Steinberg characters for Chevalley groups over finite local rings

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Abstract

We construct for a Chevalley group over a finite local ring an analogue of the Steinberg character that was defined for the general linear group by P. Lees and, independently, by G. Hill. Further, we show that this analogue has a homological origin and, when irreducible, describe it in terms of a linear character of the corresponding Hecke algebra. However, we find that the analogue is reducible in general. Thus we determine its decomposition into distinct irreducible constituents and characterise these constituents using Gelfand–Graev characters.

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Introduction

Let $G$ be a Chevalley group of type $\Sigma$ over the quotient ring $R = o/p^\ell$ where $o$ is the ring of integers of a non-archimedean local field $K$ with prime ideal $p$ and $\ell > 1$ is a fixed integer. For example, using the terminology from Section 1, the adjoint and extended Chevalley groups in type $A_n$ are $\text{PSL}_{n+1}(R)$ and $\text{PGL}_{n+1}(R)$ respectively whereas in type $C_n$ they are $\text{PSp}_{2n}(R)$ and $\text{PGSp}_{2n}(R)$. We will assume that the residue field $o/p$ is finite, and has very good characteristic, so that $R$ is a finite local ring and $G$ is a finite group. The motivation for studying the representation theory of such groups is its close connection to the representation theory of the $p$-adic group $G(K)$ via the maximal compact subgroup $G(o)$.

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We will be examining a character \( St_\ell \) which is an analogue of the Steinberg character of the Chevalley group over the residue field \([20–22]\). This analogue was first defined by Lees \([15]\) for the general linear group as the character afforded by the top homology space of a simplicial complex similar to the combinatorial building \([19]\). Hill \([11]\) independently described the same analogue as the unique common constituent of the permutation character \((1_B)^G\) over the subgroup \(B\) of upper triangular matrices and a version of the Gelfand–Graev character \(\Gamma_0\) (see \([4]\)).

Our approach will be to adapt the construction and techniques used for \(\text{GL}_n(R)\) in \([2]\).

Let \(\Pi\) be a base for \(\Sigma\) and denote by \(B\) the integral points of a Borel subgroup of \(G(K)\) reduced modulo \(p\). We associate to each subset \(J\) of \(S = -\Pi\) a “parabolic” subgroup \(H_J\) of \(G\) where, in particular, \(H_\emptyset = B\). Then, imitating Curtis’ formula \([5]\), we define the analogue to be

\[
St_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G
\]

and describe a simplicial complex whose top homology space gives rise to \(St_\ell\).

Although the Steinberg character for the Chevalley group over a finite field and its analogue for \(\text{GL}_n(R)\) are both irreducible, we find that the analogue for \(G\) is reducible in general. Indeed, the inner product of \(St_\ell\) with itself is \(d\), the index of \(G\) in the extended Chevalley group. To prove this we need information about the \((H_J, H_J)\)-double cosets of \(G\). While the double cosets contained in \(H_S\) are relatively straightforward, those lying in \(G - H_S\) have a more complicated structure. Fortunately, the following result, whose proof occupies the final three sections of the paper, is sufficient for our purposes.

**Theorem A.** Each \((B, B)\)-double coset in \(G - H_S\) has the form \(BgB = H_\alpha gB\) for some \(\alpha \in S\) depending only on \(HgH\).

When \(G\) is the extended Chevalley group, and so when \(St_\ell\) is irreducible, it transpires that the analogue is induced from the Steinberg character of \(H_S\). Further, it corresponds to a linear character \(\psi\) of the Hecke algebra \(H(G, B)\) defined by setting \(\psi(\beta_\alpha) = -1\) for each \(\alpha \in S\) where \(\beta_\alpha\) is the basis element obtained from the double coset \(H_\alpha - B\). In general, we find that \(St_\ell\) decomposes into the sum of \(d\) irreducible constituents each appearing with multiplicity 1. Following the definition in \([11]\), we consider certain Gelfand–Graev characters of \(G\) and show that each irreducible constituent of \(St_\ell\) appears as the unique common constituent of the permutation character over \(B\) and exactly one of these Gelfand–Graev characters.

We introduce some notation. If \(G\) is a finite group then its identity element will be denoted by 1 and we will consider the commutator \([x, y] = xyx^{-1}y^{-1}\) for \(x, y \in G\). If \(\phi\) and \(\psi\) are characters of \(G\) then \((\phi, \psi)\) will be their inner product. Finally, if \(\rho\) is a character of a subgroup \(H\) of \(G\) then for each \(g \in G\) the conjugate character \(\rho^g\) of \(gHg^{-1}\) is defined by \(\rho^g(gxg^{-1}) = \rho(g)\) for all \(x \in H\).

1. Chevalley groups

We begin by outlining the construction of the Chevalley groups over a finite local ring. Following \([3]\) they are defined as groups of automorphisms of Lie rings corresponding to non-abelian semisimple Lie algebras over the complex numbers.
1.1. Root system

Let \( g \) be a non-abelian semisimple Lie algebra over \( \mathbb{C} \) with Cartan subalgebra \( h \) and rank \( n \). Let \( \Sigma \) be the set of non-zero roots of \( g \) with respect to \( h \) and fix a base \( \Pi \) for \( \Sigma \). We will denote by \( \Sigma^+ \) and \( \Sigma^- \) the sets of positive and negative roots in \( \Sigma \) respectively. Further, if \( \beta \in \Sigma \) is expressed as the linear combination of simple roots

\[
\beta = \sum_{\alpha \in \Pi} k_{\alpha} \alpha,
\]

(1)

then the height of \( \beta \) is defined to be \( \text{ht}(\beta) = \sum_{\alpha \in \Pi} k_{\alpha} \). The set of roots of height \( a \) in \( \Sigma \) will be denoted by \( \Sigma_a \).

Let \( h^* \mathbb{R} \) be the \( \mathbb{R} \)-span of the roots in the dual space \( h^* \). The Killing form on \( g \) restricts to an inner product on \( h^* \mathbb{R} \) which we will write as \( (x,y) \) for each \( x, y \in h^* \mathbb{R} \). The Weyl group \( W \) of \( g \) is then the group generated by \( \{ w_\alpha : \alpha \in \Sigma \} \) where \( w_\alpha \) is the reflection in the hyperplane orthogonal to the root \( \alpha \in \Sigma \).

Let \( \Lambda_r \) denote the root lattice in \( h^* \mathbb{R} \). This is a free abelian group of rank \( n \) generated by the simple roots. In addition, let \( \Lambda_w \) denote the weight lattice in \( h^* \mathbb{R} \); that is, the set

\[
\Lambda_w = \{ \lambda \in h^* \mathbb{R} : (\lambda, \check{\alpha}) \in \mathbb{Z} \text{ for any } \alpha \in \Sigma \}
\]

where \( \check{\alpha} = 2\alpha/(\alpha, \alpha) \) is the co-root corresponding to \( \alpha \). For each \( \alpha \in \Pi \) define \( \lambda_\alpha \) to be the unique element of \( \Lambda_w \) such that \( (\lambda_\alpha, \check{\alpha}) = 1 \) and \( (\lambda_\alpha, \check{\beta}) = 0 \) for any \( \beta \in \Pi - \{ \alpha \} \). Then \( \Lambda_w \) is a free abelian group of rank \( n \) generated by the fundamental weights \( \{ \lambda_\alpha : \alpha \in \Pi \} \) and \( \Lambda_r \) is a subgroup of \( \Lambda_w \) where for each \( \alpha \in \Pi \)

\[
\alpha = \sum_{\beta \in \Pi} (\alpha, \check{\beta}) \lambda_\beta.
\]

(2)

1.2. Finite local ring

Let \( K \) be a non-archimedean local field with ring of integers \( o \) and let \( p \) denote the prime ideal of \( o \). We will assume that the residue field \( \kappa = o/p \) is finite of order \( q \) and has good characteristic; that is,

(i) \( \text{char} \, \kappa \neq 2 \) if \( \Sigma \) has irreducible components of type \( B_n, C_n \) or \( D_n \);
(ii) \( \text{char} \, \kappa \neq 2 \) or \( 3 \) if \( \Sigma \) has irreducible components of type \( F_4, G_2, E_6 \) or \( E_7 \);
(iii) \( \text{char} \, \kappa \neq 2, 3 \) or \( 5 \) if \( \Sigma \) has irreducible components of type \( E_8 \).

If in addition \( \text{char} \, \kappa \) does not divide \( n + 1 \) whenever \( \Sigma \) has irreducible components of type \( A_n \) then we say that the characteristic of \( \kappa \) is very good.

Fix an integer \( \ell > 1 \) and set \( R = o/p^\ell \). This is a finite, local, principal ideal ring with unique maximal ideal \( m = p/p^\ell \) and if \( \pi \) is a generator of \( m \) then \( \pi^\ell = 0 \). The ideals of \( R \) are of the form \( m^i = \pi^i R \) for \( 0 \leq i \leq \ell \), giving the filtration

\[
0 = m^\ell \subset m^{\ell-1} \subset \cdots \subset m^1 \subset m^0 = R,
\]
and $rs = 0$ whenever $r \in m^i$ and $s \in m^{\ell-i}$. We will also consider the subgroup $u^i = (1 + m^i) \cap R^\times$ of $R^\times$ for each $0 \leq i \leq \ell$ so that we obtain the filtration

$$\{1\} = u^\ell \subset u^{\ell-1} \subset \cdot \cdot \cdot \subset u^1 \subset u^0 = R^\times.$$  

1.3. Chevalley groups

Denote by ad the adjoint representation of $\mathfrak{g}$ and let $\mathfrak{g}_Z$ be the $\mathbb{Z}$-span of a fixed Chevalley basis $\{e_\alpha, h_\beta : \alpha \in \Sigma, \beta \in \Pi\}$. For each $\xi \in \mathbb{C}$ and $\alpha \in \Sigma$ the co-ordinates of the automorphism $x_\alpha(\xi) = \exp(\xi \text{ ad}(e_\alpha))$ of $\mathfrak{g}$ are polynomials in $\xi$ with coefficients in $\mathbb{Z}$. If we consider the Lie ring $\mathfrak{g}_R = \mathfrak{g}_Z \otimes \mathbb{R}$ then for each $r \in \mathbb{R}$ and $\alpha \in \Sigma$ we may define a corresponding automorphism $x_\alpha(r)$ of $\mathfrak{g}_R$ by taking its co-ordinates to be these same polynomials evaluated at $r$. The adjoint Chevalley group $G'(R)$ is therefore the subgroup of $\text{Aut}(\mathfrak{g}_R)$

$$G' = G'(R) = \langle x_\alpha(r) : \alpha \in \Sigma, r \in \mathbb{R} \rangle.$$  

Further, for each homomorphism $\mu \in \text{Hom}(\Lambda_r, R^\times)$ there is a corresponding diagonal automorphism $h(\mu)$ of $\mathfrak{g}_R$ given by setting $h(\mu) \cdot e_\alpha = \mu(\alpha)e_\alpha$ for $\alpha \in \Sigma$ and $h(\mu) \cdot h_\beta = h_\beta$ for $\beta \in \Pi$. The extended Chevalley group is then

$$\overline{G} = \overline{G}(R) = \langle x_\alpha(r), h(\mu) : \alpha \in \Sigma, r \in \mathbb{R}, \mu \in \text{Hom}(\Lambda_r, R^\times) \rangle.$$  

More generally, if we fix a lattice $\Lambda$ with $\Lambda_r \leq \Lambda \leq \Lambda_w$ then any homomorphism $\mu \in \text{Hom}(\Lambda_r, R^\times)$ restricts to a homomorphism $\mu \in \text{Hom}(\Lambda_r, R^\times)$. Thus we may consider the associated Chevalley group

$$G = G(R) = \langle x_\alpha(r), h(\mu) : \alpha \in \Sigma, r \in \mathbb{R}, \mu \in \text{Hom}(\Lambda, R^\times) \rangle$$

and we see that we have $G'(R) \leq G(R) \leq \overline{G}(R)$.

**Remark 1.1.** Consider the Chevalley groups $G(K)$ over $K$ and $G(\sigma)$ over $\sigma$ defined as above. Identifying $\mathfrak{g}_\sigma = \mathfrak{g}_Z \otimes_{\mathbb{Z}} \sigma$ as a subring of $\mathfrak{g}_K = \mathfrak{g}_Z \otimes K$, so that $\text{Aut}(\mathfrak{g}_\sigma)$ is the subgroup of $\text{Aut}(\mathfrak{g}_K)$ consisting of all automorphisms of $\mathfrak{g}_K$ which preserve $\mathfrak{g}_\sigma$, we have $G(\sigma) = G(K) \cap \text{Aut}(\mathfrak{g}_\sigma)$ (see [14,23]). Further, the natural projection $\eta : \sigma \to R$ gives rise to a homomorphism of Lie rings $\eta : \mathfrak{g}_\sigma \to \mathfrak{g}_R$ and thus a corresponding homomorphism of automorphism groups $\eta : \text{Aut}(\mathfrak{g}_\sigma) \to \text{Aut}(\mathfrak{g}_R)$. The Chevalley group $G$ over $R$ is then the image of $G(\sigma)$ under this homomorphism.

**Remark 1.2.** Abe [1] uses an alternative definition for the Chevalley group over $R$ as the $R$-points of a semisimple algebraic group which is defined and split over $\mathbb{Z}$. In this context, $\overline{G}$ is obtained from the adjoint algebraic group whereas $G'$ is the quotient of the simply-connected group by its centre. Our results still hold in this situation and can in fact be extended to split reductive groups. Unfortunately, this is not the case for quasi-split groups.
2. Structure

We now examine the structure of $G$, noting that Remark 1.1 allows us to apply immediately in our situation many of the results for Chevalley groups over fields found in [3,23]. For the remainder of the paper we will require that the characteristic of the residue field is very good, although if $G$ is the extended Chevalley group it will be sufficient to assume that char $\kappa$ is good.

2.1. Unipotent subgroups

For any $r, s \in R$ and $\alpha \in \Sigma$

$$x_\alpha(r)x_\alpha(s) = x_\alpha(r + s).$$

Thus for each $0 \leq i \leq \ell$ the root subgroup $U_\alpha(i) = \{x_\alpha(r) : r \in m^i\}$ is isomorphic to the additive group $m^i$ and we obtain the filtration

$$\{1\} = U_\alpha(\ell) \subset U_\alpha(\ell - 1) \subset \cdots \subset U_\alpha(1) \subset U_\alpha(0) = U_\alpha.$$

The relation between the generators $x_\alpha(r)$ and $x_\beta(s)$ for $\alpha \neq \pm \beta$ is given by the usual Chevalley commutator formula.

Lemma 2.1. For any $\alpha, \beta \in \Sigma$ with $\alpha \neq \pm \beta$ and $r, s \in R$

$$[x_\alpha(r), x_\beta(s)] = \prod_{i,j > 0} x_{i\alpha + j\beta}(c_{i,j,\alpha,\beta}(-r)^i s^j)$$

where the product is taken over all positive $i$ and $j$ such that $i\alpha + j\beta \in \Sigma$.

Remark 2.2. The constants $c_{i,j,\alpha,\beta}$ are independent of $r$ and $s$. Moreover, $c_{i,j,\alpha,\beta} \in \{\pm 1\}$ except when $\alpha$ and $\beta$ belong to an irreducible component of type $B_n$, $C_n$ or $F_4$, in which case $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2\}$, or an irreducible component of type $G_2$, in which case $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$. In particular, as the characteristic of $\kappa$ is assumed to at least be good, the constants will be invertible in $R$.

To find an expression for the commutator in the case where $\alpha = -\beta$ we use the homomorphism $\phi_\alpha : SL_2(R) \to G'(R)$ defined by setting

$$\phi_\alpha \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = x_\alpha(r) \quad \text{and} \quad \phi_\alpha \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = x_{-\alpha}(r)$$

for every $r \in R$ (see [23, Lemma 48]). For each $\alpha \in \Sigma$ and $r \in R^\times$ let

$$n_\alpha(r) = \phi_\alpha \begin{bmatrix} 0 & r \\ -r^{-1} & 0 \end{bmatrix} \quad \text{and} \quad h_\alpha(r) = \phi_\alpha \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}.$$

Lemma 2.3. Let $r, s \in R$ be such that $r^2 s^2 = 0$, then for any $\alpha \in \Sigma$

$$[x_\alpha(r), x_{-\alpha}(s)] = h_\alpha(1 + rs)x_\alpha(-r^2 s)x_{-\alpha}(rs^2).$$
Proof. This follows immediately from the fact that in $\text{SL}_2(R)$ we have
\[
\begin{bmatrix}
1 & r \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
s & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -r \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & rs \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & r^2 \\
0 & 1
\end{bmatrix}.
\]
\[\Box\]

Let $U(i) = \langle U_\alpha(i) \rangle_{\alpha \in \Sigma^+}$ and $U^-(i) = \langle U_\alpha(i) \rangle_{\alpha \in \Sigma^-}$ for each $0 \leq i \leq \ell$. In particular, we will write $U = U(0)$ and $U^- = U^-(0)$. The following result is an immediate consequence of the Chevalley commutator formula.

Lemma 2.4.

(i) Each $g \in U(i)$ can be expressed uniquely as $g = \prod_{\alpha \in \Sigma^+} x_\alpha(r_\alpha)$ for some $r_\alpha \in m^i$ where the product is taken over the positive roots in an arbitrary, but fixed, order.

(ii) Similarly, each $g \in U^-(i)$ can be expressed uniquely as $g = \prod_{\alpha \in \Sigma^-} x_\alpha(r_\alpha)$ for some $r_\alpha \in m^i$ where the product is taken over the negative roots in an arbitrary, but fixed, order.

2.2. Diagonal subgroups

For any $\mu, \nu \in \text{Hom}(\Lambda_r, R^\times)$
\[
h(\mu)h(\nu) = h(\mu + \nu)
\]
so we may define the diagonal subgroups $T'(i) = \langle h(\mu) : \mu \in \text{Hom}(\Lambda_r, u^i) \rangle$ and $T(i) = \langle h(\mu) : \mu \in \text{Hom}(\Lambda, u^i) \rangle$ for $0 \leq i \leq \ell$.

