrt{i}\,\beta}(\mathbf{j}) - \mathrm{div}\,\mathscr{E}.\end{aligned}$$

If we apply $(\Delta_{\mathrm{T}} + i\beta^2)^{-1}$ to both sides we have

$$V_{\sqrt{i}\,\beta}(m - M) = C_1(M) + C_2(\mathbf{J}) + C_3(\mathbf{j}) - f. \quad (5.19)$$

In this equation $C_1 = (\Delta_{\mathrm{T}} - i\beta^2)^{-1}\, \{(\Delta_{\mathrm{T}} + \alpha^2)\, \tilde{\mathscr{V}}_{\alpha\beta} + (\alpha^2 - i\beta^2)\, V_{\sqrt{i}\,\beta}\}$, $C_2 = (\Delta_{\mathrm{T}} + i\beta^2)^{-1}\, \mathscr{E}_\alpha$, and $C_3 = -(\Delta_{\mathrm{T}} + i\beta^2)^{-1}\, \mathscr{E}_{\sqrt{i}\,\beta}$, and one checks that all three are of order minus three. Also $f = (\Delta_{\mathrm{T}} + i\beta^2)^{-1}\, \mathrm{div}\,\mathscr{E}$ and for $\mathscr{E} \in \mathbf{H}^r(S)$ this is in $H^{r+1}(S)$.

To obtain our next equation we want to invert (5.16). In analogy to (5.18) we observe that Lemma 4.4 yields

$$V_\alpha(\mathbf{J})_{\mathrm{T}} = V_{\sqrt{i}\,\beta}(\mathbf{J})_{\mathrm{T}} + \mathscr{V}_{\alpha\beta}(\mathbf{J}), \quad (5.20)$$

where $\mathscr{V}_{\alpha\beta}$ is again of order minus three. Now we write $(5.16)_1$ as

$$\begin{aligned}(V_{\sqrt{i}\,\beta}(\mathbf{j}))_{\mathrm{T}} = {} & V_{\sqrt{i}\,\beta}(\mathbf{J})_{\mathrm{T}} + \mathscr{V}_{\alpha\beta}(\mathbf{J}) + \mathrm{grad}_{\mathrm{T}}\, V_{\sqrt{i}\,\beta}(M - m) \\ & + \mathrm{grad}_{\mathrm{T}}\, \mathscr{V}_{\alpha\beta}(M) - \mathscr{E},\end{aligned}$$

or

$$\begin{aligned}\mathbf{j} = \mathbf{J} + (V_{\sqrt{i}\,\beta})_{\mathrm{T}}^{-1}\, \{ & \mathscr{V}_{\alpha\beta}(\mathbf{J}) + \mathrm{grad}_{\mathrm{T}}\, V_{\sqrt{i}\,\beta}(M - m) \\ & + \mathrm{grad}_{\mathrm{T}}\, \mathscr{V}_{\alpha\beta}(M) - \mathscr{E}\}.\end{aligned} \quad (5.21)$$

We substitute (5.19) into (5.21) for $V_{\sqrt{i}\,\beta}(M-m)$. The result has the form

$$\mathbf{j}=\mathbf{J}+D_1(M)+D_2(\mathbf{J})+D_3(\mathbf{j})+\mathbf{g}. \tag{5.22}$$

Here

$$D_1=(V_{\sqrt{i}\,\beta})_{\mathrm{T}}^{-1}\,\mathrm{grad}_{\mathrm{T}}\,(\mathscr{V}_{\alpha\beta}(M)-C_1(M))$$

$$D_2=(V_{\sqrt{i}\,\beta})_{\mathrm{T}}^{-1}(\mathscr{V}_{\alpha\beta}(\mathbf{J})-\mathrm{grad}_{\mathrm{T}}\,C_2(\mathbf{J}))$$

$$D_3=-(V_{\sqrt{i}\,\beta})_{\mathrm{T}}^{-1}\,\mathrm{grad}\,C_3(\mathbf{j})$$

and all three operators are of order at most minus one. Also $\mathbf{g}=(V_{\sqrt{i}\,\beta})_{\mathrm{T}}^{-1}(\mathrm{grad}_{\mathrm{T}}f-\mathscr{E})$. Since $f\in H^{r+1}(S)$ and $\mathscr{E}\in\mathbf{H}^r(S)$ we have $\mathbf{g}\in\mathbf{H}^{r-1}(S)$.

