A class of Frattini-like subgroups of a finite group

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Abstract


Let $G$ be a finite group and $\pi$ a set of primes. We consider the families of subgroups of $G$:

$\mathcal{F}_1 = \{ M \leq G, |G:M|_{\pi} = 1 \}$,

$\mathcal{F}_2 = \{ M \leq G, |G:M|_{\pi} = 1, |G:M| \text{ is composite} \}$.

Denote $Q_{\pi}(G) = \bigcap \{ M : M \in \mathcal{F}_1 \}$ if $\mathcal{F}_1$ is nonempty, otherwise $Q_{\pi}(G) = G$ and $S_{\pi}(G) = \bigcap \{ M : M \in \mathcal{F}_2 \}$ if $\mathcal{F}_2$ is nonempty, otherwise $S_{\pi}(G) = G$. The purpose of this paper is to investigate these subgroups further.

1. Introduction

Of late there has been considerable interest in the study of analogs of the Frattini subgroup of a finite group and investigation of their properties, particularly, their influence on the structure of the group (see [1-4]). In [2], Bhattacharya and Mukherjee introduce the subgroups $Q_{\pi}(G)$ and $S_{\pi}(G)$ and exhibit their relationship with the given group $G$ under the hypothesis of $G$ being $\pi$-solvable. In [3], Guo gets a result with the same hypothesis. The objective of this paper is to investigate these groups further and to show that the $\pi$-solvable assumption is unnecessary in their main result. All the main results in [2] and [3] have been generalized.

The main results of this paper are as follows:

Theorem. Let $G$ be a finite $\pi$-separable group. Then:

(i) $\Phi_{\pi}(G)/O_{\pi}(G) = \Phi(G/O_{\pi}(G))$ is a nilpotent $\pi'$-group and $\Phi_{\pi}(G)$ is $\pi$-closed.

(ii) Both $S_{\pi}(G)/O_{\pi}(G)$ and $S_{\pi}(G)/O_{\pi}(G)$ are supersolvable.

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Corollary. Let $G$ be a finite $\pi$-separable group. Suppose that both $N$ and $M$ are normal subgroups of $G$ with $N \leq \Phi_{\pi}(G)$. Then $M$ is $\pi$-closed if and only if $MN/N$ is $\pi$-closed.

For convenience, we give some notations and definitions first.

Let $\pi$ be any set of primes and $\pi'$ the complementary set of primes. Let $G$ be a finite group. Then we denote $N \leq G$ to indicate that $N$ is a maximal subgroup of $G$. Also, $|G : M|_\pi$ denotes the $\pi$-part of $|G : M|$. Consider the following families of subgroups:

- $\mathcal{F}_1 = \{M: M \leq G, |G : M|_\pi = 1\}$,
- $\mathcal{F}_2 = \{M: M \leq G, |G : M|_\pi = 1, |G : M|$ is composite $\}$,
- $\mathcal{F}_3 = \{M: M \leq G, |G : M|$ is composite $\}$.

Definition. $\Phi_{\pi}(G) = \bigcap \{M: M \in \mathcal{F}_1\}$ if $\mathcal{F}_1$ is nonempty, otherwise $\Phi_{\pi}(G) = G$.

$S_{\pi}(G) = \bigcap \{M: M \in \mathcal{F}_2\}$ if $\mathcal{F}_2$ is nonempty, otherwise $S_{\pi}(G) = G$.

$L(G) = \bigcap \{M: M \in \mathcal{F}_3\}$ if $\mathcal{F}_3$ is nonempty, otherwise $L(G) = G$.

A group $G$ is called $\pi$-separable if every composition factor of $G$ is either a $\pi$-group or $\pi'$-group. A group $G$ is called $\pi$-solvable if every composition factor of $G$ is either a $\pi'$-group or a $p$-group with $p \in \pi$. Clearly, $G$ is $\pi$-separable if and only if $G$ is $\pi'$-separable. Each $\pi$-solvable group is $\pi$-separable.

When $G$ is $\pi$-solvable, one can easily show that both $\Phi_{\pi}(G)$ and $S_{\pi}(G)$ are solvable. If we only assume that $G$ is $\pi$-separable, then $S_{\pi}(G)$ need not to be solvable. For instance, let $G$ be a nonabelian simple group and $\pi = \pi(G)$. Then $G$ is $\pi$-separable but not $\pi$-solvable, both $\mathcal{F}_1$ and $\mathcal{F}_2$ are empty and $\Phi_{\pi}(G) = S_{\pi}(G) = G$.