Further, for each $\alpha \in \Sigma$ and $r \in R^\times$, the element $h_\alpha(r)$ acts on the Chevalley basis by $h_\alpha(r) \cdot e_\beta = r^{(\alpha, \beta)} e_\beta$ for every $\beta \in \Sigma$ and $h_\alpha(r) \cdot h_\gamma = h_\gamma$ for every $\gamma \in \Pi$. Thus it is clearly the diagonal automorphism corresponding to the homomorphism $v_{\alpha, r} \in \text{Hom}(\Lambda_r, R^\times)$ defined by setting $v_{\alpha, r}(\beta) = r^{(\alpha, \beta)}$ for each $\beta \in \Sigma$. Consequently, $T'(i) = \langle h_\alpha(r) : \alpha \in \Sigma, r \in u^i \rangle$ is contained in $T(i)$ for each $0 \leq i \leq \ell$. Indeed, since the proof of [3, Theorem 7.1.1] remains valid in our situation, $T'(i)$ has the following description.

Proposition 2.5. $T'(i)$ is the subgroup of $T(i)$ consisting of all diagonal automorphisms corresponding to homomorphisms $\mu \in \text{Hom}(\Lambda_r, u^i)$ that extend to $\Lambda_w$.

This means that the adjoint group $G'$ corresponds to the weight lattice $\Lambda_w$ whereas the extended group $\overline{G}$ is obtained by taking the root lattice $\Lambda_r$.

Lemma 2.6. For any $\beta \in \Sigma$, $r \in R$ and $\mu \in \text{Hom}(\Lambda, R^\times)$
\[
h(\mu) x_\alpha(r) h(\mu)^{-1} = x_\alpha(\mu(\alpha) r).
\]

In particular, as $T(0)$ normalises the root subgroups $U_\alpha$ and the diagonal subgroup $T(0)$, we see that $G$ is normal in $\overline{G}$. Further, from [3, Proposition 8.4.6] we know that $T(0) \cap G' = T'(0)$ and so $\overline{T}(0) \cap G = \overline{T}(0) \cap T(0)G' = T(0)$. Hence
\[
\overline{G}/G = \overline{T}(0)G/G \simeq \overline{T}(0)/(\overline{T}(0) \cap G) \simeq \overline{T}(0)/T(0).
\]
Moreover, if for each $0 \leq i \leq \ell$ we define the subgroup $B(i) = (T(i), U(i))$ then it has $U(i)$ as a normal subgroup and so $B(i) = T(i)U(i)$. We will again write $T = T(0)$ and $B = B(0)$ so that $B = TU$.

2.3. Monomial subgroup

For each $w \in W$ choose a reduced expression $w = w_{\alpha_1} \cdots w_{\alpha_k}$ with $\alpha_i \in \Pi$ and set

$$n_w = n_{\alpha_1}(1) \cdots n_{\alpha_k}(1).$$

**Lemma 2.7.** For any $w \in W$, $\alpha \in \Sigma$ and $r \in R$

$$n_w x_\alpha(r)n_w^{-1} = x_{w(\alpha)}(\eta_w, \alpha r)$$

where $\eta_w, \alpha \in \{\pm 1\}$ is independent of $r$. Further, for any $\mu \in \text{Hom}(\Lambda, R^\times)$

$$n_w h(\mu)n_w^{-1} = h(\mu')$$

where $\mu' \in \text{Hom}(\Lambda, R^\times)$ is such that $\mu'(\lambda) = \mu(w^{-1}(\lambda))$ for every $\lambda \in \Lambda$.

Although $\{n_w: w \in W\}$ is far from being a complete set of $(B, B)$-double coset representatives in $G$, from the Bruhat decomposition of the Chevalley group over the residue field we are able to give the following result (see [14, Proposition 2.4]).

**Lemma 2.8.** $G$ can be expressed as the disjoint union

$$G = \bigcup_{w \in W} Bn_w U^-(1)B.$$

2.4. Congruence subgroups

Consider the natural projection $\eta_i : R \to R/m_i$ for each $1 \leq i \leq \ell$. This induces a homomorphism $\eta_i : G \to G(R/m_i)$ and the corresponding congruence subgroup is the normal subgroup $K_i = \ker \eta_i$ of $G$.

**Lemma 2.9.** $K_i = U^-(i)T(i)U(i)$ for each $1 \leq i \leq \ell$.

**Proof.** From [1, Corollary 3.3] we know that $\overline{K}_i = U^-(i)\overline{T}(i)U(i)$ and so we obtain $K_i = K_i \cap G = U^-(i)\overline{T}(i)U(i) \cap G = U^-(i)T(i)U(i)$. \(\square\)

It transpires that when the characteristic of the residue field $\kappa$ is very good the congruence subgroups for $G'$, $G$ and $\overline{G}$ agree. Before proving this we need the following result.

**Lemma 2.10.** If $a$ is a positive integer not divisible by $\text{char} \kappa$ and $i \geq 1$ then for every $r \in u^i$ there is an $s \in u^i$ with $r = s^a$. 
Proof. We will actually show that the only element \( s \in u^i \) which has \( s^a = 1 \) is \( s = 1 \). This would imply that the map sending \( s \in u^i \) to \( s^a \in u^i \) is injective and thus, since \( u^i \) is finite, surjective. Consequently, suppose that \( s \in u^i \) has \( s^a = 1 \) but that \( s \neq 1 \). Then \( s^a - 1 = 0 \) implies that \( (s^a - 1 + \cdots + s + 1)(s - 1) = 0 \) and therefore \( s^a - 1 + \cdots + s + 1 \in m \). However, this means we must have \( 0 = \eta_1(s)^a - 1 + \cdots + \eta_1(s) + 1 = a \) which is a contradiction. \( \square \)

Proposition 2.11. If \( \text{char} \, \kappa \) is very good then \( K'_i = K_i = \overline{K}_i \) for each \( i \geq 1 \).

Proof. It suffices to show that any homomorphism \( \mu \in \text{Hom}(\Lambda_r, u^i) \) extends to a homomorphism \( \mu' \in \text{Hom}(\Lambda_w, u^i) \) since this would imply that \( T'(i) = T(i) = \overline{T}(i) \) by Proposition 2.5 and so \( K'_i = K_i = \overline{K}_i \) by Lemma 2.9.

Let \( A = [(\alpha, \tilde{\beta})]_{\alpha, \beta \in \Pi} \) denote the Cartan matrix of \( \Sigma \) and \( A^{-1} = [a_{\alpha, \beta}]_{\alpha, \beta \in \Pi} \) its inverse. By (2), for each fundamental weight \( \lambda_\beta \) we have

\[
\lambda_\beta = \sum_{\alpha \in \Pi} a_{\alpha, \beta} \alpha.
\]

The determinant of \( A \) does not divide \( \text{char} \, \kappa \), since it is very good, and so Lemma 2.10 implies that for each \( \alpha \in \Pi \) we may choose an element \( s_\alpha \in u^i \) with \( \mu(\alpha) = s_\alpha^{\det(A)} \). Further, since \( \det(A)a_{\alpha, \beta} \) is an integer, for each \( \beta \in \Pi \) we are able to define

\[
r_\beta = \prod_{\alpha \in \Pi} s_\alpha^{\det(A)a_{\alpha, \beta}}.
\]

Thus the homomorphism \( \mu' \in \text{Hom}(\Lambda_w, u^i) \) given by setting \( \mu'(\lambda_\beta) = r_\beta \) for every \( \beta \in \Pi \) is such that for each \( \alpha \in \Pi \)

\[
\mu'(\alpha) = \mu' \left( \sum_{\beta \in \Pi} (\alpha, \tilde{\beta}) \lambda_\beta \right) = \prod_{\beta \in \Pi} r_\beta^{(\alpha, \tilde{\beta})} = s_\alpha^{\det(A)} = \mu(\alpha).
\]

Hence \( \mu' \) extends \( \mu \) and \( K'_i = K_i = \overline{K}_i. \) \( \square \)

2.5. Orders of the groups

For each \( \alpha \in \Pi \) and \( r \in u^i \) define the homomorphism \( \mu_{\alpha, r} \in \text{Hom}(\Lambda_r, u^i) \) by setting

\[
\mu_{\alpha, r}(\beta) = \begin{cases} 
  r & \text{if } \alpha = \beta; \\
  1 & \text{if } \alpha \neq \beta
\end{cases}
\]

for every \( \beta \in \Pi \). We may express any homomorphism \( \mu \in \text{Hom}(\Lambda_r, u^i) \) uniquely as the sum

\[
\mu = \sum_{\alpha \in \Pi} \mu_{\alpha, \mu(\alpha)} \quad \text{and so, if we let } y_\alpha(r) = h(\mu_{\alpha, r}), \text{ every } h(\mu) \in \overline{T}(i) \text{ uniquely as the product}
\]

\[
h(\mu) = \prod_{\alpha \in \Pi} y_\alpha(\mu(\alpha)).
\]
Table 1
The index $d = [G : G']$ in very good characteristic

<table>
<thead>
<tr>
<th>$\Sigma$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
<th>$D_{2k+1}$</th>
<th>$D_{2k}$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$(n+1, q-1)$</td>
<td>$(2, q-1)$</td>
<td>$(2, q-1)$</td>
<td>$(4, q-1)^2$</td>
<td>$(3, q-1)$</td>
<td>$(2, q-1)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

For later use we also note that for any $\alpha \in \Pi$, $\beta \in \Sigma$, $r \in R \times R$ and $s \in R$

$$y_\alpha(r)x_\beta(s)y_\alpha(r)^{-1} = x_\beta(k_\alpha s),$$

where $k_\alpha$ is as in (1).

From Lemma 2.4 we have $|U(1)| = |U^-(1)| = q^{N(\ell-1)}$, where $N = |\Sigma^+|$, and by (4) we know that $|\overline{T}(1)| = q^{n(\ell-1)}$. Thus the order of the congruence subgroup $K_1$ is

$$|K_1| = |U^-(1)||\overline{T}(1)||U(1)| = q^{(2N+n)(\ell-1)}$$

and the order of the extended Chevalley group can be calculated from the order of the corresponding group over the finite field $\kappa$.

Lemma 2.12.

(i) $|\overline{T}| = q^{n(\ell-1)}(q-1)^n$;

(ii) $|\overline{B}| = q^{(n+N)\ell-n}(q-1)^n$;

(iii) $|\overline{G}| = q^{(2N+n)\ell-n}(q-1)^n \sum_{w \in W} q^{\ell(w)}$.

Moreover, by (3) we know that $[\overline{G} : G] = [\overline{T} : T]$ which in turn is equal to the number of homomorphisms $\mu \in \text{Hom}(\Lambda, R^\times)$ that are trivial on $\Lambda_r$.

Lemma 2.13.

(i) $|T| = q^{n(\ell-1)}(q-1)^n/d$;

(ii) $|B| = q^{(n+N)\ell-n}(q-1)^n/d$;

(iii) $|G| = q^{(2N+n)\ell-n}(q-1)^n \sum_{w \in W} q^{\ell(w)}/d$

where $d = |\text{Hom}(\Lambda/\Lambda_r, R^\times)|$.

Finally, if the characteristic of $\kappa$ is very good then by the proof of Proposition 2.11 we know that the only homomorphism $\mu \in \text{Hom}(\Lambda_r, 1 + m)$ which is trivial on $\Lambda_r$ is $\mu = 1$ and so $d = |\text{Hom}(\Lambda/\Lambda_r, \kappa^\times)|$. Specific values of the index of $G'$ in $\overline{G}$ for irreducible root systems in very good characteristic are given in Table 1.

3. Analogue

As in [2], the analogue of the Steinberg character will be described as an alternating sum of permutation characters over certain “parabolic” subgroups. This is similar to both Steinberg’s original construction [20] for the general linear group and Curtis’ later definition [5] for finite groups with $BN$-pair.
3.1. Parabolic subgroups

A subgroup \( P \) of \( G \) is parabolic if it is of the form

\[
P = \langle U_\alpha(i_\alpha), B : \alpha \in \Sigma^- \rangle
\]

for some \( 0 \leq i_\alpha \leq \ell \).

Remark 3.1. When \( \Sigma = A_n \), this reduces to the definition given in [2]. Further, Suzuki [24] showed that every subgroup of \( G \) containing \( B \) is of this form provided that \( \kappa \) has odd characteristic; \( q \neq 3 \) for types \( A_3, B_n, C_n, D_n \) and \( F_4 \); and \( \text{char} \kappa \neq 3 \) for type \( G_2 \).

Let \( S = \{ \alpha : -\alpha \in \Pi \} \) and for each \( \alpha \in S \) define \( X_\alpha = U_\alpha(\ell - 1) \). We will be interested in the parabolic subgroups \( H_\alpha = \langle X_\alpha, B \rangle \) for \( \alpha \in S \).

Lemma 3.2. The minimal parabolic subgroups of \( G \) strictly containing \( B \) are precisely the parabolic subgroups \( H_\alpha \) for \( \alpha \in S \).

Proof. We need to show that any parabolic subgroup \( P \) strictly containing \( B \) also contains \( H_\alpha \) for some \( \alpha \in S \). As \( P \neq B \) we know that \( P \) contains a root subgroup \( U_\beta(i) \) for some \( \beta \in \Sigma^- \) and \( i < \ell \). In particular, this means that \( U_\beta(\ell - 1) \leq P \) for some \( \beta \in \Sigma^- \). If \( \beta \in S \) then \( H_\beta \leq P \) so we will assume that \( \beta \notin S \).

Let \( \gamma \in \Sigma^+ \) be such that \( \beta + \gamma \in S \) and set \( \alpha = \beta + \gamma \). For any \( r \in m^{\ell - 1} \) and \( s \in R \) the Chevalley commutator formula gives

\[
[x_\beta(r), x_\gamma(s)] = \sum_{j > 0} x_{\beta + j\gamma}(-c_{1,j,\beta,\gamma rs^j})
\]

since \((-r)^i s^j = 0 \) for \( i > 1 \). Further, \( \beta + \gamma \in \Sigma^- \) and if \( \beta + j\gamma \in \Sigma \) for \( j > 1 \) then \( \beta + j\gamma \in \Sigma^+ \) since \( \text{ht}(\beta + j\gamma) > \text{ht}(\beta + \gamma) = -1 \). Thus

\[
[x_\beta(r), x_\gamma(s)] = x_\alpha(-c_{1,1,\beta,\gamma rs}) v
\]

where

\[
v = \prod_{j > 1} x_{\beta + j\gamma}(-c_{1,j,\beta,\gamma rs^j}) \in B(\ell - 1).
\]

Hence, if we fix \( t \in m^{\ell - 1} \) and choose \( r \in m^{\ell - 1} \) with \( -c_{1,1,\beta,\gamma rs} = t \), then we obtain \( x_\alpha(t) = [x_\beta(r), x_\gamma(1)] v^{-1} \in P \) and so must have \( H_\alpha \leq P \). □

More importantly for our construction it transpires that \( X_\alpha \) forms a left, and also right, transversal for \( B \) in \( H_\alpha \).

Proposition 3.3. \( H_\alpha = X_\alpha B = BX_\alpha \).
Proof. It suffices to prove that \( X_\alpha B \subseteq BX_\alpha \) since this would imply that they are equal, as they contain the same number of elements, and thus both equal to \( H_\alpha \). We already know that \( X_\alpha T \subseteq BX_\alpha \) by Lemma 2.6 and to show that we also have \( X_\alpha U \subseteq BX_\alpha \) we will consider the commutator \([x_\alpha(r), x_\beta(s)]\) for every \( r \in m^{\ell-1}, s \in R \) and \( \beta \in \Sigma^+ \).

If \( \alpha \neq -\beta \) then, as in the proof of Lemma 3.2,

\[
[x_\alpha(r), x_\beta(s)] = \prod_{j>0} x_{\alpha+j\beta}(-c_{1,j,\alpha,\beta}rs^j).
\]

However, on this occasion we see that if \( \alpha + j\beta \in \Sigma \) for \( j > 0 \) then \(\alpha + j\beta \in \Sigma^+ \) since \( \text{ht}(\alpha + j\beta) \geq (j - 1)\text{ht}(\beta) \geq 0 \). Thus

\[
[x_\alpha(r), x_\beta(s)] = \prod_{j>0} x_{\alpha+j\beta}(-c_{1,j,\alpha,\beta}rs^j) \in B(\ell - 1).
\]

Similarly, if \( \alpha = -\beta \) then by Lemma 2.3

\[
[x_\alpha(r), x_\beta(s)] = h_\alpha(1 + rs)x_{-\alpha}(rs^2) \in B(\ell - 1).
\]

Now, fix \( u \in U \) and express it as \( u = x_{\beta_1}(s_1)\cdots x_{\beta_k}(s_k) \) for some ordering \( \beta_1, \ldots, \beta_k \) of \( \Sigma^+ \). If we set \( u_i = x_{\beta_1}(s_1)\cdots x_{\beta_i}(s_i) \) for each \( i \) then

\[
[x_\alpha(r), u] = \left[ x_\alpha(r), x_{\beta_1}(s_1)\cdots x_{\beta_k}(s_k) \right]
\]

\[
= \left[ x_\alpha(r), x_{\beta_1}(s_1) \right] \left( u_1 \left[ x_\alpha(r), x_{\beta_2}(s_2) \right] u_1^{-1} \right) \cdots \left( u_{k-2} \left[ x_\alpha(r), x_{\beta_{k-1}}(s_{k-1}) \right] u_{k-2}^{-1} \right) \left( u_{k-1} \left[ x_\alpha(r), x_{\beta_k}(s_k) \right] u_{k-1}^{-1} \right)
\]

where each \( u_{i-1}[x_\alpha(r), x_{\beta_i}(s_i)]u_{i-1}^{-1} \in B(\ell - 1) \). Thus \([x_\alpha(r), u] \in B(\ell - 1)\) and so \( x_\alpha(r)u = [x_\alpha(r), u]x_\alpha(r) \in BX_\alpha \). Hence we have \( X_\alpha U \subseteq BX_\alpha \) which implies that \( X_\alpha B = X_\alpha TU \subseteq BX_\alpha U \subseteq BX_\alpha \).