We turn next to $(5.16)_3$. We have $\mathscr{K}_\alpha(\mathbf{J})=\mathscr{K}_{\sqrt{i}\,\beta}(\mathbf{J})+\mathscr{L}_{\alpha\beta}(\mathbf{J})$ where $\mathscr{L}_{\alpha\beta}$ is of order minus two. Equation (5.16) yields, then,

$$\mathbf{j}+\mathbf{J}=\mathscr{K}_{\sqrt{i}\,\beta}(\mathbf{j}-\mathbf{J})-\mathscr{L}_{\alpha\beta}(\mathbf{J})+\mathscr{H}. \tag{5.23}$$

We substitute for $\mathbf{j}-\mathbf{J}$ from (5.22) to obtain

$$\mathbf{j}+\mathbf{J}=E_1(M)+E_2(\mathbf{J})+E_3(\mathbf{j})+\mathbf{h}-\mathscr{L}_{\alpha\beta}[\mathbf{J}]. \tag{5.24}$$

Here $E_i=\mathscr{K}_{\sqrt{i}\,\beta}D_i$, $i=1,2,3$, and these are of order minus two at most and $\mathbf{h}=\mathscr{H}+\mathscr{K}_{\sqrt{i}\,\beta}\,\mathbf{g}\in\mathbf{H}^r(S)$.

In (5.22) and (5.24) we have two equations which feature $\mathbf{J}$ and $\mathbf{j}$. We obtain two more which feature M and m. The first comes from applying $V_{\sqrt{i}\,\beta}^{-1}$ to (5.19) and is

$$m-M=F_1(M)+F_2(\mathbf{J})+F_3(\mathbf{j})+k. \tag{5.25}$$

Here $F_i=V_{\sqrt{i}\,\beta}^{-1}C_i$ are all of order minus two and $k=V_{\sqrt{i}\,\beta}^{-1}f\in H^r$. To get a second equation we first conclude from $(5.16)_4$ and Lemma 5.3 that $i\beta^2 m=\mathrm{div}_{\mathrm{T}}\,\mathbf{j}$. Then we write $(5.16)_2$ as

$$V_{\sqrt{i}\,\beta}(\mathrm{div}_{\mathrm{T}}\,\mathbf{J})+\mathscr{V}_{\alpha\beta}(\mathrm{div}_{\mathrm{T}}\,\mathbf{J})-\alpha^2V_{\sqrt{i}\,\beta}(M)-\alpha^2\mathscr{V}_{\alpha\beta}(M)=0$$

and apply $V_{\sqrt{i}\,\beta}^{-1}$ to obtain

$$\alpha^2M=\mathrm{div}_{\mathrm{T}}\,\mathbf{J}+G_1(M)+G_2(J). \tag{5.26}$$

Here $G_1=-\alpha^2V_{\sqrt{i}\,\beta}^{-1}\mathscr{V}_{\alpha\beta}$ is of order minus two and $G_2=\tilde{V}_{\sqrt{i}\,\beta}^{-1}\mathscr{V}_{\alpha\beta}(\mathrm{div}\,\mathbf{J})$ is of order minus one. We compute $\mathrm{div}_{\mathrm{T}}(\mathbf{j}+\mathbf{J})$ from (5.24) and substitute into (5.26). This yields

$$i\beta^2m+\alpha^2M=H_1(M)+H_2(\mathbf{J})+H_3(\mathbf{j})+e. \tag{5.27}$$

Here $H_1=\mathrm{div}_{\mathrm{T}}\,E_1+G_1$, $H_2=\mathrm{div}_{\mathrm{T}}\,E_2+G_2-\mathrm{div}_{\mathrm{T}}\,\mathscr{L}_{\alpha\beta}$, and $H_3=\mathrm{div}_{\mathrm{T}}\,E_3$ are all of order minus one and $e=\mathrm{div}\,\mathbf{h}\in H^r(S)$.