All the groups in this paper are finite.

2. Preliminary results

Property 2.1. $G$ is $\pi$-separable $\iff$ every chief factor of $G$ is either a $\pi$-group or $\pi'$-group.

$G$ is $\pi$-solvable $\iff$ every chief factor of $G$ is either a $\pi'$-group or a $p$-group with $p \in \pi$.

Lemma 2.2. Let $K \leq G$. Then:

1. $\Phi_{\pi}(G)/K \leq \Phi_{\pi}(G/K)$; consequently, if $K \leq \Phi_{\pi}(G)$, it follows that $\Phi_{\pi}(G/K) = \Phi_{\pi}(G/K)$.
2. $S_{\pi}(G/K) = S_{\pi}(G/K)$; consequently, if $K \leq S_{\pi}(G)$, it follows that $S_{\pi}(G/K) = S_{\pi}(G/K)$.
3. $O_{\pi}(G) \Phi(G) \leq \Phi_{\pi}(G) \leq S_{\pi}(G)$ and $O_{\pi'}(G) \Phi(G) \leq \Phi_{\pi}(G) \leq S_{\pi}(G)$.
4. $1 = O_{\pi}(G/O_{\pi}(G)) = O_{\pi}(G/O_{\pi}(G)) = O_{\pi}(G/S_{\pi}(G))$. 


Proof. (1) and (2) are clear by definition (see [2, Lemma 1]). \( \forall M \leq G \): if \( O_\pi(G) \neq M \), then \( G = MO_\pi(G) \) and so \( |G : M| \) is a \( \pi \)-number. Hence \( O_\pi(G) = \Phi_\pi(G) \leq S_\pi(G) \). Since \( \Phi(G) \leq \Phi_\pi(G) \), (3) follows. \( O_\pi(G/\Phi_\pi(G)) \leq \Phi_\pi(G)/\Phi_\pi(G) = 1 \) by (1) and (3). The same proof for \( S_\pi(G) \) yields (4).

Lemma 2.3. Let \( G \) be a finite \( \pi \)-separable group. Then:

1. Every maximal subgroup of \( G \) has index which is either \( \pi \)-number or a \( \pi' \)-number.
2. \( \Phi_\pi(G) \cap \Phi_\pi(w) = \Phi(w) \) and \( S_\pi(G) \cap S_\pi(w) = L(G) \) is solvable.
3. If \( L \trianglelefteq G \) and \( L \) is a \( \pi' \)-subgroup, then \( L \leq \Phi_\pi(G) \Leftrightarrow L \leq \Phi(G) \) and \( L \leq S_\pi(G) \Leftrightarrow L \leq L(G) \).

Proof. (1) We use induction on \( |G| \). Since \( G \) is \( \pi \)-separable, \( O_\pi(G)O_\pi(G) \neq 1 \). Without loss of generality, we assume that \( O_\pi(G) \neq 1 \). If \( G = O_\pi(G) \), there is nothing to prove. Assume that \( G \neq O_\pi(G) \). \( \forall M \leq G \): if \( O_\pi(G) \neq M \), then \( G = O_\pi(G)M \) and so \( |G : M| \) is a \( \pi \)-number. If \( O_\pi(G) \subseteq M \), then \( |G : M| = |G/O_\pi(G) : M/O_\pi(G)| \) is either a \( \pi \)-number or a \( \pi' \)-number. We are done.