3.2. The analogue of the Steinberg character

As an immediate consequence of Proposition 3.3 we see that for any \( \alpha, \beta \in S \)

\[ H_\alpha H_\beta = BX_\alpha X_\beta B = BX_\beta X_\alpha B = H_\beta H_\alpha. \]

Thus, if for each non-empty \( J = \{\alpha_1, \ldots, \alpha_k\} \subseteq S \) we define

\[ H_J = (H_{\alpha_1}, \ldots, H_{\alpha_k}), \]

then \( H_J = H_{\alpha_1} \cdots H_{\alpha_k} \). In particular, this means that \( H_J = X_J B = BX_J \) where

\[ X_J = \{ x_{\alpha_1}(r_1)\cdots x_{\alpha_k}(r_k) : r_1, \ldots, r_k \in m^{\ell-1} \}. \]

Similarly, we let \( H_\emptyset = B \) and \( X_\emptyset = \{1\} \) so that \( H_\emptyset = X_\emptyset B = BX_\emptyset \).
Lemma 3.4. Let $I, J \subseteq S$, then

(i) $|H_J| = q^{|J|} |B|$;
(ii) $H_I H_J = H_J H_I$;
(iii) $H_I \cap H_J = H_I \cap J$;
(iv) $\langle H_I, H_J \rangle = H_I \cup J$.

Definition 3.5. Let $S_\ell$ be the alternating sum of permutation characters

$$S_\ell = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G.$$

Remark 3.6.

(i) When $\ell = 1$ we see that $X_\alpha = U_\alpha$ for every $\alpha \in S$ and so $H_\alpha = \langle U_\alpha, B \rangle$ is the standard parabolic subgroup of $G(\kappa)$ corresponding to the generator $w_{-\alpha}$ of $W$. Similarly, $H_J = \langle H_\alpha : \alpha \in J \rangle$ is a standard parabolic subgroup for each $J \subseteq S$ and the expression for $S_\ell$ is exactly Curtis’ formula [5] for the Steinberg character of the Chevalley group over the residue field.

(ii) If $\ell > 1$ and $\Sigma = A_n$ then the parabolic subgroups $H_J$ are the same as the subgroups described in [2], at least modulo their centres. In this case $S_\ell$ is identical to the analogue considered in [2] and by Lees in [15].

We have only defined $S_\ell$ as a virtual character of $G$, not as a true character. However, since $H_\alpha H_\beta = H_\beta H_\alpha$ for each $\alpha, \beta \in S$, we may apply the argument in [2, Section 4] to show the following.

Theorem 3.7. $S_\ell$ is the character afforded by the module $\mathbb{C}Ge$ where $e$ is the idempotent

$$e = \sum_{J \subseteq S} (-1)^{|J|} e_{H_J}$$

and $e_H = |H|^{-1} \sum_{h \in H} h$ for each subgroup $H$ of $G$.

Moreover, if we let $H = H_S$ then $S_\ell$ is induced from the character

$$\chi = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^H$$

of $H$ that is afforded by the module $\mathbb{C}He$.

Remark 3.8. For each choice of $\Lambda$ we potentially have a different Chevalley group and so a different analogue of the Steinberg character. However, if $\tilde{S}_\ell$ denotes the analogue for the extended Chevalley group $\tilde{G}$ then Mackey theory implies that

$$(\tilde{S}_\ell)_G = \sum_{J \subseteq S} (-1)^{|J|} ((1_{\tilde{H}_J})^{\tilde{G}})_G = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^G = S_\ell$$
since $\overline{H} J G = \overline{G}$ with $\overline{H} J \cap G = X_J \overline{B} \cap G = X_J B = H_J$. Similarly,

$$\chi_H = \sum_{J \subseteq S} (-1)^{|J|} \left( \prod_{i=1}^{|J|} H_{J_i} \right)^H = \sum_{J \subseteq S} (-1)^{|J|} (1_{H_J})^H = \chi.$$  

4. Homology

Solomon [19] demonstrated that Curtis’ formula was a consequence of the Steinberg character

of a Chevalley group over a finite field arising from the top homology space of the combinatorial

building. Moreover, the original definition of $\text{St}_t$ for the general linear group given by Lees [15]

was in terms of a similar simplicial complex. We will show that the same is true for the analogue

of the Steinberg character for $G$ by adapting the approach used by Lees. Throughout this section

we will assume that $n > 1$.

4.1. An analogue of the combinatorial building

Fix an ordering of the roots in $S$ and for each $\alpha \in S$ let $M_\alpha = H_{S - \{\alpha\}}$ so that the maximal

proper parabolic subgroups of $H$ are precisely the $M_\alpha$. Define $\Delta$ to be the simplicial com-

plex of dimension $n - 1$ whose vertices are the left cosets $\{g M_\alpha : g \in G, \alpha \in S\}$ and where

$(g_0 M_{\alpha_0}, \ldots, g_k M_{\alpha_k})$ is a $k$-simplex if and only if $\alpha_0 < \cdots < \alpha_k$ and $g_0 M_{\alpha_0} \cap \cdots \cap g_k M_{\alpha_k} \neq \emptyset$. The action of $G$ on the left cosets gives rise to an action on $\Delta$ sending $k$-simplices to $k$-simplices. This in turn induces an action on the integral homology spaces $H_k(\Delta)$ of $\Delta$.

Lemma 4.1. Each $k$-simplex is of the form $g \cdot (M_{\alpha_0}, \ldots, M_{\alpha_k})$ for some $g \in G$ and $\{\alpha_0, \ldots, \alpha_k\} \subseteq S$.

Proof. Let $(g_0 M_{\alpha_0}, \ldots, g_k M_{\alpha_k})$ be a $k$-simplex and choose an element $g \in g_0 M_{\alpha_0} \cap \cdots \cap g_k M_{\alpha_k}$. For each $i$ we have $g \in g_i M_{\alpha_i}$ and thus $g M_{\alpha_i} = g_i M_{\alpha_i}$. Hence $g \cdot (M_{\alpha_0}, \ldots, M_{\alpha_k}) = (g M_{\alpha_0}, \ldots, g M_{\alpha_k}) = (g_0 M_{\alpha_0}, \ldots, g_k M_{\alpha_k})$. \qed

Lemma 4.2. $\text{Stab}_G(M_{\alpha_0}, \ldots, M_{\alpha_k}) = H_{S - \{\alpha_0, \ldots, \alpha_k\}}$.

Proof. It is clear that $g \cdot (M_{\alpha_0}, \ldots, M_{\alpha_k}) = (M_{\alpha_0}, \ldots, M_{\alpha_k})$ precisely when $g M_{\alpha_i} = M_{\alpha_i}$ for each $i$. However, this is equivalent to requiring that $g \in M_{\alpha_i}$ for each $i$ and thus that $g \in M_{\alpha_0} \cap \cdots \cap M_{\alpha_k} = H_{S - \{\alpha_0, \ldots, \alpha_k\}}$. \qed

Let $C_k(\Delta)$ denote the $\mathbb{Z}$-span of the $k$-simplices in $\Delta$, then by Lemma 4.1

$$C_k(\Delta) = \bigoplus_{\alpha_0 < \cdots < \alpha_k} \mathbb{Z} G \cdot (M_{\alpha_0}, \ldots, M_{\alpha_k})$$

and Lemma 4.2 implies that the character afforded by $C_k(\Delta) \otimes_{\mathbb{Z}} \mathbb{C}$ is

$$\sum_{|J|=n-k-1} (1_{H_J})^G.$$
4.2. Induction of homology spaces

Let $\Delta'$ denote the subcomplex of $\Delta$ whose vertices $\{gM_\alpha : g \in H, \alpha \in S\}$ are the left cosets in $H$ and where $(g_0M_{\alpha_0}, \ldots, g_kM_{\alpha_k})$ is again a $k$-simplex if and only if $\alpha_0 < \cdots < \alpha_k$ and $g_0M_{\alpha_0} \cap \cdots \cap g_kM_{\alpha_k} \neq \emptyset$. As the first step to determining the homology spaces of $\Delta$, Lees [15] showed that it could be expressed as the disjoint union of certain subcomplexes equivalent to $\Delta'$. In fact, there is a stronger connection between $\Delta'$ and $\Delta$.

**Lemma 4.3.** $\tilde{C}_k(\Delta) = \text{Ind}_H^G \tilde{C}_k(\Delta')$ for each $k$.

**Proof.** Let $T$ be a left transversal for $H$ in $G$. From Lemma 4.1 we know that every $k$-simplex of $\Delta$ can be expressed as $t \cdot (hM_{\alpha_0}, \ldots, hM_{\alpha_k})$ for some $t \in T$ and $k$-simplex $(hM_{\alpha_0}, \ldots, hM_{\alpha_k})$ of $\Delta'$. Further, if $t, t' \in T$ are such that $t \cdot (h_0M_{\alpha_0}, \ldots, h_kM_{\alpha_k}) = t' \cdot (h'_0M_{\alpha_0'}, \ldots, h'_kM_{\alpha_k'})$ for $k$-simplices $(h_0M_{\alpha_0}, \ldots, h_kM_{\alpha_k})$ and $(h'_0M_{\alpha_0'}, \ldots, h'_kM_{\alpha_k'})$ of $\Delta'$ then we have $t \cdot h_iM_{\alpha_i} = t' \cdot h'_iM_{\alpha'_i}$ for each $i$. In particular, this implies that $tH = t'H$ and so that $t = t'$. Hence $C_k(\Delta) = \bigoplus_{t \in T} t \cdot C_k(\Delta') = \text{Ind}_H^G C_k(\Delta')$. □

The boundary map $\partial : C_k(\Delta) \rightarrow C_{k-1}(\Delta)$ commutes with the action of $G$ and its restriction to $C_{k-1}(\Delta')$ is the boundary map $\partial' : C_k(\Delta') \rightarrow C_{k-1}(\Delta')$. Thus the homology spaces of $\Delta$ must be induced from the homology spaces of $\Delta'$.

**Proposition 4.4.** $\tilde{H}_k(\Delta) = \text{Ind}_H^G \tilde{H}_k(\Delta')$ for each $k$.

4.3. Homology spaces of $\Delta'$

Proposition 4.4 implies that to determine the homology spaces of $\Delta$ it is sufficient to describe the homology spaces of $\Delta'$. The approach used by Lees [15] was to show that $\Delta'$ could be constructed from simpler simplicial complexes.

Recall that the join of simplicial complexes $\Omega$ and $\Omega'$ is the simplicial complex $\Omega * \Omega'$ whose vertices are the vertices of $\Omega$ and $\Omega'$ and whose $k$-simplices are of the form $(\omega_0, \ldots, \omega_i, \omega_{i+1}, \ldots, \omega_k)$ where $(\omega_0, \ldots, \omega_i)$ and $(\omega_{i+1}, \ldots, \omega_k)$ are simplices of $\Omega$ and $\Omega'$ respectively. The reduced homology spaces of the join are given by the formula (see [17])

$$\tilde{H}_k(\Omega * \Omega') = \bigoplus_{i+j=k-1} \tilde{H}_i(\Omega) \otimes \tilde{H}_j(\Omega').$$

(5)

For each $\alpha \in S$ let $\Delta'_\alpha$ be the subcomplex of $\Delta'$ consisting only of the $q$ distinct vertices $\{gM_\alpha : g \in H\}$ and note that its reduced homology spaces are

$$\tilde{H}_k(\Delta'_\alpha) = \begin{cases} \mathbb{Z}^{q-1} & \text{if } k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

(6)

**Lemma 4.5.** $\Delta'$ is the join of the $\Delta'_\alpha$ as $\alpha$ runs through $S$ in ascending order.
Proof. Each $k$-simplex of the join has the form $(g_0 M_{\alpha_0}, \ldots, g_k M_{\alpha_k})$ for some $g_0, \ldots, g_k \in H$ and $\alpha_0 < \cdots < \alpha_k$. Thus we only need to show that $g_0 M_{\alpha_0} \cap \cdots \cap g_k M_{\alpha_k} \neq \emptyset$. For each $i$ choose $x_i \in X_{\alpha_i}$ with $g_i M_{\alpha_i} = x_i M_{\alpha_i}$ and consider $g = x_0 \cdots x_k \in H$. Then $g M_{\alpha_i} = x_0 \cdots x_k M_{\alpha_i} = x_i M_{\alpha_i} = g_i M_{\alpha_i}$ which implies that $g \in g_i M_{\alpha_i}$ for every $i$. \hfill \Box

By repeatedly applying (5) we find that the reduced homology spaces of $\Delta'$ are

$$\tilde{H}_k(\Delta') = \bigoplus_{i_\alpha} \bigotimes_{\alpha \in S} \tilde{H}_{i_\alpha}(\Delta'_{\alpha})$$

where the direct sum runs over all $\{i_\alpha : \alpha \in S\}$ such that $\sum_{\alpha \in S} i_\alpha = k - n + 1$. Hence, using (6) this gives

$$\tilde{H}_k(\Delta') = \begin{cases} \mathbb{Z}(q-1)^n & \text{if } k = n - 1; \\ 0 & \text{otherwise} \end{cases}$$

and the homology spaces of $\Delta'$ are as follows.

**Proposition 4.6.** For each $k$

$$H_k(\Delta') = \begin{cases} \mathbb{Z} & \text{if } k = 0; \\ \mathbb{Z}(q-1)^n & \text{if } k = n - 1; \\ 0 & \text{otherwise}. \end{cases}$$

4.4. Characters

Having described the homology spaces of $\Delta'$ we would like to show that $St_\ell$ is afforded by the $(n - 1)$st homology space of $\Delta$. We begin by examining the character afforded by the 0th homology space of $\Delta'$.

**Lemma 4.7.** $H_0(\Delta') \otimes \mathbb{C}$ affords the trivial character of $H$.

**Proof.** Let $\alpha$ be the minimal root in $S$. For each $g \in H$ and $\beta \in S - \{\alpha\}$ we see that $(M_{\alpha}, gM_\beta)$ is a 1-simplex with $\partial_1(M_{\alpha}, gM_\beta) = (M_{\alpha}) - (gM_\beta)$ and so $[gM_\beta] = [M_{\alpha}]$ in $H_0(\Delta')$. Further, $(gM_{\alpha}, gM_\beta)$ is a 1-simplex such that $\partial_1(gM_{\alpha}, gM_\beta) = (gM_{\alpha}) - (gM_\beta)$ which implies that $[gM_\beta] = [gM_{\alpha}] = [M_{\alpha}]$. Hence $H_0(\Delta') \otimes \mathbb{C} = \mathbb{C}[M_{\alpha}]$ and $g \cdot [M_{\alpha}] = [gM_{\alpha}] = [M_{\alpha}]$ for every $g \in H$. \hfill \Box

Now, Proposition 4.4 implies that $H_0(\Delta) \otimes \mathbb{C}$ affords the permutation character $(1_H)^G$ while the Hopf trace formula states that

$$\sum_{k=0}^{n-1} (-1)^k \text{tr}(g, H_k(\Delta) \otimes \mathbb{C}) = \sum_{k=0}^{n-1} (-1)^k \text{tr}(g, C_k(\Delta) \otimes \mathbb{C}).$$

However, we already know that $C_k(\Delta) \otimes \mathbb{C}$ affords the character

$$\sum_{|J|=n-k-1} (1_{H_J})^G$$
so if \( \zeta \) denotes the character afforded by \( H_{n-1}(\Delta) \otimes \mathbb{C} \) then, by Proposition 4.6,

\[
(-1)^{n-1} \zeta + (1_H)^G = \sum_{J \subseteq S} (-1)^{n-1-|J|}(1_{H_J})^G.
\]

Rearranging this gives

\[
\zeta = \sum_{J \subseteq S} (-1)^{|J|}(1_{H_J})^G
\]

and we have shown the following.

**Theorem 4.8.**

(i) \( H_0(\Delta) \otimes \mathbb{C} \) affords the permutation character \( (1_H)^G \);
(ii) \( H_{n-1}(\Delta) \otimes \mathbb{C} \) affords the character \( St_\ell = \sum_{J \subseteq S} (-1)^{|J|}(1_{H_J})^G \);
(iii) \( H_k(\Delta) = 0 \) if \( k \neq 0 \) or \( n - 1 \).

5. Reducibility

Both the Steinberg character for the Chevalley group over the residue field and its analogue for the general linear group are irreducible. However, it transpires that this is not generally true for \( St_\ell \) and so we would like to determine exactly when it is irreducible. Using the expression of \( St_\ell \) as an alternating sum of permutation characters we obtain

\[
(St_\ell, St_\ell) = \sum_{I, J \subseteq S} (-1)^{|I|+|J|}((1_{H_I})^G, (1_{H_J})^G) = \sum_{I, J \subseteq S} (-1)^{|I|+|J|}|D_G(H_I, H_J)|
\]

where \( D_G(H_I, H_J) \) denotes the set of \((H_I, H_J)\)-double cosets in \( G \). We therefore need to examine the double coset structure of \( G \) and we begin by describing the double cosets belonging to \( H \).

5.1. \( T \)-orbits in \( X \)

Let \( X = X_S = \{ \prod_{\alpha \in S} x_\alpha(r_\alpha) : r_\alpha \in \mathfrak{m}^{\ell-1} \} \), then \( X \) forms a left transversal for \( B \) in \( H \) whose elements are permuted by \( T \) and for any \( x \in X \) we have

\[
TxB = \bigcup_{t \in T} txB = \bigcup_{t \in T} t(x^{-1}B) = \bigcup_{x' \in T \cdot x} x'B.
\]

Consequently, the \((T, B)\)-double cosets in \( H \) are given by the \( T \)-orbits in \( X \).

Now, for each \( I \subseteq S \) consider

\[
\chi_I = X_I - \bigcup_{J \subset I} X_J;
\]
that is, \( \mathcal{X}_\emptyset = \{1\} \) and \( \mathcal{X}_I = \{ \prod_{\alpha \in I} x_\alpha(r_\alpha) : r_\alpha \in \pi^{\ell - 1} R^\times \text{ for } \alpha \in I \} \) for each non-empty \( I \subseteq S \).