Equations (5.22), (5.24), (5.25), and (5.27) form a Riesz–Schauder system on $H^{r-1}(S) \times H^{r-1}(S) \times H^r(S) \times H^r(S)$. Each of the operators occurring on the right sides is of order at most minus one. The forcing terms **g** and **h** belong to $\mathbf{H}^r(S)$, hence to $\mathbf{H}^{r-1}(S)$, while the forcing terms k and e belong to $H^r(S)$. Once again a reversal of the steps shows that if $(\mathbf{J}, M, \mathbf{j}, m)$ satisfy (5.22), (5.24), (5.25), and (5.27) they also satisfy (5.16). By an argument which has become familiar the uniqueness result for $(\mathrm{P}_{\alpha\beta})$ shows that the only solution of the corresponding homogeneous equations has all functions zero. Thus the proof of Theorem 3.2 is complete.

6. The Skin-Effect Approximation

In this section we describe our asymptotic solution for the eddy current problem. We carry out the calculations for the half-space problem described in Section 3. We do this mainly to make the procedure clearer but we observe that if one uses the orthonormal co-ordinate systems as described in Section 4 then in fact our calculations are locally exact.

Let us study the half-space problem, then. This is

$$\begin{aligned} &\operatorname{curl} \mathbf{E} = \mathbf{H}, \qquad \operatorname{curl} \mathbf{H} = \alpha^2 \mathbf{E} \qquad \text{in} \quad x_3 > 0 \\ &\operatorname{curl} \mathbf{E} = \mathbf{H}, \qquad \operatorname{curl} \mathbf{H} = i\beta^2 \mathbf{E} \qquad \text{in} \quad x_3 < 0. \end{aligned} \tag{6.1}$$

We have a prescribed incident field $\mathbf{E}^0, \mathbf{H}^0$ and the interface conditions are

$$\mathbf{E}_{\mathrm{T}}^{+} = \mathbf{E}_{\mathrm{T}}^{-}, \qquad \mathbf{H}_{\mathrm{T}}^{+} = \mathbf{H}_{\mathrm{T}}^{-} \qquad \text{on} \quad x_3 = 0. \tag{6.2}$$

The asymptotic form we want is described in (A_Ω), $(A_{\Omega'})$ of Section 1. For the half-space these assume the form

$$\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix} \sim \begin{matrix}\mathbf{E}^0\\ \mathbf{H}^0\end{matrix} + \sum_{n=0}^{\infty} \begin{matrix}\mathbf{E}_n\\ \mathbf{H}_n\end{matrix}\, \beta^{-n} \qquad \text{in} \quad x_3 > 0, \tag{6.3}$$

$$\begin{matrix}\mathbf{E}\\ \mathbf{H}\end{matrix} \sim e^{\sqrt{-i}\,\beta x_3} \sum_{n=0}^{\infty} \begin{matrix}\mathbf{E}_n\\ \mathbf{H}_n\end{matrix}\, \beta^{-n} \qquad \text{in} \quad x_3 < 0. \tag{6.4}$$

The idea, then, is to substitute (6.3) and (6.4) into (6.1) and (6.2) and equate coefficients of like powers of β.