(2) The equations follow from (1). We show that \( L(G) \) is solvable by induction on \( |G| \). We assume that \( L(G) > 1 \). Let \( p = \max\{q : q \in \pi(L(G))\} \) and \( P \in \text{Syl}_p(L(G)) \). If \( N_{L(G)}(P) = G \), then \( 1 \neq P \leq G \) and \( P \leq S_\pi(G) \cap S_\pi(w) \). Then \( L(G/P) = L(G)/P \) by Lemma 2.2(2). Hence both \( P \) and \( L(G)/P \) are solvable and so is \( L(G) \). If \( N_{L(G)}(P) \neq G \), then \( \exists M \leq G \) with \( N_{L(G)}(P) \leq M \). Since \( L(G) \leq G \), the Frattini argument yields that \( G = L(G)N_{L(G)}(P) = L(G)M \). By (1), \( |G : M| \) is either a \( \pi \)-number or a \( \pi' \)-number. If \( |G : M| \) is composite, then \( L(G) \leq M \), a contradiction. Therefore, \( |G : M| = q \) is a prime which divides \( |L(G)| \). Since \( N_{L(G)}(P) \leq M, N_{L(G)}(P) \leq M \cap L(G), |G : M| = |L(G) : M \cap L(G)| = 1 \) (mod \( p \)) which is contrary to the choice of \( p \). The result now follows.

(3) It follows directly from (2) and Lemma 2.2(3).

3. Properties of \( \Phi_\pi(G) \)

We call an element \( x \) in \( G \) a \( \pi \)-non-generator if for any subset \( T \subseteq G \) with \( |G : \langle T \rangle|_\pi = 1 \), \( G = \langle T, x \rangle \) implies that \( G = \langle T \rangle \).

Theorem 3.1. \( \Phi_\pi(G) = \langle x : x \in G, x \text{ is a } \pi \text{-non-generator of } G \rangle \).

Proof. Let \( x \) be a \( \pi \)-non-generator of \( G \). If \( x \not\in \Phi_\pi(G) \), then there exists a maximal subgroup \( M \) of \( G \) with \( |G : M|_\pi = 1 \) such that \( x \not\in M \). Hence \( G = \langle M, x \rangle \neq M \), contrary to the fact that \( x \) is a \( \pi \)-non-generator. Conversely, \( \forall x \in \Phi_\pi(G) \), if \( x \) is not a \( \pi \)-non-generator, then there exists a subset \( T \) of \( G \) with \( |G : \langle T \rangle|_\pi = 1 \) and \( G = \langle T, x \rangle \) but \( G \neq \langle T \rangle \). Take \( M \) to be a maximal subgroup of \( G \) containing \( \langle T \rangle \), then \( M \) has \( \pi' \)-index and \( x \in M \), a contradiction.
Corollary 3.2. (1) Suppose that \( N \leq G \) and \( U \leq G \). If \( N \leq \Phi_\pi(U) \), then \( N \not\leq \Phi_\pi(G) \).

(2) If \( N \not\leq G \), then \( \Phi_\pi(N) \leq \Phi_\pi(G) \).

Proof. (1) If \( N \not\leq \Phi_\pi(G) \), then there is \( M \leq G \) with \( |G : M|_\pi = 1 \) and \( N \leq M \).

Hence \( G = NM = UM, U = U \cap G = N(U \cap M) \), since

\[
|U : U \cap M| = |N(U \cap M)|/|U \cap M| = |N|/|N \cap M| = |NM|/|M| = |G : M|.
\]

Hence \( N \leq \Phi_\pi(U) \leq U \cap M \), a contradiction.

(2) If \( N \not\leq G \), then \( \Phi_\pi(N) \) char \( N \not\leq G \). Hence \( \Phi_\pi(N) \not\leq G \). By (1), \( \Phi_\pi(N) \not\leq \Phi_\pi(G) \). Assume that \( N \not\leq N_1 \leq \cdots \leq N_k = G \), then \( \Phi_\pi(N_i) \leq \Phi_\pi(N_i) \leq \cdots \leq \Phi_\pi(G) \). \( \square \)

We call a group \( G \) \( \pi \)-closed if \( O_\pi(G) \) is a Hall \( \pi \)-subgroup of \( G \).

Theorem 3.3. Let \( G \) be a finite \( \pi \)-separable group. Then:

(1) \( \Phi_\pi(G)/O_\pi(G) = \Phi(G/O_\pi(G)) \) is a nilpotent \( \pi' \)-group and \( \Phi_\pi(G) \) is \( \pi \)-closed.

(2) Let \( K \leq G \) and \( K \leq \Phi_\pi(G) \). Then \( M \) is \( \pi \)-closed if and only if \( MK/K \) is \( \pi \)-closed for every normal subgroup \( M \) of \( G \).

(3) If \( G = G_1 \times \cdots \times G_k \), then \( \Phi_\pi(G) = \Phi_\pi(G_1) \times \cdots \times \Phi_\pi(G_k) \).