In particular, \( T \) must permute the elements in \( \mathcal{X}_I \) since it normalises each \( X_J \). Thus we will consider the decomposition

\[
\mathcal{X}_I = \bigcup_{i=1}^{d_I} \mathcal{X}_{I,i}
\]

of \( \mathcal{X}_I \) into its distinct \( T \) orbits \( \mathcal{X}_{I,1}, \ldots, \mathcal{X}_{I,d_I} \) and choose a representative \( x_{I,i} \in \mathcal{X}_{I,i} \) for each \( i \).

We then obtain the disjoint union

\[
H = \bigcup_{I \subseteq S} \bigcup_{i=1}^{d_I} TX_{I,i}B
\]

where \( TX_{I,i}B = X_{I,i}B \).

Although it is difficult to calculate \( d_I \) in general, when \( I = S \) it is equal to the index \( d \) of \( G \) in \( \overline{G} \).

**Lemma 5.1.** \( d_S = d \).

**Proof.** Suppose that \( x \in \mathcal{X}_S \) with \( x = \prod_{\alpha \in S} x_\alpha(r_\alpha) \) for some \( r_\alpha \in \pi^{\ell - 1} R^\times \). Then \( h(\mu) \in \text{Stab}_T(x) \) if and only if \( x = h(\mu)xh(\mu)^{-1} = \prod_{\alpha \in S} x_\alpha(\mu(\alpha)r_\alpha) \). However, this is equivalent to requiring that \( \mu(\alpha)r_\alpha = r_\alpha \) and so \( \mu(\alpha) \in 1 + m \) for each \( \alpha \in S \). Thus, \( \text{Stab}_T(x) = T(1) \) and the size of the orbit containing \( x \) is \( [T : \text{Stab}_T(x)] = [T : T(1)] = (q - 1)^n/d \) implying that the number of distinct orbits in \( \mathcal{X}_S \) is \( d \). \( \square \)

Further, for the extended Chevalley group we have \( d_I = 1 \) for each \( I \subseteq S \).

**Lemma 5.2.** \( \overline{T} \) transitively permutes the elements in \( \mathcal{X}_I \) for each \( I \subseteq S \).

**Proof.** Let \( x, x' \in \mathcal{X}_I \) so that \( x = \prod_{\alpha \in I} x_\alpha(r_\alpha) \) and \( x' = \prod_{\alpha \in I} x_\alpha(r'_\alpha) \) where \( r_\alpha, r'_\alpha \in \pi^{\ell - 1} R^\times \). For each \( \alpha \in I \) let \( s_\alpha \in R^\times \) be such that \( r'_\alpha = s_\alpha r_\alpha \) and set \( s_\alpha = 1 \) for \( \alpha \notin I \). If we define \( \mu \in \text{Hom}(\Lambda_r, R^\times) \) with \( \mu(\alpha) = s_\alpha \) for each \( \alpha \in S \) then \( h(\mu)xh(\mu)^{-1} = \prod_{\alpha \in I} h(\mu)x_\alpha(r_\alpha)h(\mu)^{-1} = \prod_{\alpha \in I} x_\alpha(s_\alpha r_\alpha) = x' \). Hence the action of \( \overline{T} \) on \( \mathcal{X}_I \) is transitive. \( \square \)

### 5.2. Double coset structure of \( H \)

We have described the \( (T, B) \)-cosets in \( H \) in terms of the \( T \)-orbits in \( X \) and so to determine the \( (B, B) \)-double cosets we need to examine the action of \( U \). In fact, \( U \) acts trivially on the left cosets of \( B \) in \( H \).

**Lemma 5.3.** \( Ux^kB = x^kB \) for any \( x \in X \).
Lemma 5.4. Obtain the following result from (7).

Writing \( x \in X \) as \( x = \prod_{\alpha \in S} x_\alpha(r_\alpha) \) we find that, since \( x_\alpha(-r_\alpha) \) commutes with each commutator \([x_\alpha(-r_\alpha), u]\),

\[
[x^{-1}, u] = \prod_{\alpha \in S} [x_\alpha(-r_\alpha), u] \in B(\ell - 1).
\]

Hence \( ux = x[x^{-1}, u]u^{-1} \in xB \) for each \( u \in U \) and so \( UxB = xB \). □

In particular, this means that \( BxB = TUxB = TBx \) for each \( x \in X \) and we immediately obtain the following result from (7).

**Lemma 5.4.** H can be expressed as the disjoint union

\[
H = \bigcup_{I \subseteq S} \bigcup_{i=1}^{d_I} Bx_{I,i}B
\]

where \( Bx_{I,i}B = X_{I,i}B \).

Further, we can use this to describe the \((HJ, HJ')\)-double cosets in \( H \).

**Proposition 5.5.** For each \( J, J' \subseteq S \) we can express \( H \) as the disjoint union

\[
H = \bigcup_{I \subseteq S - (J \cup J')} \bigcup_{i=1}^{d_I} H_{Jx_{I,i}H_{J'}}.
\]

**Proof.** Let \( x \in X \) and express it as \( x = \prod_{\alpha \in S} x_\alpha(r_\alpha) \) for some \( r_\alpha \in m^{\ell - 1} \). Further, define \( x_1, x_2 \) and \( x_3 \) to be the products of the \( x_\alpha(r_\alpha) \) as \( \alpha \) runs over the sets \( J, S - (J \cup J') \) and \( J' - J \) respectively. Then \( x = x_1x_2x_3 \) with \( x_1 \in H_J \) and \( x_3 \in H_{J'} \) and so \( H_{Jx_{I,i}H_{J'}} = H_{Jx_{1}H_{J'}} \). Thus each \((HJ, HJ')\)-double coset representative may be chosen to lie in \( X_{S - (J \cup J')} \) and, by Lemma 5.4, we may express \( H \) as the union (8).

To show that this union is disjoint suppose that \( H_{Jx_{I,i}H_{J'}} = H_{Jx_{I',i}H_{J'}} \) for some \( I, I' \subseteq S - (J \cup J') \). Then \( x_{I,i} = gx_{I',i}g' \) for some \( g \in H_J \) and \( g' \in H_{J'} \). If we write \( g = xb \) and \( g' = b'x' \) where \( b, b' \in B, x \in X_J \) and \( x' \in X_{J'} \) then \( bx_{I',i}b' = x^{-1}x_{I,i}(x')^{-1} = x^{-1}(x')^{-1}x_{I,i} \). This implies that \( x^{-1}(x')^{-1}x_{I,i} \in X_I \) and thus that \( x^{-1}(x')^{-1} \in X_{I\cup J'} \). Since \( x_{I',i} \in X_{J'} \). However, we also have \( x^{-1}(x')^{-1} \in X_{I\cup J'} \) which means that \( x^{-1}(x')^{-1} \in X_{I\cup J'} \cap X_{J\cup J'} = \{1\} \). Hence \( Bx_{I,i}B = Bx_{I',i}B \) and so \( x_{I,i} = x_{I',i} \). □

5.3. Counting the cosets

From Lemma 5.1 we know that there are \( d \) double cosets of the form \( Bx_{S,i}B \) but in general we are not able to count explicitly the number of \((HJ, HJ')\)-double cosets in \( H \). However, it suffices to show that the remainder can be paired up in a particular way.

For each \( J, J' \subseteq S \) define

\[
\mathcal{E}(H_J, H_{J'}) = \begin{cases} D_H(B, B) - \{Bx_{S,1}B, \ldots, Bx_{S,d}B\} & \text{if } J = J' = \emptyset; \\ D_H(H_J, H_{J'}) & \text{otherwise} \end{cases}
\]
and for a fixed $J' \subseteq S$ let

$$\mathcal{E} = \bigcup_{J \subseteq S} \mathcal{E}(H_J, H_{J'}) .$$

Then $\mathcal{E}$ is clearly the disjoint union of the subsets

$$\mathcal{E}_0 = \bigcup_{|J| \text{ even}} \mathcal{E}(H_J, H_{J'}) \text{ and } \mathcal{E}_1 = \bigcup_{|J| \text{ odd}} \mathcal{E}(H_J, H_{J'}) .$$

**Proposition 5.6.** $|\mathcal{E}_0| = |\mathcal{E}_1| .

**Proof.** Fix an ordering of the roots in $S$ and consider $H_J x_{l,i} H_{J'} \in \mathcal{E}$ with $I \subseteq S - (J \cup J')$. We have excluded the double cosets with $I = S$ so we may choose a minimal root $\alpha \in S - I$. Thus we are able to define a map $\Phi : \mathcal{E} \rightarrow \mathcal{E}$ by setting

$$\Phi(H_J x_{l,i} H_{J'}) = \begin{cases} H_J - \{\alpha\} x_{l,i} H_{J'} & \text{if } \alpha \in J; \\ H_J \cup \{\alpha\} x_{l,i} H_{J'} & \text{if } \alpha \notin J. \end{cases}$$

Note that $I \subseteq S - (J - \{\alpha\} \cup J')$ if $\alpha \in J$ and $I \subseteq S - (J \cup \{\alpha\} \cup J')$ if $\alpha \notin J$. Consequently $\Phi$ is well defined since we are using the particular double coset representatives $x_{l,i}$ from Proposition 5.5.

Suppose that $H_J x_{l,i} H_{J'} \in \mathcal{E}$ with $\alpha \in S - I$ minimal. If $\alpha \in J$ then

$$\Phi^2(H_J x_{l,i} H_{J'}) = \Phi(H_J - \{\alpha\} x_{l,i} H_{J'}) = H_J - \{\alpha\} \cup \{\alpha\} x_{l,i} H_{J'} = H_J x_{l,i} H_{J'}$$

whereas if $\alpha \notin J$ then

$$\Phi^2(H_J x_{l,i} H_{J'}) = \Phi(H_J \cup \{\alpha\} x_{l,i} H_{J'}) = H_J \cup \{\alpha\} - \{\alpha\} x_{l,i} H_{J'} = H_J x_{l,i} H_{J'} .$$

Thus $\Phi^2$ is the identity and $\Phi$ is a bijection. Further, $\Phi$ sends double cosets in $\mathcal{E}_0$ to double cosets in $\mathcal{E}_1$ and vice versa. Hence, $\Phi$ must restrict to a bijection between $\mathcal{E}_0$ and $\mathcal{E}_1$. \qed

### 5.4. Reducibility of $\chi$

The pairing described in Proposition 5.6 allows us to calculate the inner product of $\chi$ with the permutation character over $H_J$.

**Lemma 5.7.** For each $J \subseteq S$,

$$(\chi, (1_{H_J})^H) = \begin{cases} d & \text{if } J = \emptyset; \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** We have

$$(\chi, (1_{H_J})^H) = \sum_{I \subseteq S} (-1)^{|I|}(1_{H_I})^H, (1_{H_J})^H) = \sum_{I \subseteq S} (-1)^{|I|} |D_H(H_I, H_J)| .$$
If \( J \neq \emptyset \) then this gives \( (\chi, (1_{H_J})^H) = |E_0| - |E_1| = 0 \) whereas if \( J = \emptyset \) then we obtain \( (\chi, (1_B)^H) = (|E_0| + d) - |E_1| = d \). \( \square \)

**Proposition 5.8.** \( \chi \) is a constituent of \( (1_B)^H \) of degree \( \chi(1) = (q - 1)^n \) which is irreducible exactly when \( G = \overline{G} \).

**Proof.** By Lemma 5.7

\[
(\chi, \chi) = \sum_{J \subseteq S} (-1)^{|J|}(\chi, (1_{H_J})^H) = (\chi, (1_B)^H) = d
\]

and so \( (\chi, \chi) = 1 \) if and only if \( d = [\overline{G} : G] = 1 \). Further,

\[
\chi(1) = \sum_{J \subseteq S} (-1)^{|J|}[H : H_J] = \sum_{J \subseteq S} (-1)^{|J|}q^{|J| - |J|} = (q - 1)^n.
\]

\( \square \)

5.5. A BN-pair for \( H \)

If \( G \) is the extended Chevalley group then Proposition 5.8 implies that \( \chi \) is an irreducible character of \( H \). In fact, we will show that in this situation \( B \) forms part of a BN-pair for \( H \) and \( \chi \) is actually the Steinberg character of \( H \), at least when \( q \neq 2 \).

For each \( \alpha \in S \) define

\[
\sigma_\alpha = x_\alpha (\pi^{\ell-1})y_{-\alpha}(-1)
\]

then

\[
\sigma_\alpha^2 = x_\alpha (\pi^{\ell-1})y_{-\alpha}(-1)x_\alpha (\pi^{\ell-1})y_{-\alpha}(-1) = x_\alpha (\pi^{\ell-1})x_\alpha (-\pi^{\ell-1}) = 1.
\]

Further, for each \( \alpha, \beta \in S \) we have \( \sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha \), since \( x_\alpha (\pi^{\ell-1}) \) and \( y_{-\alpha}(-1) \) commute with \( x_\beta (\pi^{\ell-1}) \) and \( y_{-\beta}(-1) \). Consequently, if we let \( \sigma_\emptyset = 1 \) and, for each non-empty subset \( J = \{\alpha_1, \ldots, \alpha_k\} \subseteq S \),

\[
\sigma_J = \sigma_{\alpha_1} \cdots \sigma_{\alpha_k}
\]

then the group generated by \( \{\sigma_\alpha: \alpha \in S\} \) is \( N = \{\sigma_J: J \subseteq S\} \).

Now, for each non-empty \( J \subseteq S \)

\[
\sigma_J = x_{\alpha_1} (\pi^{\ell-1}) \cdots x_{\alpha_k} (\pi^{\ell-1}) y_{-\alpha_1} (-1) \cdots y_{-\alpha_k} (-1) \in X_J B
\]

and so, by Lemma 5.2, we must have \( B\sigma_J B = X_J B \). Thus \( N \) forms a complete set of \((B, B)\)-double coset representatives in \( H \). Indeed, the multiplication of the double cosets is closely related to the multiplication in \( N \).

**Lemma 5.9.** For each \( \alpha \in S \) and \( J \subseteq S \)

\[
(B\sigma_\alpha B)(B\sigma_J B) \subseteq B\sigma_J B \cup B\sigma_\alpha \sigma_J B.
\]

Moreover, if \( \alpha \in J \) and \( q \neq 2 \) then this is an equality.
**Proof.** First note that \((B \sigma_\alpha B)(B \sigma_J B) = (X_\alpha B)(X_J B) = X_\alpha X_J B\). If \(\alpha \notin J\) then \(X_\alpha X_J = X_{J \cup \{\alpha\}}\) and \(X_\alpha X_J B = X_{J \cup \{\alpha\}} B = B \sigma_J \cup \{\alpha\} B = B \sigma_\alpha \sigma_J B\). Further, if \(\alpha \in J\) then \(X_\alpha X_J = X_{\alpha J - \{\alpha\}} \subseteq X_{\alpha J - \{\alpha\}} = X_J \cup X_{J - \{\alpha\}}\) and \(X_\alpha X_J B \subseteq X_J B \cup X_{J - \{\alpha\}} B = B \sigma_J B \cup B \sigma_\alpha \sigma_J B\). Indeed, if \(q \neq 2\) then \(X_\alpha X_\alpha = X_\alpha\) and so this is an equality.  

**Proposition 5.10.**  
(i) If \(q \neq 2\) then \(B\) and \(N\) form a BN-pair for \(H\) with elementary abelian Weyl group.  
(ii) If \(q = 2\) then \(B\) is normal in \(H\) with \(H/B \cong N\).  

**Proof.** (i) It is clear that \(H = \langle B, N \rangle\) and \(B \cap N = \{1\}\) is trivially normal in \(N\) with \(N/(B \cap H) \cong N\) generated by the involutions \(\{\sigma_\alpha\}_{\alpha \in S}\). Further, by Lemma 5.9 we know that for any \(\alpha \in S\) and \(J \subseteq S\)

\[\sigma_\alpha B \sigma_J \subseteq B \sigma_J B \cup B \sigma_\alpha \sigma_J B\]

and if \(q \neq 2\) then \((B \sigma_\alpha B)(B \sigma_\alpha B) = B \cup B \sigma_\alpha B\) implies that \(\sigma_\alpha B \sigma_\alpha \neq B\).

(ii) Suppose that \(q = 2\) and note that in this case \(N\) forms a left transversal for \(B\) in \(H\). For each \(\alpha \in S\) we have \([H_\alpha : B] = 2\) and so \(\sigma_\alpha B \sigma_\alpha^{-1} = B\). Thus \(\sigma_J B \sigma_J^{-1} = B\) for each \(J \subseteq S\) implying that \(B\) is normal in \(H\) with \(H/B \cong N\).  

**Remark 5.11.**

(i) If \(q \neq 2\) then for each \(J \subseteq S\) we have \(H_J = B \langle \sigma_\alpha : \alpha \in J \rangle B\) and so these are the parabolic subgroups associated to the BN-pair in \(H\). Consequently, the expression for \(\chi\) as an alternating sum is identical to Curtis’ formula [5] for the Steinberg character of \(H\) and \(\Delta'\) is the combinatorial building for \(H\) [19].

(ii) Similarly, if \(q = 2\) then \(H_J /B = \langle \sigma_\alpha : \alpha \in J \rangle\) for each \(J \subseteq S\) and so these are the parabolic subgroups in the Coxeter group \(N\). The expression for \(\chi\) as an alternating sum is therefore Solomon’s formula [18] for the sign character of \(N\) inflated to \(H\) and \(\Delta'\) is the Coxeter complex for \(N\).

6. Reducibility continued

Having determined the reducibility of \(\chi\) we now turn our attention to \(\text{St}_\ell\). In particular, we need to examine the remaining double cosets in \(G\).

6.1. Double coset structure of \(G\)

**Theorem 6.1.** Each double coset \(BgB\) in \(G - H\) is of the form \(BgB = H_\alpha gB\) for some \(\alpha \in S\) depending only on the double coset \(HgH\).

To prove Theorem 6.1 we need to show that for every \(r \in m^{\ell - 1}\) we have \(BgB = Bx_\alpha(r)gB\) since then

\[H_\alpha gB = \bigcup_{r \in m^{\ell - 1}} Bx_\alpha(r)gB = BgB.\]
Indeed, it is enough to prove that \([b, g] = b' x_α(r)\) for some \(b, b' \in B\) since this gives \(g = b^{-1}[b, g]g b = b'^{-1} b x_α(r) g b \in B x_α(r) g B\). In fact, we will show the following generalisation of [2, Proposition 27] where \(V'\) denotes the normal subgroup \(V' = T(1)U\) of \(B\). However, due to its complexity we delay the proof until Section 10.