We introduce some notation. We set $\chi = e^{\sqrt{-i}\,\beta x_3}$. It is convenient to decompose fields into tangential and normal components so for any field $\mathbf{F}$ we write

$$\mathbf{F} = \mathscr{F} + f\mathbf{e}_3, \qquad \mathscr{F} = \mathscr{F}^1 \mathbf{e}_1 + \mathscr{F}^2 \mathbf{e}_2. \tag{6.5}$$

For any tangential vector $\mathscr{F}$ as in (6.5) we set

$$\mathscr{F}^{\perp} = -\mathscr{F}^2 \mathbf{e}_1 + \mathscr{F}^1 \mathbf{e}_2 \equiv \mathbf{e}_3 \times \mathscr{F}. \tag{6.6}$$

Note that $(\mathscr{F}^{\perp})^{\perp} = -\mathscr{F}$. We also use the notation

$$\operatorname{grad}_{\mathrm{T}} f = f_{x_1}\mathbf{e}_1 + f_{x_2}\mathbf{e}_2, \qquad \operatorname{div}_{\mathrm{T}} \mathbf{F} = \operatorname{div} \mathscr{F} = \mathscr{F}^1_{x_1} + \mathscr{F}^2_{x_2} \tag{6.7}$$

and be observe that, for the decomposition (6.5),

$$\operatorname{curl} \mathbf{F} = \mathscr{F}^{\perp}_{x_3} - (\operatorname{grad}_{\mathrm{T}} f)^{\perp} - (\operatorname{div} \mathscr{F}^{\perp})\,\mathbf{e}_3. \tag{6.8}$$

We have $\operatorname{grad} \chi = \sqrt{-i}\,\beta\chi\mathbf{e}_3$ and

$$\operatorname{curl} \chi\mathbf{F} = \chi\{\sqrt{-i}\,\beta\mathscr{F}^{\perp} + \mathscr{F}^{\perp}_{x_3} - (\operatorname{grad} f)^{\perp} - (\operatorname{div} \mathscr{F}^{\perp})\,\mathbf{e}_3\}. \tag{6.9}$$

We are now ready for the substitution of (6.3) and (6.4) into (6.1) and (6.2). The first observation is that we have

$$\operatorname{curl} E_n = H_n, \qquad \operatorname{curl} H_n = \alpha^2 E_n \qquad \text{in} \quad x_3 > 0. \tag{6.10}$$

For $x_3 < 0$ the situation is complicated by the presence of the term χ. From (6.9) and (6.4) we have

$$\operatorname{curl} \mathbf{E} \sim \chi \Big\{ \sqrt{-i}\,\beta\mathscr{E}^{\perp}_0 + \sum_{n=0}^{\infty} [\sqrt{-i}\,\mathscr{E}^{\perp}_{n+1} + \mathscr{E}^{\perp}_{n,x_3} - (\operatorname{grad} e_n)^{\perp} - (\operatorname{div} \mathscr{E}^{\perp}_n)\,\mathbf{e}_3]\,\beta^{-n} \Big\}, \tag{6.11}$$

$$\operatorname{curl} \mathbf{H} \sim \chi \Big\{ \sqrt{-i}\,\beta\mathscr{H}^{\perp}_0 + \sum_{n=0}^{\infty} [\sqrt{-i}\,\mathscr{H}^{\perp}_{n+1} + \mathscr{H}^{\perp}_{n,x_3} - (\operatorname{grad} h_n)^{\perp} - (\operatorname{div} \mathscr{H}^{\perp}_n)\,\mathbf{e}_3]\,\beta^{-n} \Big\}, \tag{6.12}$$

where $\mathbf{E}_n = \mathscr{E}_n + \mathbf{e}_3 e_n$, $H_n = \mathscr{H}_n + \mathbf{e}_3 h_n$.

We equate (6.11) to

$$\mathbf{H} = \chi \sum_{n=0}^{\infty} (\mathscr{H}_n + \mathbf{e}_3 h_n)\,\beta^{-n}$$

and (6.12) to

$$i\beta^2\mathbf{E} = \chi \Big\{ i\beta^2\mathscr{E}_0 + i\beta^2\mathbf{e}_0 + i\beta\mathscr{E}_1 + i\beta\mathbf{e}_1 + \sum_{n=0}^{\infty} (i\mathscr{E}_{n+2} + \mathbf{e}_3 i\mathbf{e}_{n+2})\,\beta^{-n} \Big\}.$$