Proof. (1) By Lemma 2.2(4), we only need to show that

\[
\Phi_\pi(G)/O_\pi(G) = \Phi(G/O_\pi(G)).
\]

Assume that the result is false and consider a counterexample \( G \) with minimal order. Then:

(i) \( O_\pi(G) = 1 \).

In fact, if \( O_\pi(G) \neq 1 \), consider \( \tilde{G} = G/O_\pi(G) \), by Lemma 2.2(1), \( \Phi(G/O_\pi(G)) = \Phi_\pi(G)/O_\pi(G) \) and \( O_\pi(G/O_\pi(G)) = 1 \). Hence

\[
\Phi_\pi(G)/O_\pi(G) \cong \Phi_\pi(\tilde{G})/O_\pi(\tilde{G}) = \Phi(\tilde{G}/O_\pi(\tilde{G})) \cong \Phi(G/\Phi_\pi(G)),
\]

a contradiction.

(ii) \( \Phi(G) = 1 \).

If \( \Phi(G) \neq 1 \), we consider \( \tilde{G} = G/\Phi(G) \). Since \( \Phi(G) \) is nilpotent and \( O_\pi(\Phi(G)) \leq \Phi_\pi(G) = 1 \) by (i), we have that \( \Phi(G) \) is a \( \pi' \)-subgroup. Let \( L/\Phi(G) = O_\pi(G/\Phi(G)) \not\leq G/\Phi(G) \). Then \( L \not\leq G \) and \( \Phi(G) \) is a Hall \( \pi' \)-subgroup of \( L \). By the Schur–Zassenhaus Theorem, [5. 9.1.2], there is a Hall
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Let \( L \) be a \( \pi \)-subgroup of \( L \) such that \( L = L_1 \Phi(G) \) and all the Hall \( \pi \)-subgroups of \( L \) are conjugate to \( L_1 \) in \( L \). A direct generalization of the Frattini argument yields that \( G = N_G(L_1) \Phi(G) = N_G(L_1) \), i.e., \( L_1 \subseteq \Phi_\pi(G) \). This implies that \( O_\pi(G) = 1 \), since \( \Phi_\pi(G) - \Phi_\pi(G)/O_\pi(G) \cong \Phi(G)/O_\pi(G) \). By the choice of \( G \),

\[ \Phi_\pi(G)/O_\pi(G) \cong \Phi_\pi(G)/O_\pi(G) = \Phi(G)/O_\pi(G) \cong \Phi(G/O_\pi(G)), \]

contrary to our assumption.

(iii) The conclusion.

If \( \Phi(G) \neq 1 \), then there exists a minimal normal subgroup \( N \) of \( G \) which is contained in \( \Phi(G) \). Since \( G \) is \( \pi \)-separable, \( N \) is either a \( \pi \)-group or a \( \pi' \)-group. From (i) follows that \( N \) is a \( \pi' \)-group. Now from Lemma 2.3(3) it follows that \( 1 \neq N \leq \Phi(G) = 1 \), a contradiction. This shows that \( \Phi(G) = \Phi(G) = 1 \), contrary to the choice of \( G \). The result now follows.

(2) If \( MK/K \) is \( \pi \)-closed, then \( L/K \leq G/K \), where \( L/K \) is a Hall \( \pi \)-subgroup of \( MK/K \). We prove that there exists a Hall \( \pi \)-subgroup \( L_1 \) of \( L \) and every Hall \( \pi \)-subgroup of \( L \) conjugates to \( L_1 \) in \( L \) by use induction on \( |L| \). In fact, since \( K \subseteq \Phi_\pi(G) \) is \( \pi \)-closed, if \( O_\pi(K) \neq 1 \), the induction yields the result. If \( O_\pi(K) = 1 \), then \( K \) is a \( \pi' \)-subgroup and hence \( K \) is a normal Hall \( \pi' \)-subgroup of \( L \). Our result now follows from the Schur–Zassenhaus Theorem [5, 9.1.2]. Let \( L_1 \) be a Hall \( \pi \)-subgroup of \( L \). Then \( L_1 \) is also a Hall \( \pi \)-subgroup of \( MK \). Since \( G \leq L \),

\[ |G : N_G(L_1)| = |L : N_L(L_1)| \leq |L : L_1| \]

is a \( \pi' \)-number, Lemma 2.3(1) yields that \( G = N_G(L_1) \). Hence \( L_1 \leq G \). Therefore, \( MK \) is \( \pi \)-closed and so is \( M \).