**Theorem 6.2.** If \(B g B\) is a double coset in \(G - H\) then there exists an \(α \in S\) so that for every \(r \in m^{ℓ-1}\) we have \([v, g] = v' x_α(r)\) for some \(v, v' \in V'\). Moreover, \(α\) depends only on the double coset \(H g H\).

From Theorem 6.1 we obtain a similar result regarding the \((H J, H J')\)-double cosets in \(G - H\).

**Lemma 6.3.** Let \(H J g H J'\) be a double coset contained in \(G - H\) then there is an \(α \in S\), depending only on the double coset \(H g H\), with

\[
H J g H J' = \begin{cases} H J_{-α} g H J' & \text{if } α \in J; \\ H J_{∪α} g H J' & \text{if } α ∉ J. \end{cases}
\]

**Proof.** First note that if \(H J g H J'\) belongs to \(G - H\) then clearly so must \(B g B\). Thus by Theorem 6.1 there is an \(α \in S\), which depends only on the double coset \(H g H\), such that \(B g B = H α g B H J' = H J_{-α} g H J'\) whereas if \(α ∉ J\) then \(H J g H J' = H J B g B H J' = H J H α g B H J' = H J_{∪α} g H J'\).

### 6.2. Counting the cosets

Although we are not able to describe explicitly the double cosets belonging to \(G - H\), Lemma 6.3 enables us to pair them up in a manner similar to Section 5.3.

Fix \(J' ⊆ S\) and consider

\[
D = \bigcup_{J ⊆ S} D_{G - H}(H J, H J')
\]

where \(D_{G - H}(H J, H J')\) denotes the \((H J, H J')\)-double cosets lying in \(G - H\). Then \(D\) decomposes as the disjoint union of subsets

\[
D_0 = \bigcup_{|J| \text{ even}} D_{G - H}(H J, H J') \quad \text{and} \quad D_1 = \bigcup_{|J| \text{ odd}} D_{G - H}(H J, H J').
\]

**Proposition 6.4.** \(|D_0| = |D_1|\).

**Proof.** Fix an ordering of the roots in \(S\) and for each \(H J g H J'\) choose the minimal \(α \in S\) satisfying the conclusion of Lemma 6.3. We then define a map \(Ψ : D → D\) by setting

\[
Ψ(H J g H J') = \begin{cases} H J_{-α} g H J' & \text{if } α \in J; \\ H J_{∪α} g H J' & \text{if } α ∉ J. \end{cases}
\]

This is well defined since if \(H J g H J' = H J' g H J'\) then the minimal \(α \in S\) must be the same for both representatives and Lemma 6.3 implies that, as sets, we have \(Ψ(H J g H J') = H J g H J' = H J' g H J' = H J g' H J' = Ψ(H J g' H J')\).
Now, suppose that $H_J g H_J' \in \mathcal{D}$ and that $\alpha \in S$ is the minimal root satisfying the conclusion of Lemma 6.3. If $\alpha \in J$ then $\alpha$ is also minimal for the double coset $H_{J-\{\alpha\}} g H_J$ and therefore

$$\Psi^2(H_J g H_J') = \Psi(H_J-\{\alpha\} g H_J') = H_{J-\{\alpha\}} g H_J' = H_J g H_J'.$$

Similarly, if $\alpha \notin J$ then $\alpha$ is minimal for $H_{J \cup \{\alpha\}} g H_J'$ which gives

$$\Psi^2(H_J g H_J') = \Psi(H_{J \cup \{\alpha\}} g H_J') = H_{J \cup \{\alpha\}-\{\alpha\}} g H_J' = H_J g H_J'.$$

Consequently, $\Psi^2$ is the identity and $\Psi$ is a bijection. Further, $\Psi$ clearly sends double cosets from $\mathcal{D}_0$ to double cosets in $\mathcal{D}_1$ and vice versa. Hence $\Psi$ must restrict to a bijection between $\mathcal{D}_0$ and $\mathcal{D}_1$. □

6.3. Reducibility of $\text{St}_\ell$

Using Proposition 6.4 we see that the inner product of the analogue with the permutation character over $H_J$ can be obtained from the corresponding inner product for $\chi$.

**Proposition 6.5.** For each $J \subseteq S$ we have $(\text{St}_\ell, (1_{H_J})^G) = (\chi, (1_{H_J})^H)$.

**Proof.** Fix $J \subseteq S$, then

$$(\text{St}_\ell, (1_{H_J})^G) = \sum_{I \subseteq S} (-1)^{|I|} ((1_{H_I})^G, (1_{H_J})^G)$$

$$= \sum_{I \subseteq S} (-1)^{|I|} |\mathcal{D}_G(H_I, H_J)|$$

$$= \sum_{I \subseteq S} (-1)^{|I|} |\mathcal{D}_H(H_I, H_J)| + \sum_{I \subseteq S} (-1)^{|I|} |\mathcal{D}_G-H(H_I, H_J)|$$

$$= (\chi, (1_{H_J})^H) + (|\mathcal{D}_0| - |\mathcal{D}_1|)$$

$$= (\chi, (1_{H_J})^H)$$

as required. □

**Corollary 6.6.** For each $J \subseteq S$,

$$(\text{St}_\ell, (1_{H_J})^G) = \begin{cases} d & \text{if } J = \emptyset; \\ 0 & \text{otherwise}. \end{cases}$$

**Theorem 6.7.** $\text{St}_\ell$ is a constituent of $(1_B)^G$ of degree

$$\text{St}_\ell(1) = q^{N\ell-n}(q-1)^n \sum_{w \in W} q^\ell(w)$$

which is irreducible if and only if $G = \overline{G}$. 
Proof. By Corollary 6.6

\[(St_\ell, St_\ell) = \sum_{J \subseteq S} (-1)^{|J|}(St_\ell, (1_{H_J})^G) = (St_\ell, (1_B)^G) = d\]

and so \((St_\ell, St_\ell) = 1\) if and only if \(d = [\overline{G} : G] = 1\). Further, since \(St_\ell\) is induced from the degree \((q - 1)^n\) character \(\chi\) of \(H\), we obtain

\[St_\ell(1) = \chi(1)[G : H] = \frac{\chi(1)}{[H : B]}[G : B] = \frac{(q - 1)^n}{q^n} q^{N_\ell} \sum_{w \in W} q^{\ell(w)}. \]

In particular, when \(G = \overline{G}\) we obtain the following characterisation for the analogue which is similar to Curtis’ characterisation [5] of the Steinberg character.

Proposition 6.8. \(\overline{St}_\ell\) is the unique irreducible constituent of \((1_B)\overline{G}\) which is not an irreducible constituent of \((1_P)\overline{G}\) for any parabolic subgroup \(P > B\).

Proof. The argument is identical to [2, Theorem 30]. \(\Box\)

7. Hecke algebras

Iwahori [13], and Matsumoto [16] more generally, proved that the Hecke algebra of a Chevalley group over a finite field has a presentation similar to that of its Weyl group as a Coxeter group. Thus it is possible to define a linear character of the Hecke algebra that is analogous to the sign character of the Weyl group. This linear character then corresponds to a unique irreducible character of the Chevalley group which Curtis, Iwahori and Kilmoyer [7] showed was the Steinberg character.

Before examining the decomposition of the analogue of the Steinberg character into its irreducible constituents we describe a similar construction of \(St_\ell\) for the extended Chevalley group in terms of its Hecke algebra with respect to \(B\). Throughout this section we will assume that \(G = \overline{G}\) and therefore that \(St_\ell\) is irreducible.

7.1. Definition and standard results

The Hecke algebra of \(G\) with respect to \(B\) is the subalgebra \(\mathcal{H}(G, B)\) of \(\mathbb{C}G\) defined by

\[\mathcal{H}(G, B) = e_B \mathbb{C}Ge_B,\]

and a basis is given by the set \(\{\beta_g\}_{B \subseteq G, B \in \mathcal{D}_G(B, B)}\) where

\[\beta_g = |B|^{-1} \sum_{x \in BgB} x.\]

The representation theory of \(G\) is related to the representation theory of the Hecke algebra in the following way (see [8]).
Proposition 7.1. If $\zeta$ is an irreducible constituent of $(1_B)^G$ then its restriction to $\mathcal{H}(G, B)$ is an irreducible character of $\mathcal{H}(G, B)$ whose degree is the multiplicity of $\zeta$ in $(1_B)^G$. Conversely, each irreducible character $\psi$ of $\mathcal{H}(G, B)$ is the restriction of a unique irreducible constituent of $(1_B)^G$.

In particular, linear characters of $\mathcal{H}(G, B)$ correspond uniquely to irreducible characters of $G$ appearing as multiplicity 1 constituents of $(1_B)^G$.

7.2. Hecke algebra of $H$

We know from Lemma 5.4 that the $(B, B)$-double cosets of $H$ are of the form $BxJB = X_J B$ for $J \subseteq S$ and some $x_J \in X_J$. Thus, if $\beta_J$ denotes the basis element corresponding to $Bx_JB$, then

$$\beta_J = e'_{X_J} e_B$$

where $e'_{X_J} = \sum_{x \in X_J} x$. In particular, $\beta_\alpha = q e_{H_\alpha} - e_B$ for each $\alpha \in S$.

Although the following presentation for the Hecke algebra is a consequence of $H$ possessing a $BN$-pair with elementary abelian Weyl group, we will prove it directly using the above description of the basis elements.

Proposition 7.2. $\mathcal{H}(H, B)$ is generated by the elements $\{\beta_\alpha : \alpha \in S\}$ together with the quadratic relations

$$\beta_\alpha^2 = (q - 1)\beta_\emptyset + (q - 2)\beta_\alpha$$

(9)

and homogeneous relations

$$\beta_\alpha \beta_{\alpha'} = \beta_{\alpha'} \beta_\alpha.$$  

(10)

Proof. Let $\alpha \in S$ and $J \subseteq S$, then $\beta_\alpha \beta_J = e'_{X_\alpha} e_B e'_{X_J} e_B = e'_{X_\alpha} e'_{X_J} e_B$. If $\alpha \notin J$ then $e'_{X_\alpha} e'_{X_J} = e'_{X_{J\cup \{\alpha\}}}$ which gives

$$\beta_\alpha \beta_J = e'_{X_{J\cup \{\alpha\}}} e_B = \beta_{J\cup \{\alpha\}}$$

(11)

whereas if $\alpha \in J$ then $e'_{X_\alpha} e'_{X_J} = e'_{X_\alpha} e'_{X_\alpha} e'_{X_J - \{\alpha\}} = (q - 1)e'_{X_J - \{\alpha\}} + (q - 2)e'_{X_J}$ and

$$\beta_\alpha \beta_J = (q - 1)e'_{X_J - \{\alpha\}} e_B + (q - 2)e'_{X_\alpha} e'_{X_J - \{\alpha\}} e_B = (q - 1)\beta_{J - \{\alpha\}} + (q - 2)\beta_J.$$  

(12)

Now, (11) implies that $\mathcal{H}(H, B)$ is generated by $\{\beta_\alpha : \alpha \in S\}$ and is commutative, since $\beta_\alpha \beta_{\alpha'} = \beta_{\alpha', \alpha'} = \beta_{\alpha'} \beta_\alpha$ for $\alpha \neq \alpha'$. Further, it is clear that (9) is simply a special case of (12). Indeed, (12) can be obtained from (9) and (11) since if $\alpha \in J$ then

$$\beta_\alpha \beta_J = \beta_\alpha^2 \beta_{J - \{\alpha\}} = (q - 1)\beta_\emptyset \beta_{J - \{\alpha\}} + (q - 2)\beta_\alpha \beta_{J - \{\alpha\}} = (q - 1)\beta_{J - \{\alpha\}} + (q - 2)\beta_J.$$  

Hence (9) and (10) are the defining relations for $\mathcal{H}(H, B)$.  \qed
Rewriting (9) as
\[(\beta_\alpha - (q-1)\beta_\phi)(\beta_\alpha + \beta_\phi) = 0\]
we see that we may define a homomorphism \(\phi : \mathcal{H}(H, B) \to \mathbb{C}\) by setting \(\phi(\beta_\alpha) = -1\) for each \(\alpha \in S\) and so \(\phi(\beta_J) = (-1)^{|J|}\) for each \(J \subseteq S\).

7.3. Extension to \(\mathcal{H}(G, B)\)

Let \(\mathcal{K}\) denote the subspace of \(\mathcal{H}(G, B)\) spanned by the basis elements corresponding to \((B, B)\)-double cosets in \(G - H\). As vector spaces we have the decomposition \(\mathcal{H}(G, B) = \mathcal{H}(H, B) \oplus \mathcal{K}\). Moreover, \(\mathcal{K}\) is clearly a left and right \(\mathcal{H}(H, B)\)-module.

**Theorem 7.3.** \(\phi\) extends uniquely to the linear character \(\psi\) of \(\mathcal{H}(G, B)\) defined by setting \(\psi(\beta) = 0\) for any \(\beta \in \mathcal{K}\).

**Proof.** We begin by showing that \(\psi\) is a homomorphism. Let \(\beta\) and \(\beta'\) be basis elements of \(\mathcal{H}(G, B)\). If \(\beta\) and \(\beta'\) both belong to \(\mathcal{H}(H, B)\) then we have \(\psi(\beta\beta') = \phi(\beta\beta') = \phi(\beta)\phi(\beta') = \psi(\beta)\psi(\beta')\) since \(\psi\) extends the linear character \(\phi\) of \(\mathcal{H}(H, B)\). Thus we will assume that \(\beta\) lies in \(\mathcal{K}\) and note that the argument in the case where \(\beta' \in \mathcal{K}\) is similar. In particular, this means that \(\psi(\beta) = 0\) and so we would like to show that \(\psi(\beta\beta') = 0\).

Let \(BgB\) be the double coset in \(G - H\) corresponding to \(\beta\). By Theorem 6.1 we know that \(BgB = H_\alpha gB\) for some \(\alpha \in S\) and so \(e_{H_\alpha}\beta = \beta\). Thus, if we write \(\beta\beta' = \gamma + \gamma'\) for some \(\gamma \in \mathcal{H}(H, B)\) and \(\gamma' \in \mathcal{K}\) then we have \(\beta\beta' = e_{H_\alpha}\beta\beta' = e_{H_\alpha}\gamma + e_{H_\alpha}\gamma'\) where \(e_{H_\alpha}\gamma \in \mathcal{H}(H, B)\) and \(e_{H_\alpha}\gamma' \in \mathcal{K}\). This means that \(e_{H_\alpha}\gamma = \gamma\) and so
\[\beta_\alpha\gamma = (qe_B - e_{H_\alpha})\gamma = qe_B\gamma - e_{H_\alpha}\gamma = (q-1)\gamma.\]

However, since \(\phi\) is a homomorphism, we obtain
\[(q-1)\phi(\gamma) = \phi(\beta_\alpha\gamma) = \phi(\beta_\alpha)\phi(\gamma) = (-1)\phi(\gamma).\]

Hence we must have \(\phi(\gamma) = 0\) and so \(\psi(\beta\beta') = \psi(\gamma) + \psi(\gamma') = \phi(\gamma) = 0\).

Now, suppose that \(\psi'\) is a homomorphism of \(\mathcal{H}(G, B)\) which extends \(\phi\). If \(\beta \in \mathcal{K}\) is a basis element corresponding to the double coset \(BgB\) contained in \(G - H\) then we again have \(BgB = H_\alpha gB\) for some \(\alpha \in S\) and so \(\beta_\alpha\beta = (q-1)\beta\). Hence
\[(q-1)\psi'(\beta) = \psi'(\beta_\alpha\beta) = \psi'(\beta_\alpha)\psi'(\beta) = (-1)\psi'(\beta),\]

implying that \(\psi'(\beta) = 0\) and thus \(\psi' = \psi\). \(\square\)

**Lemma 7.4.** \(\psi\) is the restriction of \(\text{St}_\ell\) to \(\mathcal{H}(G, B)\).

**Proof.** We know that \((\text{St}_\ell, (1_B)^G) = 1\) and \((\text{St}_\ell, (1_{H_\alpha})^G) = 0\) for \(\alpha \in S\). Thus \(\text{St}_\ell\) restricts to a linear character of \(\mathcal{H}(G, B)\) with
\[\text{St}_\ell(\beta_\alpha) = q \text{ St}_\ell(e_{H_\alpha}) - \text{ St}_\ell(e_B) = q (\text{St}_\ell, H_\alpha) - (\text{St}_\ell, B) = -1\]
for each \(\alpha \in S\) and so, by the uniqueness in Theorem 7.3, this must be \(\psi\). \(\square\)
7.4. Idempotent

Suppose that \( \psi \) is a linear character of \( \mathcal{H}(G, B) \) corresponding to the irreducible character \( \zeta \) of \( G \), then by [6]

\[
e = \frac{\zeta(1)}{|G:B|} \sum_{B g B \in \mathcal{D}_G(B, B)} \frac{|B|}{|B g B|} \psi(\beta g^{-1}) \beta g
\]  

(13)

is a primitive idempotent in \( CG \) such that \( CG \varepsilon \) affords \( \zeta \). It transpires that the idempotent \( e \) used in [2] to define a module affording \( St_\ell \) is exactly this primitive idempotent. To show this we use an alternative description of the basis element \( \beta I \) as

\[
\beta I = \sum_{J \subseteq S} (-1)^{|I|-|J|} q^{|J|} e_{HJ}.
\]

Proposition 7.5. The primitive idempotent corresponding to \( \psi \) is

\[
e = \sum_{J \subseteq S} (-1)^{|J|} e_{HJ}.
\]

Proof. If \( \beta g \) belongs to \( \mathcal{K} \) then so does \( \beta g^{-1} \) which implies that \( \psi(\beta g^{-1}) = 0 \). Further, \( St_\ell(1)/[G : B] = (q - 1)^n / q^n \) since \( St_\ell \) is induced from a degree \((q - 1)^n\) character of \( H \). Thus (13) reduces to

\[
e = \frac{(q - 1)^n}{q^n} \sum_{I \subseteq S} \frac{|B|}{|B x_I B|} \phi(\beta_{x_I^{-1}}) \beta_I.
\]

Now, for each \( I \subseteq S \) we have \( |B x_I B| = (q - 1)^{|I|} |B| \) and \( B x_I^{-1} B = B x_I B \) which means that \( \phi(\beta_{x_I^{-1}}) = (-1)^{|I|} \). Hence we obtain

\[
e = \frac{(q - 1)^n}{q^n} \sum_{I \subseteq S} \frac{1}{(q - 1)^{|I|}} (-1)^{|I|} \beta_I
\]

\[
= \sum_{I \subseteq S} (q - 1)^{|I|-|J|} q^{-|S|} (-1)^{|I|} \sum_{J \subseteq I} (-1)^{|I|-|J|} q^{|J|} e_{HJ}
\]

\[
= \sum_{J \subseteq S} (-1)^{|J|} q^{|J|-|S|} \sum_{J \subseteq I} (q - 1)^{|I|-|J|} e_{HJ}
\]

\[
= \sum_{J \subseteq S} (-1)^{|J|} e_{HJ}
\]

as required. \( \square \)

8. Decomposition

We will now determine the decomposition of the analogue into its distinct irreducible constituents.
8.1. Clifford theory for $\chi$

We begin by examining the restriction of $\chi$ to the subgroup $V = XT(1)U$ of $H$. It is clear that $V$ is a normal subgroup of $H$ since $H = TV$ and $T$ normalises $X$, $T(1)$ and $U$. Indeed, $V' = T(1)U = B \cap V$ is also normal in $H$.