Then we equate tangential and normal components of coefficients of like powers of β. This shows us first that

$$\mathscr{E}_0 = -\mathscr{E}^0 = -\mathbf{E}^0_{\mathrm{T}}, \qquad e_0 \equiv 0, \quad e_1 \equiv 0. \tag{6.13}$$

The next powers yield the equations

$$\sqrt{-i}\,\mathscr{E}_1^{\perp} = \mathscr{H}_0; \qquad \sqrt{-i}\,\mathscr{H}_0^{\perp} = i\mathscr{E}_1 \tag{6.14}$$

$$h_0 = \operatorname{div} \mathscr{E}_0^{\perp} = 0. \tag{6.15}$$

Observe that the two equations (6.14) are the same. The next set of equations is (with $e_1 = 0$, $h_0 = 0$, $\mathscr{E}_0 = \mathbf{0}$)

$$\sqrt{-i}\,\mathscr{E}_2^{\perp} + \mathscr{E}_{1,x_3}^{\perp} = \mathscr{H}_1; \qquad \sqrt{-i}\,\mathscr{H}_1^{\perp} + \mathscr{H}_{0,x_3}^{\perp} = i\mathscr{E}_2 \tag{6.16}$$

$$-\operatorname{div} \mathscr{E}_1^{\perp} = h_1, \qquad -\operatorname{div} \mathscr{H}_0^{\perp} = ie_2. \tag{6.17}$$

Before considering the general case let us pause to see what we can do so far. We can eliminate $\mathscr{E}_2$ from Eqs. (6.16), $(6.16)_2$ to obtain

$$-\sqrt{-i}\,\mathscr{H}_1 - \mathscr{H}_{0,x_3} = i\mathscr{E}_2^{\perp} = -\sqrt{-i}\,\sqrt{-i}\,\mathscr{E}_2^{\perp}$$

or

$$\sqrt{-i}\,\mathscr{E}_2^{\perp} = \mathscr{H}_1 + \sqrt{i}\,\mathscr{H}_{0,x_3}.$$

Substituting in $(6.16)_1$ gives

$$\mathscr{H}_1 + i\mathscr{H}_{0,x_3} + \mathscr{E}_{1,x_3}^{\perp} = \mathscr{H}_1 \qquad \text{or} \qquad \sqrt{i}\,\mathscr{H}_{0,x_3} + \mathscr{E}_{1,x_3}^{\perp} = 0.$$

But (6.14) yields $\sqrt{i}\,\mathscr{H}_{0,x_3} - \mathscr{E}_{1,x_3}^{\perp} = 0$. Hence we conclude that

$$\begin{gathered} \mathscr{H}_{0,x_3} \equiv \mathscr{E}_{1,x_3} \equiv 0 \qquad \text{or} \qquad \mathscr{E}_1(x_1, x_2, x_3) \equiv \mathscr{E}_1(x_1, x_2, 0^-), \\ \mathscr{H}_0 \equiv \sqrt{-i}\,\mathscr{E}_1^{\perp} \text{ in } x_3 < 0. \end{gathered} \tag{6.18}$$

We can now start the recursion process. First we use $(6.10)_0$ and (6.13) to conclude that

$$\operatorname{curl} \mathbf{E}_0 = \mathbf{H}_0, \qquad \operatorname{curl} \mathbf{H}_0 = \alpha^2 E_0 \qquad \text{in} \quad x_3 > 0,$$

$$\mathbf{E}_0^+ = -\mathbf{E}_\mathrm{T}^0 \qquad \text{on} \quad x_3 = 0.$$

Thus $(\mathbf{E}_0, \mathbf{H}_0)$ is just the solution of $(\mathbf{P}_{\alpha\infty})$ which we can solve. But from (6.2) we obtain

$$\mathscr{H}_0^- = \mathscr{H}_0^+ = (\mathbf{H}_0)_\mathrm{T}^+ \qquad \text{on} \quad x_3 = 0. \tag{6.19}$$