(3) Since \( O_\pi(G_1 \times \cdots \times G_k) = O_\pi(G_1) \times \cdots \times O_\pi(G_k) \) and \( \Phi(G_1 \times \cdots \times G_k) = \Phi(G_1) \times \cdots \times \Phi(G_k) \), by (1), \( \Phi_\pi(G/O_\pi(G)) = \Phi(G/O_\pi(G)) \); this yields our result. \( \square \)

4. Properties of \( S_\pi(G) \)

Theorem 4.1. Let \( G \) be a finite group. Let \( \mathcal{F} \) be a solvable saturated formation containing the formation of finite nilpotent group. Suppose that \( M \) is a normal subgroup of \( G \) with \( \Phi(G) \leq M \). Then \( M \in \mathcal{F} \) if and only if \( M/\Phi(G) \in \mathcal{F} \).

Proof. Assume that the result is false and consider a counterexample \( G \) with minimal order. Then \( \Phi(G) \neq 1 \). Let \( N \) be a minimal normal subgroup of \( G \) with \( N \leq \Phi(G) \). Then:

(1) \( M \) is solvable and \( M/N \in \mathcal{F} \).
In fact, $\mathcal{F}$ is solvable and $M/\Phi(G) \in \mathcal{F}$ yields that $M$ is solvable. Since $N \cong \Phi(G)$,

$$\frac{(M/N)/\Phi(G/N)}{(M/N)/(\Phi(G)/N)} \cong M/\Phi(G) \in \mathcal{F},$$

the minimal choice of $G$ yields that $M/N \in \mathcal{F}$.

(2) There exists $V \in \mathcal{F}$ such that $M = NV$ with $V \leq G$.

From (1) it follows that $M$ is solvable and $M_{\pi} \leq N$. By the Gaschütz Theorem [5, 9.5.4], there exists an $\mathcal{F}$-covering subgroup $V$ of $M$ such that $M = M_{\pi}V = NV$.

Since $M \leq G$, $\forall g \in G$, $NV = M = M^{g} = NV^{g}$ and $V^{g}$ is also a $\mathcal{F}$-covering subgroup of $M$, the same theorem asserts that $V^{g} = V^{h}$ for an element $h \in H$. That is, $G = NN_{G}(V) = \Phi(G)N_{G}(V) = N_{G}(V)$ and so $V \leq G$.

(3) $M \in \mathcal{F}$.

Since $N$ is a minimal normal subgroup of $G$ and $V \leq G$, $N \cap V \leq G$. Hence $N \cap V = 1$ or $N \cap V = N$. If $N \cap V = N$, then $N \leq V$ and $M - NV - V \in \mathcal{F}$ by (2).

If $N \cap V = 1$, then $M = N \times V \in \mathcal{F}$ since $N$ is nilpotent and $N \in \mathcal{F}$ by our assumption. Hence $M \in \mathcal{F}$.

The last contradiction yields our result. $\square$

**Theorem 4.2.** Let $G$ be a finite $\pi$-separable group. Then $S_{\pi}(G)/O_{\pi}(G)$ is supersolvable.

**Proof.** Assume that the result is false and consider a counterexample $G$ with minimal order. Then:

(i) $O_{\pi}(G) = 1$ and $S_{\pi}(G) \neq 1$.

It is a trivial fact as in the proof of Theorem 3.3.

(ii) $\Phi(G) = 1$.

As proved in Theorem 3.3, we can show that $O_{\pi}(\tilde{G}) = 1$. Since $S_{\pi}(\tilde{G}) = S_{\pi}(\tilde{G})/O_{\pi}(\tilde{G}) \cong S_{\pi}(G)/O_{\pi}(G)$, by the choice of $G$,

$$S_{\pi}(G)/O_{\pi}(G) \cong S_{\pi}(\tilde{G})/O_{\pi}(\tilde{G}) = S_{\pi}(G)/\Phi(G)$$

is supersolvable. By Theorem 4.1, $S_{\pi}(G)$ is supersolvable and so is $S_{\pi}(G)/O_{\pi}(G)$, contrary to our assumption.