**Lemma 8.1.** $V'$ is normal in $H$.

**Proof.** We only need to show that $xvx^{-1} \in V'$ for each $x \in X$ and $v \in V'$ since $V'$ is clearly normal in $B$. From the proof of Lemma 5.3 we know that $[x,u] \in B(\ell - 1)$ for any $u \in U$. Thus writing $v = h(\mu)u$ for some $h(\mu) \in T(1)$ and $u \in U$ we see that $xvx^{-1} = xh(\mu)ux^{-1} = h(\mu)[x,u]u \in V'$. \(\square\)

Each linear character $\rho$ of $X$ extends to a linear character of $V$ which we will again denote by $\rho$. Further, since $X$ is generated by the root subgroups $X_\alpha$ for $\alpha \in S$, any linear character of $X$ is determined by its restriction to each $X_\alpha$. Let $X$ denote the set of linear characters $\rho$ of $V$ that are obtained from linear characters of $X$ with $\rho_{X_\alpha} \neq 1_{X_\alpha}$ for each $\alpha \in S$.

**Proposition 8.2.** $\chi_V = \sum_{\rho \in X} \rho$.

**Proof.** Mackey theory implies that for any $\rho \in X$

$$\langle \rho, \chi_V \rangle = \sum_{J \subseteq S} (-1)^{|J|} \langle \rho, ((1_{HJ})^H)_V \rangle = \sum_{J \subseteq S} (-1)^{|J|} \langle \rho_{HJ \cap V}, 1_{HJ \cap V} \rangle$$

(14)

since $H_J V = H$ for each $J \subseteq S$. If $J \neq \emptyset$ then for any $\alpha \in J$ we have

$$0 \leq \langle \rho_{HJ \cap V}, 1_{HJ \cap V} \rangle \leq \langle \rho_{X_\alpha}, 1_{X_\alpha} \rangle = 0.$$

Consequently the $J \neq \emptyset$ terms in (14) disappear and we obtain

$$\langle \rho, \chi_V \rangle = \langle \rho_{V'}, 1_{V'} \rangle = 1.$$

Thus each $\rho \in X$ appears as a constituent of $\chi_V$ with multiplicity 1. However, there are $q - 1$ possible choices of non-trivial character $\rho_{X_\alpha}$ of $X_\alpha$ and so a total of $(q - 1)^n$ possibilities for each $\rho \in X$. Hence these must be all of the irreducible constituents of $\chi_V$. \(\square\)

**Lemma 8.3.** $X$ decomposes into $d$ orbits under the action of $T$.

**Proof.** Let $\rho \in X$, then $h(\mu) \in \text{Stab}_T(\rho)$ if and only if for each $r \in m^{\ell-1}$ and $\alpha \in S$ we have $\rho(x_\alpha(r)) = \rho^{h(\mu)}(x_\alpha(r)) = \rho(x_\alpha(\mu(\alpha)^{-1}r))$. However, since $\rho_{X_\alpha} \neq 1_{X_\alpha}$ this is true exactly when $\mu(\alpha)^{-1}r = r$ for every $r \in m^{\ell-1}$ and so $\mu(\alpha) \in 1 + m$ for each $\alpha \in S$. Thus $\text{Stab}_T(\rho) = T(1)$ and the size of the orbit containing $\rho$ is $[T : \text{Stab}_T(\rho)] = [T : T(1)] = (q - 1)^n / d$. Hence the number of distinct orbits in $X$ must be $d$. \(\square\)

Let $X_1, \ldots, X_d$ denote the distinct $T$-orbits in $X$ and for each $i$ set $\chi_i = \rho_i^H$ where $\rho_i$ is a representative from $X_i$. 
Theorem 8.4. \( \chi = \sum_{i=1}^{d} \chi_i \) where \( \chi_1, \ldots, \chi_d \) are distinct irreducible constituents of \( \chi \) of degree \( \chi_i(1) = (q - 1)^n / d \).

**Proof.** From the proof of Lemma 8.3 we see that \( \text{Stab}_H(\rho_i) = V \) for each \( i \). Clifford theory then implies that each \( \chi_i \) is irreducible with \( (\chi_i)_V = \sum_{\rho \in \mathcal{X}_i} \rho \) and \( (\chi, \chi_i) = (\chi_V, \rho_i) = \sum_{\rho \in \mathcal{X}} (\rho, \rho_i) = 1 \). Thus each \( \chi_i \) appears as a constituent of \( \chi \) with multiplicity 1. Indeed, the \( \chi_i \) are distinct since the \( \mathcal{X}_i \) are disjoint and \( \chi = \sum_{i=1}^{d} \chi_i \) since \( \mathcal{X} = \bigcup_{i=1}^{d} \mathcal{X}_i \). Finally \( \chi_i(1) = |\mathcal{X}_i| = (q - 1)^n / d \).

8.2. Decomposition of \( \text{St}_\ell \)

If we consider \( \zeta_i = \chi_i^G \) for each \( i \) then it is clear that \( \text{St}_\ell = \sum_{i=1}^{d} \zeta_i \). We would therefore like to show that \( \zeta_1, \ldots, \zeta_d \) are distinct and irreducible.

**Proposition 8.5.** For each \( i, j \)

\[
(\zeta_i, \zeta_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** The Intertwining number theorem gives

\[
(\zeta_i, \zeta_j) = \sum_{H \in \mathcal{D}_G(H, H)} ((\chi_i)_g H g^{-1} \cap H, (\chi_j)_g H g^{-1} \cap H).
\]

(15)

By Theorem 6.2 we know that if \( H g H \) is contained in \( G - H \) then there is an \( \alpha \in S \) so that for every \( r \in m^{\ell-1} \) we have \( g v_r g^{-1} = v'_r x_\alpha(r) \) for some \( v_r, v'_r \in V' \). Let \( Y = (g v_r g^{-1}: r \in m^{\ell-1}) \) and note that \( Y \) is a subgroup of both \( g H g^{-1} \cap H \) and \( V \).

For each \( \rho \in \mathcal{X} \) we have \( \rho Y \neq 1_Y \) since \( \rho(g v_r g^{-1}) = \rho(v'_r x_\alpha(r)) = \rho(x_\alpha(r)) \) for every \( r \in m^{\ell-1} \). Further, \( \rho_Y^g = 1_Y \) since \( \rho^g(g v_r g^{-1}) = \rho(v_r) = 1 \) for every \( r \in m^{\ell-1} \). Thus

\[
0 \leq ((\chi_i)_g H g^{-1} \cap H, (\chi_j)_g H g^{-1} \cap H) \\
= \sum_{\rho \in \mathcal{X}_i, \rho' \in \mathcal{X}_j} (\rho_Y, (\rho'_Y)^g) \\
= \sum_{\rho \in \mathcal{X}_i, \rho' \in \mathcal{X}_j} (\rho_Y, 1_Y) \\
= 0.
\]

Hence the \( g \neq 1 \) terms in (15) disappear and we are left with

\[
(\zeta_i, \zeta_j) = (\chi_i, \chi_j)
\]

which is 1 if \( i = j \) and 0 otherwise. \( \square \)
Theorem 8.6. \( \text{St}_\ell = \sum_{i=1}^d \zeta_i \) where \( \zeta_1, \ldots, \zeta_d \) are distinct irreducible constituents of \( \text{St}_\ell \) of degree
\[
\zeta_i(1) = \frac{1}{d} q^{N\ell-n}(q-1)^n \sum_{w \in W} q^{\ell(w)}.
\]

Remark 8.7. If \( G \) is equal to the extended Chevalley group \( \overline{G} \) then \( T \) transitively permutes the linear characters in \( X \) and Theorem 8.6 gives an alternative proof of the irreducibility of the analogue in this case.

8.3. Modular representations

Let \( p \) be a prime different from the characteristic of the residue field \( \kappa \) and \( F \) a field of characteristic \( p \). From Theorem 8.6 we see that the \( p \)-part of the degree of \( \zeta_i \) is the same as the \( p \)-part of the order of \( G \). In particular, this means that \( \zeta_i \) is an irreducible character of \( p \)-defect zero and is thus the \( F \)-character afforded by some projective irreducible \( FG \)-module, provided that \( F \) is sufficiently large (see [12]).

More specifically, let \( \xi \) denote a primitive \( q \)th root of unity. Each linear character \( \rho \in X \) can be considered as a homomorphism \( \rho : V \to \mathbb{Z}[\xi,q^{-1}] \) and we may define
\[
e_{\rho} = \frac{1}{|V|} \sum_{x \in V} \rho(x^{-1})x \in \mathbb{Z}[\xi,q^{-1}] V.
\]

Let \( L'_i \) be the \( \mathbb{Z}[\xi,q^{-1}] \)-span of the linearly independent set \( \{e_{\rho}e_T' : \rho \in \mathcal{X}_i \} \) where \( e_T' = \sum_{t \in T} t \).

Then for each \( v \in V \)
\[
v e_{\rho}e_T' = \rho(v)e_{\rho}e_T'
\]

and for each \( t \in T \)
\[
t e_{\rho}e_T' = \frac{1}{|V|} \sum_{x \in V} \rho(x^{-1})tx^{-1}e_T' = \frac{1}{|V|} \sum_{x \in V} \rho'(x^{-1})xe_T' = e_{\rho'}e_T'.
\]

Thus \( H \) preserves \( L'_i \) and \( L'_i \otimes_{\mathbb{Z}[\xi,q^{-1}]} \mathbb{C} \) affords the character \( \chi_i \). Further, let \( L_i \) be the \( \mathbb{Z}[\xi,q^{-1}] \)-span of the linearly independent set \( \{ge_{\rho}e_T' : \rho \in \mathcal{X}_i, g \in T \} \) where \( T \) denotes a left transversal for \( H \) in \( G \). Then \( G \) preserves \( L_i \) and \( L_i \otimes_{\mathbb{Z}[\xi,q^{-1}]} \mathbb{C} \) affords the character \( \zeta_i \). Hence, if \( F \) contains a primitive \( q \)th root of unity, then \( L_i \otimes_{\mathbb{Z}[\xi,q^{-1}]} F \) must be an irreducible projective \( FG \)-module whose \( F \)-character is \( \zeta_i \).

9. Gelfand–Graev characters

Hill [11] defined the analogue for the general linear group to be the unique common constituent of the permutation character over \( B \) and a version \( \Gamma_0 \) of the Gelfand–Graev character. We will show that the irreducible constituents of \( \text{St}_\ell \) have a similar characterisation.
9.1. Non-degenerate characters

A linear character $\theta$ of $U$ is non-degenerate if its restriction to $U_\alpha(\ell - 1)$ is non-trivial for each $\alpha \in \Pi$. Let $\Theta$ denote the set of all non-degenerate linear characters of $U$. In the case of the general linear group the non-degenerate linear characters of $U$ are transitively permuted by $T$. However, we will show that this is not true in general for $G$.

**Lemma 9.1.** $|\Theta| = q^n(\ell - 1)(q - 1)^n$.

**Proof.** Each $\theta \in \Theta$ is completely determined by its values on the root subgroups $U_\alpha$ for $\alpha \in \Pi$ and there are $q^\ell$ linear characters of $U_\alpha$ of which exactly $q^\ell - 1$ are trivial on $U_\alpha(\ell - 1)$. Hence for each $\alpha \in \Pi$ there are $q^\ell - 1(q - 1)$ possible choices for the restriction of $\theta$ to $U_\alpha$ and so a total of $q^n(\ell - 1)(q - 1)^n$ possibilities for $\theta$. □

**Lemma 9.2.** $\Theta$ forms $d$ orbits under the action of $T$.

**Proof.** Let $\theta \in \Theta$, then $h(\mu) \in \text{Stab}_T(\theta)$ if and only if for every $\alpha \in \Pi$ and $r \in R$ we have $\theta(x_\alpha(r)) = \theta^{h(\mu)}(x_\alpha(r)) = \theta(x_\alpha(\mu(\alpha)^{-1}r))$. However, this is equivalent to requiring that $\mu(\alpha)^{-1}r = r$ for every $r \in R$, since $\theta$ is non-degenerate, and so $\mu(\alpha) = 1$ for each $\alpha \in \Pi$. Thus $\text{Stab}_T(\theta) = 1$ and the size of the orbit containing $\theta$ is $[T : \text{Stab}_T(\theta)] = |T| = q^n(\ell - 1)(q - 1)^n / d$. Hence the number of distinct $T$-orbits in $\Theta$ must be $d$. □

**Remark 9.3.** The number of $T$-orbits in $\Theta$ is equal to the number of $T$-orbits in $X$. In fact, there is a correspondence between the orbits in $\Theta$ and $X$. Let $w_0$ denote the element of maximal length in $W$, then $n_{w_0}U_{w_0}^{-1} \cap H = X$. Further, for each $\alpha \in S$ and $r \in m^{\ell - 1}$ we have

$$\theta^{n_{w_0}}(x_\alpha(r)) = \theta(n_{w_0}^{-1}x_\alpha(r)n_{w_0}) = \theta(x_\alpha^{-1}(\eta_{w_0,\alpha}^{-1}r))$$

where $w_0^{-1}(\alpha) \in \Pi$. The non-degeneracy of $\theta$ implies that $\theta^{n_{w_0}}_{X_\alpha} \neq 1_{X_\alpha}$ for each $\alpha \in S$ and so $\theta^{n_{w_0}}_{X_\alpha} = \rho_X$ for exactly one $\rho \in X$. Thus, if for each $i$ we let

$$\Theta_i = \{ \theta \in \Theta: \theta^{n_{w_0}}_{X_\alpha} = \rho_X \text{ for some } \rho \in X \}$$

then $\Theta$ is clearly the disjoint union of the $\Theta_i$. Finally, if $\theta^{n_{w_0}}_{X_\alpha} = \rho_X$ then for each $h(\mu) \in T$

$$(\theta^{h(\mu)})^{n_{w_0}}_X = (\theta^\mu X)^{n_{w_0}^{-1}h(\mu)n_{w_0}} = \rho_X^{n_{w_0}^{-1}h(\mu)n_{w_0}}$$

where $n_{w_0}^{-1}h(\mu)n_{w_0} \in T$. Hence the action of $T$ preserves each $\Theta_i$ and they must be the distinct $T$-orbits in $\Theta$.

9.2. Gelfand–Graev characters

For each $i$ let $\Gamma_i = \theta_i^G$ where $\theta_i$ is a representative from the orbit $\Theta_i$. We will show that each $\Gamma_i$ has a unique constituent in common with $(1_B)^G$. 

Lemma 9.4. \( Un_{w_0}kB = Un_{w_0}B \) for any \( k \in U^-(1) \).

**Proof.** Let \( k \in U^-(1) \), then \( Un_{w_0}kB = Un_{w_0}kn_{w_0}^{-1}n_{w_0}B = Un_{w_0}B \) since \( n_{w_0}kn_{w_0}^{-1} \in U \). \( \square \)

Proposition 9.5. \( (\Gamma_i, (1_B)^G) = 1 \) for each \( i \).

**Proof.** By the Intertwining number theorem we have

\[
(\Gamma_i, (1_B)^G) = (\theta_i^G, (1_B)^G) = \sum_{UgB \in D_G(U,B)} ((\theta_i)_{gB^{-1} \cap U}, 1_{gB^{-1} \cap U}).
\]

Suppose that \( g = n_wk \) is a double coset representative with \( w \neq w_0 \). There must be a root \( \gamma \in \Sigma^+ \) with \( w(\gamma) \in \Pi \) so if we let \( \alpha = w(\gamma) \) then for each \( r \in m^{-1} \) we have \( gx_\gamma(r)g^{-1} = n_wx_\gamma(r)n_w^{-1} = x_\alpha(\eta_{w,\alpha}r) \). Thus \( U_\alpha(\ell - 1) \leq gB^{-1} \cap U \) and, since \( \theta \) is non-degenerate,

\[
0 \leq ((\theta_i)_{gB^{-1} \cap U}, 1_{gB^{-1} \cap U}) \leq ((\theta_i)_{U_\alpha(\ell - 1)}, 1_{U_\alpha(\ell - 1)}) = 0.
\]

Hence the \( g \neq n_w \) terms in (16) disappear and we are left with

\[
(\Gamma_i, (1_B)^G) = ((\theta_i)_{n_wBn_w^{-1} \cap U}, 1_{n_wBn_w^{-1} \cap U}) = 1
\]

since \( n_wBn_w^{-1} \cap U = \{1\} \). \( \square \)

9.3. Characterisation

We also see that each \( \zeta_i \) appears as a constituent of exactly one of the analogues of the Gelfand–Graev character.