The right side of (6.19) is known (and easily computed with our process). Then (6.14), (6.2), and (6.19) yield

$$(\mathbf{E}_1)_\mathrm{T}^+ = (\mathbf{E}_1)_\mathrm{T}^- = \mathscr{E}_1^- = -\sqrt{i}\,(\mathscr{H}_0^{\perp})^- = -\sqrt{i}\,((\mathbf{H}_0)_\mathrm{T}^+)^{\perp}. \tag{6.20}$$

Then by $(6.10)_1$ we have a new problem for $(\mathbf{E}_1, \mathbf{H}_1)$ which is just like $(P_{\alpha\infty})$ but with new boundary values for $\mathbf{E}_T$ as given by (6.20). Again this is solvable.

We have thus found completely the first two terms in the expansion for $\mathbf{E}$ and $\mathbf{H}$ in $x_3 > 0$. In $x_3 < 0$ we have $\mathscr{E}_0 = \mathbf{0}$, $e_0 \equiv e_1 \equiv h_0 = 0$. By (6.18) we can also determine $\mathscr{E}_1$ and $\mathscr{H}_0$ in $x_3 < 0$, both being independent of x_3. Equation $(6.17)_1$ then determines h_1. We see that $(6.17)_2$ also determine e_2. What we have not determined so far is $\mathscr{H}_1$ in $x_3 < 0$. For this we have to start on the next step. The next equations we obtain are

$$\begin{aligned} \sqrt{-i}\, \mathscr{E}_3^{\perp} + \mathscr{E}^{\perp}_{2,x_3} - (\operatorname{grad}_T e_2)^{\perp} &= \mathscr{H}_2, \\ \sqrt{-i}\, \mathscr{H}_2^{\perp} + \mathscr{H}^{\perp}_{1,x_3} - (\operatorname{grad} h_1)^{\perp} &= i\mathscr{E}_3. \end{aligned} \tag{6.21}$$

We eliminate $\mathscr{E}_3$ here just as we did $\mathscr{E}_2$ in (6.16) and obtain

$$\sqrt{i}\, \mathscr{H}_{1,x_3} - \sqrt{i}\, \operatorname{grad}_T h_1 + \mathscr{E}^{\perp}_{2,x_3} - (\operatorname{grad}_T e_2)^{\perp} = 0. \tag{6.22}$$

But from $(6.16)_1$ we obtain

$$\mathscr{E}^{\perp}_{2,x_3} = \sqrt{i}\, \mathscr{H}_{1,x_3} - \sqrt{i}\, \mathscr{E}^{\perp}_{1,x_3x_3}. \tag{6.23}$$

If we substitute (6.23) into (6.22) we obtain

$$2\sqrt{i}\, \mathscr{H}_{1,x_3} = \sqrt{i}\, \mathscr{E}^{\perp}_{1,x_3x_3} + \sqrt{i}\, \operatorname{grad}_T h_1 + \sqrt{i}\, (\operatorname{grad}_T e_2)^{\perp}. \tag{6.24}$$

Now everything on the right side of (6.24) is known from earlier steps and we also have $\mathscr{H}_1(x_1, x_2, 0) = \mathbf{H}_1(x_1, x_2, 0^+)_T$, also known. Hence we can solve for $\mathscr{H}_1$.

This process can be continued recursively. Let us outline the steps. We have the following equations:

$$\sqrt{-i}\, \mathscr{E}^{\perp}_{n+2} + \mathscr{E}^{\perp}_{n+1,x_3} - (\operatorname{grad}_T e_{n+1})^{\perp} = \mathscr{H}_{n+1} \qquad n = 0, 1, 2, \ldots \qquad (\mathrm{I}_n)$$

$$\sqrt{-i}\, \mathscr{H}^{\perp}_{n+1} + \mathscr{H}^{\perp}_{n,x_3} - (\operatorname{grad}_T h_n)^{\perp} = i\mathscr{E}_{n+2} \qquad n = 0, 1, 2, \ldots \qquad (\mathrm{II}_n)$$

$$h_n = -\operatorname{div} \mathscr{E}^{\perp}_n \qquad n = 0, 1, 2, \ldots \qquad (\mathrm{I}'_n)$$

$$e_{n+2} = i \operatorname{div} \mathscr{H}^{\perp}_n \qquad n = 0, 1, 2, \ldots. \qquad (\mathrm{II}'_n)$$