For simplicity, we denote $S_{\pi}(G)$ to be $S$ and the Fitting subgroup of $S$ by $F$.

(iii) $F = N_{1} \times \cdots \times N_{k}$, where $N_{i}$ are minimal normal subgroups of $G$. $G = FL$ with $L \cap F = 1$ for a subgroup of $G$.

Let $N$ be a minimal normal subgroup of $G$ which contained in $S_{\pi}(G)$. Since $G$ is $\pi$-separable and $O_{\pi}(N) \cong O_{\pi}(G) = 1$, we have that $N$ is a $\pi'$-group and so $N \leq S_{\pi}(G) \cap O_{\pi}(G)$ is solvable by Lemma 2.3(3). Hence $N$ is a solvable minimal normal subgroup of $G$ and so $N$ is an elementary abelian $p'$-subgroup with $p \in \pi'$. Certainly $N \leq F$. Since $F$ is a nilpotent normal subgroup of $G$, $O_{\pi}(F) = 1 = \Phi(F)$ by (i) and (ii). $F$ is abelian. Let $H$ be a maximal among all
subgroups of \( F \) which can be expressed as the direct product of minimal normal subgroups of \( G \). Then \( H \trianglelefteq G \) and \( H \) is abelian. Let \( L = \min\{ T : T \trianglelefteq G, HT = G \} \). Then \( H \cap L = 1 \). In fact, \( H \cap L \triangleleft G \). If \( H \cap L \neq 1 \), since \( \Phi(G) = 1 \), \( \exists M < G \) such that \( H \cap L \not\subseteq M \) and so \( G = (L \cap H)M \). Note that \( L = L \cap G = (L \cap M)(L \cap M) \) and \( L \cap M \neq L \). However, \( G = LH = (L \cap M)H \), contrary to the minimal choice of \( L \). Since \( F = H(F \cap I) \) and \( F \cap I \triangleleft F \), \( F \cap L \neq F \cap I \), then there is a minimal normal subgroup \( N \) of \( G \) with \( N \leq F \cap L \). As \( H \cap L = 1 \), we conclude that \( H < N \times H \). This contradicts the maximal choice of \( H \) and therefore \( F \cap L = 1 \). This follows that \( F = H \) and the result follows.

(iv) \(|N_i|\) is a prime for all \( i \in \{1, \ldots, k\} \), \( S' \leq C_s(F) = F \), \( S/F \) is abelian group.

In fact, (iii) implies that \(|N_i| = p_i^{t_i}\) with \( p_i \in \pi' \). Since \( \Phi(G) = 1 \) and \( 1 \neq N_i \trianglelefteq G \), \( \exists M_i < G \) with \( N_i \not\trianglelefteq M \) and \( G = N_iM_i \). It is clear that \( N_i \cap M_i = 1 \) and \( |G : M_i| = |N_i| = p_i^{t_i} \) is a \( \pi' \)-number. If \( |G : M_i| \) is composite, then \( N_i \triangleleft S \triangleleft M_i \), a contradiction. Hence \(|N_i|\) is a prime.

\[
G/C_G(N_i) = N_\pi(N_i)/C_G(N_i) \trianglelefteq \text{Aut}(N_i)
\]

is cyclic. Therefore, \( G' \leq C_G(N_i) \) and \( G' \leq \cap_{i=1}^k C_G(N_i) = C_G(F) \), \( S' \leq G' \cap S \triangleleft C_s(F) \). Since \( F \) is abelian, \( S = F(S \cap L) \) with \( L \) chosen as in (iii), \( C_s(F) = F(L \cap C_s(F)) = FL_1 \), where \( L_1 = L \cap C_s(F) \triangleleft L \) and commutes with \( F \), hence \( L_1 \triangleleft S \). If \( L_1 \neq 1 \), then there is a minimal normal subgroup \( N \) of \( S \) which is contained in \( L_1 \). \( N \) is a subnormal subgroup of \( G \) and \( \text{O}_\pi(N) \triangleleft \text{O}_\pi(G) = 1 \). This yields that \( N \) is a \( \pi' \)-group. \( N \leq \text{O}_\pi(G) \) and \( N \) is solvable by Lemma 2.3, hence \( N \) is a solvable minimal normal subgroup of \( S \) and so \( N \triangleleft S \). Now then that \( 1 \neq N \leq F \cap L = 1 \), a contradiction. The result now follows.