Proposition 9.6. For each \( i, j \)

\[
(\zeta_i, \Gamma_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** By the Intertwining number theorem

\[
(\zeta_i, \Gamma_j) = (\chi_i^G, \theta_j^G) = \sum_{HgU \in D_G(H,U)} ((\chi_i)_{gU^{-1} \cap H}, (\theta_j)_{gU^{-1} \cap H}).
\]

Again, consider a double coset representative \( g = n_wk \) with \( w \neq w_0 \). However, on this occasion let \( \gamma \in \Sigma^+ \) be such that \( w(\alpha) = \gamma \) for some \( \alpha \in \Pi \) so that \( U_\gamma(\ell - 1) \leq gU^{-1} \cap H \). Then, for each \( r \in m^{-1} \)

\[
\theta_j^g(x_\gamma(r)) = \theta_j(k^{-1}n_w^{-1}x_\gamma(r)n_wk) = \theta_j(k^{-1}x_\alpha(\eta_{w,\alpha}r)k) = \theta_j(x_\alpha(\eta_{w,\alpha}r)).
\]

Thus \( (\theta_j)_{U_\gamma(\ell - 1)}^g \neq 1_{U_\gamma(\ell - 1)} \), by the non-degeneracy of \( \theta_j \), and
\[ 0 \leq (\chi_i)_{U^{-1} \cap H}^g, (\theta_j)_{U^{-1} \cap H}^g \leq (\chi_i)_{U_{\ell-1}}^g, (\theta_j)_{U_{\ell-1}}^g = \sum_{\rho \in \mathcal{X}_i} (\rho U_{\ell-1}, (\theta_j)_{U_{\ell-1}}^g) = 0 \]

since \( \rho U_{\ell-1} = 1 \) for each \( \rho \in \mathcal{X}_i \). Hence the \( g \neq \text{nw} \) terms in (17) disappear and we obtain

\[ (\zeta_i, \Gamma_j) = ((\chi_i)_{\text{nw}^{-1} \cap H}, (\theta_j)_{\text{nw}^{-1} \cap H}) = \sum_{\rho \in \mathcal{X}_i} (\rho \chi, (\theta_j)_{\text{nw}^{-1}}) \]

which is 1 if \( i = j \) and 0 otherwise.  

Finally, Propositions 9.5 and 9.6 immediately imply that \( \zeta_i \) is the unique common constituent of the permutation character over \( B \) and exactly one of the analogues of the Gelfand–Graev character.

**Theorem 9.7.** \( \zeta_i \) is the unique common constituent of \( \Gamma_i \) and \((1_B)^G\).

**Remark 9.8.** If \( G \) is equal to the extended Chevalley group \( \overline{G} \) then \( T \) transitively permutes the non-degenerate linear characters and there is a unique Gelfand–Graev character \( \Gamma_0 = \theta^G \). In this case Theorem 9.7 implies that \( S_{\ell} \) is the unique common constituent of the permutation character \((1_B)^G\) and \( \Gamma_0 \) which is exactly the definition of the analogue for the general linear group given by Hill [11].

10. Proof of Theorem 6.2

We now return to the delayed proof of Theorem 6.2. Recall from Lemma 2.8 we know that each \((B, B)\)-double coset representative can be taken to be of the form \( g = n_wk \) for some \( k \in U^{-1}(1) \) and \( w \in W \). We will therefore prove Theorem 6.2 by following the approach used for [2, Proposition 27] and considering different cases for \( k \) and \( w \).

10.1. Case 1

As in [2, Proposition 27] the first and simplest case occurs when the double coset \( BgB \) does not belong to \( U^{-1}(1)B \).

**Lemma 10.1.** If \( w \neq 1 \) then there is an \( \alpha \in S \) so that for every \( r \in m^{\ell-1} \) we have \([v, g] = vx_{\alpha}(r)\) for some \( v \in V'\).

**Proof.** There must be an \( \alpha \in S \) with \( w^{-1}(\alpha) \in \Sigma^+ \), since \( w \neq 1 \), so for every \( r \in m^{\ell-1} \) we have \( n_w^{-1}x_{\alpha}(r)n_w = x_{w^{-1}(\alpha)}(\pm r) \in U(\ell - 1) \). Fixing \( r \in m^{\ell-1} \) we see that \( v = n_w^{-1}x_{\alpha}(-r)n_w \in V' \) gives

\[ [v, g] = vg^{-1}g^{-1} = vn_wkvg^{-1}k^{-1}n_w^{-1} = vn_wv^{-1}n_w = vx_{\alpha}(r). \]  

\[ \square \]
10.2. Case 2

Assume now that $BgB$ is contained in $U^-(1)B$ and so that $g$ can be chosen to lie in $U^-(1)$. We know that $g$ can be written as the product of terms of the form $x_\beta(r_\beta)$ with $r_\beta \in \mathfrak{m}$ where $\beta$ runs over the negative roots in $\Sigma$. However, it will be more convenient if we instead use positive roots and express $g$ as

$$g = \prod_{\beta \in \Sigma^+} x_{-\beta}(r_\beta).$$

For each $\beta \in \Sigma^+$ let $1 \leq i_\beta \leq \ell$ be such that $r_\beta \in \pi^{i_\beta}R^\times$. Further, set $i = \min\{i_\beta: \beta \in \Sigma^+\}$ and $a = \max\{\text{ht}(\beta): i_\beta = i, \beta \in \Sigma^+\}$. The second case, corresponding to case 2(ii) in [2, Proposition 27], occurs when $i_\beta$ achieves its minimum only on simple roots.

Lemma 10.2. If $a = 1$ then there is an $\alpha \in S$ so that for every $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = x_\alpha(r)$ for some $v \in V'$.

Proof. Choose $\alpha \in S$ with $i_{-\alpha} = i$. As $g \notin H$ we must have $i < \ell - 1$ and so $1 + s \in R^\times$ for every $s \in \mathfrak{m}^{\ell-i-1}$. Thus $y_{-\alpha}(1+s)^{-1} \in V'$ and we will consider the commutator $[y_{-\alpha}(1+s)^{-1}, x_{-\beta}(r_\beta)]$ for each $\beta \in \Sigma^+$.

If $\text{ht}(\beta) > 1$ then $r_\beta s = 0$, since $i_\beta > i$, and

$$[y_{-\alpha}(1+s)^{-1}, x_{-\beta}(r_\beta)] = 1.$$

Further, if $\text{ht}(\beta) = 1$, but $\beta \neq -\alpha$, then we also have

$$[y_{-\alpha}(1+s)^{-1}, x_{-\beta}(r_\beta)] = 1.$$

However, for $\beta = -\alpha$ we obtain

$$[y_{-\alpha}(1+s)^{-1}, x_\alpha(r_{-\alpha})] = x_\alpha(r_{-\alpha}s).$$

Hence, if we fix $r \in \mathfrak{m}^{\ell-1}$ and choose $s \in \mathfrak{m}^{\ell-i-1}$ with $r_{-\alpha}s = r$ then setting $v = y_{-\alpha}(1+s)^{-1} \in V'$ gives

$$[v, g] = \prod_{\beta \in \Sigma^+} [v, x_{-\beta}(r_\beta)] = x_\alpha(r_{-\alpha}s) = x_\alpha(r),$$

since $x_{-\beta}(r_\beta)$ commutes with each commutator $[v, x_{-\beta'}(r_{\beta'}')]$.

10.3. Case 3

Now suppose that $i = i_\gamma$ for some non-simple $\gamma \in \Sigma^+$ and define the set $\mathcal{S} = \{\gamma \in \Sigma_a: i_\gamma = i\}$. This corresponds to case 2(i) in [2, Proposition 27] but here we encounter an error in the proof. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and consider the root system of type $A_{n-1}$

$$\Sigma = \{\pm(e_i - e_j): 1 \leq i < j \leq n\}$$
with simple roots $\Pi = \{e_i - e_{i+1} : 1 \leq i \leq n - 1\}$. In terms of our current notation, to prove case 2(i) we chose a root $e_l - e_k \in \mathcal{S}$ and saw that for any $s \in m^{l-i-1}$

$$[x_{e_l-e_{k-1}}(s), x_{e_k-e_l}(r_{e_l-e_k})] = x_{e_k-e_{l-1}}(r_{e_l-e_k}s)$$

with $e_k - e_{k-1} \in S$. We then claimed incorrectly that for any $e_j - e_i \in \Sigma^+$ with $(i, j) \neq (k, l)$ we had $[x_{e_l-e_{k-1}}(s), x_{e_j-e_l}(r_{e_j-e_l})] \in B(\ell - 1)$ and so, for some $v' \in V'$, that

$$[x_{e_l-e_{k-1}}(s), g] = v'x_{e_k-e_{k-1}}(r_{e_l-e_k}s).$$

However, if $i = k - 1$ and $j < i$ then $i - j \leq k - l$ only implies that $j \geq l - 1$. In particular, setting $j = l - 1$ gives

$$[x_{e_l-e_{k-1}}(s), x_{e_{k-1}-e_l}(r_{e_l-e_k-1})] = x_{e_l-e_l}(r_{e_l-e_k-1}s)$$

with $e_l - e_{l-1} \in S$ and thus we actually obtain

$$[x_{e_l-e_{k-1}}(s), g] = v'x_{e_k-e_{l-1}}(r_{e_l-e_k}s)x_{e_l-e_{l-1}}(r_{e_l-e_k-1}s). \quad (18)$$

Fortunately, this is easily remedied: if we assume that $e_l - e_k \in \mathcal{S}$ was chosen with minimal $l$ then this will ensure that $r_{e_l-e_k-1}s = 0$ since $e_l-1 - e_{k-1} \notin \mathcal{S}$.

We therefore prove the equivalent version of (18) in our situation.

**Lemma 10.3.** Let $\beta \in \Sigma_{a-1}$ and $s \in m^{l-i-1}$, then

$$[x_\beta(s), g] = v_\beta \prod_{\gamma \in (\beta+\Pi) \cap \mathcal{S}} x_{\beta-\gamma}(-c_{1,1,\beta,-\gamma sr_\gamma})$$

for some $v_\beta \in V'$.

**Proof.** We will consider the commutator $[x_\beta(s), x_{-\gamma}(r_\gamma)]$ for each $\gamma \in \Sigma^+$. If $ht(\gamma) > a$ then $sr_\gamma = 0$, since $i_\gamma > i$, which gives

$$v_{\beta,\gamma} = [x_\beta(s), x_{-\gamma}(r_\gamma)] = 1$$

and if $ht(\gamma) < a$ then, from the proof of Proposition 3.3,

$$v_{\beta,\gamma} = [x_\beta(s), x_{-\gamma}(r_\gamma)] \in B(\ell - 1).$$

Further, if $ht(\gamma) = a$ with $\gamma \notin \mathcal{S}$ then $i_\gamma > i$ and so

$$v_{\beta,\gamma} = [x_\beta(s), x_{-\gamma}(r_\gamma)] = 1$$

whereas if $\gamma \in \mathcal{S}$, and $\beta - \gamma \in \Sigma$, then by the argument in the proof of Lemma 3.2,

$$[x_\beta(s), x_{-\gamma}(r_\gamma)] = v_{\beta,\gamma}x_{\beta-\gamma}(-c_{1,1,\beta,-\gamma sr_\gamma})$$
Lemma 10.4. If \( v_{\beta, \gamma} \in B(\ell - 1) \). Hence, if we let \( v_{\beta} = \prod_{\gamma \in \Sigma^+} v_{\beta, \gamma} \in V' \) then, since \( x_{-\gamma}(r_{\gamma}) \) commutes with each commutator \( [x_{\beta}(s), x_{-\gamma}(r_{\gamma})] \),

\[
[x_{\beta}(s), g] = \prod_{\gamma \in \Sigma^+} [x_{\beta}(s), x_{-\gamma}(r_{\gamma})] = v_{\beta} \prod_{\gamma} x_{-\gamma}(-c_{1, 1, \beta, -\gamma} sr_{\gamma})
\]

where the product runs over all roots \( \gamma \in \Sigma \) with \( \beta - \gamma \in S \). \( \square \)

The remedy described above relies on the fact that if we set \( \beta = e_l - e_k \) where \( \gamma = e_l - e_k \) \( \in \Sigma \) has minimal \( l \) then \( (\beta + \Pi) \cap \Sigma = \{e_{l-1} - e_k - 1, e_l - e_k\} \) and so \( (\beta + \Pi) \cap \Sigma = \{\gamma\} \) since \( e_{l-1} - e_k - 1 \notin \Sigma \). Similarly, we obtain the desired result whenever there is a choice of \( \beta \) for which \( (\beta + \Pi) \cap \Sigma \) consists of a single root.

Lemma 10.4. If \( (\beta + \Pi) \cap \Sigma \) contains a single root for some \( \beta \in \Sigma_{a-1} \) then there is an \( \alpha \in S \) so that for each \( r \in m^{\ell-1} \) we have \( [v, g] = v'x_{\alpha}(r) \) for some \( v, v' \in V' \).

Proof. Suppose that \( (\beta + \Pi) \cap \Sigma = \{\gamma\} \) and let \( \alpha = \beta - \gamma \in S \). By Lemma 10.3, for any \( s \in m^{\ell-i-1} \) we have \( [x_{\beta}(s), g] = v'x_{\alpha}(-c_{1, 1, \beta, -\gamma} sr_{\gamma}) \) for some \( v' \in V' \). Thus, if we fix \( r \in m^{\ell-1} \) and choose \( s \in m^{\ell-i-1} \) so that \( -c_{1, 1, \beta, -\gamma} sr_{\gamma} = r \) then setting \( v = x_{\beta}(s) \in V' \) gives \( [v, g] = v'x_{\alpha}(r) \). \( \square \)

10.4. Case 4

Consider the root system of type \( D_4 \)

\[
\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\}
\]

with simple roots \( \Pi = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\} \) and suppose that \( \Sigma \) is the set \( \Sigma = \{e_3 - e_4, e_1 + e_4, e_2 + e_3\} \) of roots of height 3 in \( \Sigma \). The roots of height 2 in \( \Sigma \) are \( \Sigma_2 = \{e_1 - e_3, e_2 - e_4, e_2 + e_4\} \) and we see that

\[
((e_1 - e_3) + \Pi) \cap \Sigma = \{e_1 - e_4, e_1 + e_4\};
\]
\[
((e_2 - e_4) + \Pi) \cap \Sigma = \{e_1 - e_4, e_2 + e_3\};
\]
\[
((e_2 + e_4) + \Pi) \cap \Sigma = \{e_1 + e_4, e_2 + e_3\}.
\]

Thus Lemma 10.4 is not sufficient for all situations. However, the following proposition implies that we only need to consider seven exceptional cases.

Proposition 10.5. Let \( \Sigma \) be a non-empty set of roots of height \( a > 1 \) with the exception of the following:

(i) \( \Sigma = D_{2k} \) and \( \Sigma = \Sigma_{2k-1} \);
(ii) \( \Sigma = F_4 \) and \( \Sigma = \Sigma_4 \);
(iii) \( \Sigma = E_6 \) and \( \Sigma = \Sigma_4 \);
(iv) \( \Sigma = E_7 \) and \( \Sigma = \Sigma_9 \);
(v) \( \Sigma = E_8 \) and \( \Sigma = \Sigma_6, \Sigma_{10} \) or \( \Sigma_{15} \).


together with the corresponding sets obtained when \( \Sigma \) contains a subsystem equivalent to \( D_{2k}, E_6, E_7, E_8 \) or \( F_4 \). Then there exists an \( \beta \in \Sigma_{a-1} \) so that \( (\beta + \Pi) \cap \mathcal{S} \) consists of a single root.

**Proof.** The details are contained in Section 11. \( \square \)

For the exceptional cases we need to consider a more general element of the form

\[
v = \prod_{\beta \in \Sigma_{a-1}} x_\beta(s_\beta).
\]

**Lemma 10.6.** Let \( v = \prod_{\beta \in \Sigma_{a-1}} x_\beta(s_\beta) \) with \( s_\beta \in m^{\ell - i - 1} \), then

\[
[v, g] = v' \prod_{\alpha \in S} x_\alpha(t_\alpha)
\]

for some \( v' \in V' \), where for each \( \alpha \in S \)

\[
t_\alpha = \sum_{\beta - \gamma = \alpha} -c_{1,1,\beta,-\gamma}s_\beta r_\gamma
\]

with the sum running over all \( \beta \in \Sigma_{a-1} \) and \( \gamma \in \mathcal{S} \) such that \( \beta - \gamma = \alpha \).

**Proof.** Set \( \Sigma_{a-1} = \{\beta_1, \ldots, \beta_k\} \) and \( v_j = x_{\beta_1}(s_{\beta_1}) \cdots x_{\beta_j}(s_{\beta_j}) \in V' \) for each \( j \). Then

\[
[v, g] = \left[x_{\beta_1}(s_{\beta_1}) \cdots x_{\beta_k}(s_{\beta_k}), g\right]
\]

\[
= \left(v_{k-1}\left[x_{\beta_k}(s_{\beta_k}), g\right]v_{k-1}^{-1}\right)\left(v_{k-2}\left[x_{\beta_{k-1}}(s_{\beta_{k-1}}), g\right]v_{k-2}^{-1}\right)\cdots \left(v_1\left[x_{\beta_2}(s_{\beta_2}), g\right]v_1^{-1}\right)\left[x_{\beta_1}(s_{\beta_1}), g\right]
\]

where \( [x_{\beta_i}(s_{\beta_i}), g] \in H \) for each \( i \). Since \( V' \) is normal in \( H \), we can rearrange this to give

\[
[v, g] = v' \prod_{\beta \in \Sigma_{a-1}} \left[x_\beta(s_\beta), g\right]
\]

for some \( v' \in V' \). Further, by Lemma 10.3 we obtain

\[
[v, g] = v' \prod_{\beta \in \Sigma_{a-1}} v_\beta \prod_{\gamma \in (\beta + \Pi) \cap \mathcal{S}} x_\beta - \gamma (-c_{1,1,\beta,-\gamma}s_\beta r_\gamma)
\]

for some \( v_\beta \in V' \). Finally, rearranging and combining terms we obtain

\[
[v, g] = v'' \prod_{\alpha \in S} x_\alpha(t_\alpha)
\]

for some \( v'' \in V' \) where \( t_\alpha \) is as described above. \( \square \)
Lemma 10.7. If $\mathfrak{S}$ is one of the exceptional sets from Proposition 10.5 then there is an $\alpha \in S$ so that for every $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v'x_{\alpha}(r)$ for some $v, v' \in V'$.