We eliminate $\mathscr{E}_{n+2}$ between (I_n) and (II_n) as above and obtain

$$\sqrt{i}\, \mathscr{H}_{n,x_3} - \sqrt{i}\, \operatorname{grad}_T h_n + \mathscr{E}^{\perp}_{n+1,x_3} - (\operatorname{grad}_T e_{n+1})^{\perp} = 0$$

$$n = 0, 1, 2, \ldots. \qquad (\mathrm{III}_n)$$

But (I_n) gives $\mathscr{E}^{\perp}_{n+1,x_3} = \sqrt{i}\,\mathscr{H}_{n,x_3} - \sqrt{i}\,\mathscr{E}^{\perp}_{n,x_3x_3} + \sqrt{i}\,(\text{grad}\, e_n)^{\perp}_{x_3}$ and if we substitute this into (III_n) we obtain

$$2\sqrt{i}\,\mathscr{H}_{n,x_3} = \sqrt{i}\,\text{grad}\, h_n + \sqrt{i}\,\mathscr{E}^{\perp}_{n,x_3x_3} - \sqrt{i}\,(\text{grad}\, e_n)^{\perp}_{x_3} + (\text{grad}\, e_{n+1})^{\perp}. \quad (\mathrm{IV}_n)$$

Now we can describe the recursion process. Suppose we have calculated

$$\mathscr{H}_0,\ldots,\mathscr{H}_{n-1}, h_0,\ldots, h_n, \mathscr{E}_0,\ldots,\mathscr{E}_n, e_0,\ldots, e_{n+1} \qquad \text{in} \quad x_3 < 0.$$

Then we will have also been able to calculate

$$(\mathbf{E}_0, \mathbf{H}_0),\ldots, (\mathbf{E}_n, \mathbf{H}_n) \qquad \text{in} \quad x_3 > 0.$$

For each of these satisfies a problem of the form $(\mathrm{P}_{\alpha\infty})$ with $(E_n)^+_{\mathrm{T}} = \mathscr{E}^-_n$. It follows that the right side of (IV_n) is known; hence we know $\mathscr{H}_{n,x_3}$ in $x_3 < 0$. But we have also $\mathscr{H}^-_n = (\mathbf{H}_n)^+_{\mathrm{T}}$ and thus we determine $\mathscr{H}_n$ by integration. Once $\mathscr{H}_n$ is known (II'_n) yields e_{n+2} and (II_n) yields $\mathscr{E}_{n+1}$. Then (I'_n) gives h_{n+1} and we can repeat the step.

If the co-ordinate systems are not orthonormal the equations above become somewhat more complicated once one goes beyong the first steps. The complications come in the calculation of the coefficients in Ω'. What happens is that instead of obtaining the $\mathscr{H}_n$'s by a simple integration one has first order differential equations to solve for them. It is important to note that to obtain the first order connection to the exterior field, that is, $(\mathbf{E}_1, \mathbf{H}_1)$, it is not necessary to calculate any of the terms in the inner expansion. Equation (6.20) remains valid so we have boundary values for $(\mathbf{E}_1)_{\mathrm{T}}$ which are determined solely from $\mathbf{H}_0$, the infinite conductivity approximation.

As indicated in the Introduction some more details of the asymptotic procedure, including numerical results, as it applies to two-dimensional problems, appear in [4]. Although the analysis is predicated on the assumption that β is large the numerical experiments in [4] indicate that it is in fact valid over a very large range of β's. A verification that the formal procedure is a valid asymptotic series is presented in [8] for the two dimensional problem of [4].

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