Now, \( S_\pi(G) \) is supervolvable by the definition and (iv). This contradicts the choice of \( G \) and completes the proof of the theorem. \( \square \)

We can easily get the following corollaries:

**Corollary 4.3.** Let \( G \) be a \( \pi \)-separable group. Then:

1. \( S_\pi(S_\pi(G)) = S_\pi(G) \) and \( S_\pi(S_\pi(G)) = S_\pi(G) \).
2. \( G/O_\pi(G) \) is supervolvable \( \iff \) \( G/S_\pi(G) \) is supervolvable \( \iff \) \( S_\pi(G) = G \).

**Proof.** (1) \( \forall M < S_\pi(G) \) with \( |S_\pi(G)/M| = 1 \), \( O_\pi(G) \trianglelefteq M \). Since \( S_\pi(G)/O_\pi(G) \) is supervolvable, 

\[
|S_\pi(G) : M| = |S_\pi(G)/O_\pi(G) : M/O_\pi(G)|
\]

is a prime. Hence \( S_\pi(G) \) is the empty set and \( S_\pi(S_\pi(G)) = S_\pi(G) \).

(2) It is trivial to show that \( S_\pi(G) = G \Rightarrow G/O_\pi(G) \) is supervolvable \( \Rightarrow \) \( G/S_\pi(G) \) is supervolvable by the Theorem. If \( G/S_\pi(G) \) is supervolvable, then \( G/S_\pi(G) = S_\pi(G/S_\pi(G)) = S_\pi(G)/S_\pi(G) \), hence \( G = S_\pi(G) \). \( \square \)
Corollary 4.4. Suppose that a finite group $G$ is both $\pi_1$-separable and $\pi_2$-separable with $\pi_1 \cap \pi_2$ is empty set. Then $S_{\pi_1}(G) \cap S_{\pi_2}(G)$ is supersolvable.

Proof. Let $S_{\pi_1}(G) \cap S_{\pi_2}(G) = H$. By Theorem 4.2, both $H\Phi_{\pi_1}(G)/O_{\pi_1}(G)$ and $H\Phi_{\pi_2}(G)/O_{\pi_2}(G)$ are supersolvable. So we have that

$$H \cong H\Phi_{\pi_1}(G) \cap O_{\pi_1}(G) = H\Phi_{\pi_1}(G) \cap S_{\pi_1}(G) \cap O_{\pi_1}(G) \cap S_{\pi_2}(G)$$

is supersolvable. We are done. □

Remark 1. Let $G$ be a finite group. If we set $\pi = \pi(G)$, then $G$ is $\pi$-separable with $O_{\pi}(G) = 1$ and $S_{\pi}(G) = L(G) = \{M: M \triangleleft G\}$, where $|G:M|$ is composite. Hence $L(G)$ is supersolvable in any case. This is a result from Bhatia [1].

Remark 2. Let $G$ be a finite $\pi$-solvable group or a finite $\pi$-separable group with $\pi = \{p, q\}$. Then both $\Phi_{\pi}(G)$ and $S_{\pi}(G)$ are solvable.

Proof. We use induction on $|G|$.

(1) If $G$ is $\pi$-solvable, $\forall N \triangleleft \Phi_{\pi}(G)$ (or $S_{\pi}(G)$) with that $N$ is a minimal normal subgroup of $G$, $N$ is a $p$-subgroup with $p \in \pi$ or $N$ is a $\pi'$-subgroup. If $N$ is a $\pi'$-subgroup, then $N$ is solvable by Lemma 2.3(3). Hence $N$ is solvable in any case. Now, Lemma 2.2 implies the result.

(2) If $G$ is a $\pi$-separable group with $\pi = \{p, q\}$, then $O_{\pi}(G)$ is solvable by the Burnside Theorem on $p^aq^b$-group. Theorem 3.3 and Theorem 4.1 yields the solvability of $G$. □

Remark 3. Since $G$ is $\pi$-separable if and only if $G$ is $\pi'$-separable, we can get similar results by replacing $\pi'$ in the position of $\pi$.

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References