Proof. By Lemma 10.6 we only need to prove that there is an $\alpha \in S$ so that for any $r \in \mathfrak{m}^{\ell-1}$ we may choose the $s_{\beta}$ in such a way that $t_{\alpha} = r$ and $t_{\alpha'} = 0$ for $\alpha' \neq \alpha$. For each of the exceptional cases this can be shown explicitly and the details are contained in Section 12. □

10.5. Proof of Theorem 6.2

We have shown in Lemmas 10.1, 10.2, 10.4 and 10.7 that if the double coset representative $g$ has the form $g = n_\nu k$ for $w \in W$ and $k \in U^-(1)$ then there is an $\alpha \in S$ so that for any $r \in \mathfrak{m}^{\ell-1}$ we have $[v, g] = v'x_{\alpha}(r)$ for some $v, v' \in V'$. Thus it remains to show that $\alpha$ depends only on the double coset $HgH$.

Proof of Theorem 6.2. Suppose that $Hg_1H = Hg_2H$ with $\alpha \in S$ such that for any $r_1 \in \mathfrak{m}^{\ell-1}$ we have $[v_1, g_1] = v'_1x_{\alpha}(r_1)$ for some $v_1, v'_1 \in V'$. We would like to show that for any $r_2 \in \mathfrak{m}^{\ell-1}$ we also have $[v_2, g_2] = v'_2x_{\alpha}(r_2)$ for some $v_2, v'_2 \in V'$.

Let $h_1, h_2 \in H$ be such that $g_2 = h_1g_1h_2$ and express $h_1$ as $h_1 = xh(\mu)u_1$ for some $x \in X$, $h(\mu) \in T$ and $u_1 \in U$. Fixing $r_2 \in \mathfrak{m}^{\ell-1}$ we obtain

$$
\begin{align*}
(h_1^{-1}x_{\alpha}(r_2))h_1 &= u_1^{-1}h(\mu)^{-1}x_{\alpha}(r_2)h(\mu)u_1 \\
&= u_1^{-1}x_{\alpha}(-\mu(\alpha)r_2)u_1 \\
&= u_2x_{\alpha}(-\mu(\alpha)r_2)
\end{align*}
$$

for some $u_2 \in V'$. Let $r_1 = -\mu(\alpha)r_2 \in \mathfrak{m}^{\ell-1}$ and suppose that $v_1, v'_1 \in V$ are such that $[v_1, g_1] = v'_1x_{\alpha}(r_1)$. If we set $v_2 = h_2^{-1}v_1h_2$ and $v'_2 = v_2h_1v_1^{-1}v'_1u_2^{-1}h_1^{-1}$ then $v_2, v'_2 \in V'$, since $V'$ is normal in $H$, and

$$
[v_2, g_2] = v_2g_2v_2^{-1}g_2^{-1} = v_2h_1g_1v_1^{-1}g_1^{-1}h_1^{-1} = v_2h_1v_1^{-1}[v_1, g_1]h_1^{-1} = v_2h_1v_1^{-1}v'_1x_{\alpha}(r_1)h_1^{-1} = v_2h_1v_1^{-1}v'_1u_2^{-1}h_1^{-1}x_{\alpha}(r_2) = v'_2x_{\alpha}(r_2),
$$

as required. □

11. Proof of Proposition 10.5

It suffices to prove Proposition 10.5 for irreducible root systems since if $\Sigma'$ is an irreducible component of $\Sigma$ with simple roots $\Pi' = \Pi \cap \Sigma'$ and $\mathfrak{S}' = \mathfrak{S} \cap \Sigma'$ then $(\beta + \Pi') \cap \mathfrak{S}' = \{\gamma\}$ clearly implies that we also have $(\beta + \Pi) \cap \mathfrak{S} = \{\gamma\}$. Thus we will examine each irreducible root system separately, noting that the root system of type $A_n$ was dealt with in Section 10.3.
Let $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$ and let $e_\epsilon = \frac{1}{2} \sum_{i=1}^n e_i e_i$ for each $\epsilon = (e_1, \ldots, e_n) \in \{\pm 1\}^n$. We will only write the signs involved so that, for example, $e_{+---} = \frac{1}{2} e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 - \frac{1}{2} e_4$.

11.1. Type $G_2$

Given simple roots $\Pi = \{\alpha, \beta\}$ consider the root system of type $G_2$

$$\Sigma = \{\pm \alpha, \pm \beta, \pm (\alpha + \beta), \pm (\alpha + 2\beta), \pm (\alpha + 3\beta), \pm (2\alpha + 3\beta)\}.$$ 

If $\mathcal{S}$ is a non-empty subset of $\Sigma_a$ for $a > 1$ then $\mathcal{S}$ contains exactly one root $\gamma$. Thus, for any $\beta \in \Sigma_{a-1}$, we must also have $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$.

11.2. Type $B_n$

Consider the root system of type $B_n$

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}$$

with simple roots $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n - 1\} \cup \{\alpha_n = e_n\}$.

First note that we may assume that $\mathcal{S}$ contains no roots of the form $e_i - e_j$ since if it did then we could proceed as in the case of $A_n$. Now, suppose that $\mathcal{S}$ contains the root $\gamma = e_i$ for some $i$. Then $i \neq n$, since $\gamma \notin \Pi$, and $\beta = e_i - e_n$ is a root of height $a - 1$. Further, $(\beta + \Pi) \cap \Sigma = \{e_{i-1} - e_n, e_i\}$ with $e_{i-1} - e_n \notin \mathcal{S}$ and so $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$.

On the other hand, if all the roots in $\mathcal{S}$ are of the form $e_i + e_j$ then choose the root $\gamma = e_i + e_j \in \mathcal{S}$ with minimal $i$. If $j \neq n$ then setting $\beta = e_i + e_{j+1}$ gives $(\beta + \Pi) \cap \Sigma = \{e_{i-1} + e_{j+1}, e_i + e_j\}$ with $e_{i-1} + e_{j+1} \notin \mathcal{S}$ and so $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$. Similarly, if $j = n$ then setting $\beta = e_i$ gives $(\beta + \Pi) \cap \Sigma = \{e_{i-1}, e_i + e_n\}$ with $e_{i-1} \notin \mathcal{S}$ and thus $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$.

11.3. Type $C_n$

Consider the root system of type $C_n$

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}$$

with simple roots $\Pi = \{\alpha_i = e_i - e_{i+1} : 1 \leq i \leq n - 1\} \cup \{\alpha_n = 2e_n\}$.

Assume as above that $\mathcal{S}$ does not contain a root of the form $e_i - e_j$ but suppose that there is at least one root of the form $e_i + e_j$ and choose $\gamma = e_i + e_j \in \mathcal{S}$ with $i$ minimal. If $j \neq n$ let $\beta = e_i + e_{j+1}$, then $(\beta + \Pi) \cap \Sigma = \{e_{i-1} + e_{j+1}, e_i + e_j\}$ where $e_{i-1} + e_{j+1} \notin \mathcal{S}$ which implies that $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$. Similarly, if $j = n$ let $\beta = e_i - e_n$, then $(\beta + \Pi) \cap \Sigma = \{e_{i-1} - e_n, e_i + e_n\}$ where $e_{i-1} - e_n \notin \mathcal{S}$ and so $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$.

However, if $\mathcal{S}$ only contains roots of the form $\gamma = 2e_i$ then $i \neq n$, since $\gamma \notin \Pi$, and $\beta = e_i + e_{i+1}$ is a root of height $a - 1$. Further, $(\beta + \Pi) \cap \Sigma = \{e_{i-1} + e_{i+1}, 2e_i\}$ with $e_{i-1} + e_{i+1} \notin \mathcal{S}$ and thus $(\beta + \Pi) \cap \mathcal{S} = \{\gamma\}$.
11.4. Type $D_n$

Consider the root system of type $D_n$

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$$

with simple roots $\Pi = \{\alpha_i = e_i - e_{i+1}, \alpha_n = e_{n-1} + e_n : 1 \leq i \leq n-1\}$.

We saw in Section 10.4 that there was no suitable choice of $\beta$ for $\Sigma = D_4$ and $\mathcal{G} = \Sigma_3$. Thus we need to examine the cases of even and odd height separately.

11.4.1. $\mathcal{G}$ consists of roots of even height $2k$

Suppose that $\mathcal{G}$ contains roots of the form $e_i + e_j$ and choose $\gamma = e_i + e_j \in \mathcal{G}$ so that $i$ is maximal. As $\gamma$ has even height we cannot have $j = i + 1$ so $\beta = e_{i+1} + e_j$ is a root of height $2k - 1$. Further, $(\beta + \Pi) \cap \Sigma = \{e_i + e_j, e_{i+1} + e_{j+1}\}$ with $e_{i+1} + e_{j+1} \notin \mathcal{G}$ implying that $(\beta + \Pi) \cap \mathcal{G} = \{\gamma\}$.

Now, if $\mathcal{G}$ only contains roots of the form $e_i - e_j$ choose $\gamma = e_i - e_j \in \mathcal{G}$ with $i$ minimal. Then $i \neq j - 1$, since $\gamma \notin \Pi$, and $\beta = e_i - e_{j-1}$ is a root of height $2k - 1$. If $j \neq n$ then we see that $(\beta + \Pi) \cap \Sigma = \{e_{i-1} - e_{j-1}, e_i - e_j\}$ with $e_{i-1} - e_{j-1} \notin \mathcal{G}$ and so $(\beta + \Pi) \cap \mathcal{G} = \{\gamma\}$. Similarly, if $j = n$ then $(\beta + \Pi) \cap \Sigma = \{e_{i-1} - e_{n-1}, e_i - e_n, e_i + e_n\}$ where $e_{i-1} - e_{j-1}$ and $e_i + e_n$ do not belong to $\mathcal{G}$ and thus $(\beta + \Pi) \cap \mathcal{G} = \{\gamma\}$.

11.4.2. $\mathcal{G}$ consists of roots of odd height $2k - 1$

In view of the example in Section 10.4 it is clear that $\mathcal{G}$ cannot be equal to $\mathcal{G}' = \{e_{n-2k+1} - e_n, e_{n-2k+1} + e_n, e_{n-2k+2} + e_{n-1}, \ldots, e_{n-k} + e_{n-k+1}\}$. Indeed, if $\mathcal{G}$ contains a root $e_i - e_j$ with $j < n$ then we may proceed as for $A_n$, so we may take $\mathcal{G}$ to be a proper subset of $\mathcal{G}'$. Further, the subsystem of $D_n$ spanned by the simple roots $\{\alpha_n = e_{n-2k+1}, \alpha_{n-2k+2}, \ldots, \alpha_n\}$ is equivalent to $D_{2k}$ with $\mathcal{G}'$ as its roots of height $2k - 1$. Thus, after relabelling, we may suppose that $\mathcal{G}$ consists of some, but not all, roots of height $2k - 1$ in $D_{2k}$.

Let $\Sigma_{2k-1} = \{\gamma_0 = e_1 - e_2, \gamma_1 = e_1 + e_2, \gamma_2 = e_2 + e_2, \gamma_3 = e_3 + e_{k+1}\}$ and $\Sigma_{2k-2} = \{\beta_0 = e_2 - e_2, \beta_1 = e_1 - e_{2k-1}, \beta_2 = e_2 + e_2, \ldots, \beta_k = e_k + e_{k+2}\}$. For each $i \geq 1$ we see that $(\beta_i + \Pi) \cap \Sigma = \{\gamma_{i-1}, \gamma_i\}$. Further, $\mathcal{G} \neq \mathcal{G}'$ implies that there is an $i \geq 1$ so that exactly one of $\gamma_{i-1}$ or $\gamma_i$ belongs to $\mathcal{G}$. Hence, taking $\beta = \beta_i$ we find that $(\beta + \Pi) \cap \mathcal{G}$ must contain exactly one root.

11.5. Type $F_4$

Consider the root system of type $F_4$

$$\Sigma = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\} \cup \{\pm e_i : 1 \leq i \leq 4\} \cup \{e_\epsilon : \epsilon \in \{\pm 1\}^4\}$$

with simple roots $\Pi = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = e_{+---}\}$.

In Table 2 we give a list of all possible subsets $\mathcal{G}$ of $\Sigma_d$ containing at least two roots and the appropriate choice of $\beta$ giving $(\beta + \Pi) \cap \mathcal{G} = \{\gamma\}$. Note, however, that there is no such choice when $\mathcal{G} = \Sigma_4$. 


11.6. Type $E_6$, $E_7$, $E_8$

Consider the root system of type $E_8$

$$\Sigma = \{ \pm e_i \pm e_j : 1 \leq i < j \leq 8 \} \cup \{ e_\epsilon : \epsilon \in \{ \pm 1 \}^8, \epsilon_1 \cdots \epsilon_8 = 1 \}$$

with simple roots $\Pi = \{ \alpha_1 = e_+ - - - - - - -, \alpha_2 = e_7 + e_8, \alpha_3 = e_7 - e_8, \alpha_4 = e_6 - e_7, \alpha_5 = e_5 - e_6, \alpha_6 = e_4 - e_5, \alpha_7 = e_3 - e_4, \alpha_8 = e_2 - e_3 \}$. The root systems of types $E_6$ and $E_7$ can then be realised as the subsystems of $E_8$ spanned by the simple roots $\{ \alpha_1, \ldots, \alpha_6 \}$ and $\{ \alpha_1, \ldots, \alpha_7 \}$ respectively.

Although the details have been omitted, it is possible to check each possible subset $\mathcal{G}$ of $\Sigma$ explicitly and show that there is a suitable choice of $\beta$ giving $(\beta + \Pi) \cap \mathcal{G} = \{ \gamma \}$ with the exception of the following sets:

(i) $\mathcal{G} = \{ e_6 + e_7, e_5 - e_8, e_5 + e_8 \} \subseteq \Sigma_3$;
(ii) $\mathcal{G} = \{ e_+ - - - - - - -, e_+ - - - - - - -, e_5 + e_7, e_4 + e_8, e_4 - e_8 \} \subseteq \Sigma_4$;
(iii) $\mathcal{G} = \{ e_3 - e_8, e_4 + e_7, e_5 + e_6, e_3 + e_8 \} \subseteq \Sigma_5$;
(iv) $\mathcal{G} = \Sigma_6$;
(v) $\mathcal{G} = \{ e_+ - - - - - - -, e_+ - - - - - - -, e_+ - - - - - - -, e_3 + e_4 \} \subseteq \Sigma_9$;
(vi) $\mathcal{G} = \Sigma_{10}$;
(vii) $\mathcal{G} = \Sigma_{15}$.

Note that (ii) is the set of roots of height 4 in $E_6$ and (v) is the set of roots of height 9 in $E_7$. Further, the subsystem of $E_8$ spanned by the simple roots $\{ \alpha_2, \alpha_3, \alpha_4, \alpha_5 \}$ is equivalent to $D_4$ with (i) as the set of roots of height 3. Similarly, the subsystem of $E_8$ spanned by the simple roots $\{ \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \}$ is equivalent to $D_6$ with (iii) as the set of roots of height 5.

12. Proof of Lemma 10.7

Write $\mathcal{G} = \{ \gamma_1, \ldots, \gamma_m \}$, $\Sigma_{a-1} = \{ \beta_1, \ldots, \beta_m \}$ and $\Pi = \{ \alpha_1, \ldots, \alpha_9 \}$. Further, let $r_j \in R^\times$ and $s_j \in R$ be such that $r_{\gamma_j} = \pi^j r_j$ and $s_{\beta_j} = \pi^{\ell - i - 1} s_j$. Then Lemma 10.6 implies that $t_{-\alpha_l} = \pi^{\ell - 1} t_l$ where

$$t_l = \sum_{j,k} -c_{l,1,\beta_j,\gamma_k} s_j r_k$$
with the sum running over all \( j \) and \( k \) with \( \gamma_k - \beta_j = \alpha_i \).

We want to be able to find an \( l \) so that for every \( r \in R \) we can choose the \( s_j \) in such a way that \( t_l = r \) and \( t_k = 0 \) for \( k \neq l \). In particular, this means that we need to know the constants \( c_{1,1,\beta,-\gamma} \) for each \( \beta \) and \( \gamma \). From [3] we see that \( c_{1,1,\beta,-\gamma} = N_{\beta,-\gamma} \) where \([e_\beta, e_{-\gamma}] = N_{\beta,-\gamma} e_{\beta-\gamma} \) in \( g \) and if \( \gamma - \beta = \alpha \) then \( N_{\beta,-\gamma}(\gamma,\gamma) = N_{\alpha,\beta}(\alpha,\alpha) \). The structure constants \( N_{\alpha,\beta} \) for the Lie algebra of type \( \Sigma = D_{2k} \) were then obtained using the description of the Lie algebra in [3]; for \( \Sigma = F_4 \), \( E_6 \) and \( E_7 \) they were taken from [10]; and for \( \Sigma = E_8 \) they were calculated using GAP [9].

12.1. \( \mathfrak{g} = \Sigma_{2k-1} \) in \( D_{2k} \)

As before, let \( \gamma_0 = e_1 - e_2 \), \( \gamma_1 = e_1 + e_2 \), \( \gamma_2 = e_2 + e_2 - 1 \), \ldots, \( \gamma_k = e_k + e_{k+1} \) and \( \beta_0 = e_2 - e_2 \), \( \beta_1 = e_1 - e_{2-1} \), \( \beta_2 = e_2 + e_{2-1} \), \ldots, \( \beta_k = e_k + e_{k+2} \). Then

\[
\begin{align*}
t_1 &= -r_0 s_0 - 1 s_2, & t_2 &= -r_2 s_3, & \ldots, & t_{k-1} &= -r_{k-1} s_k, & t_k &= 0, \\
t_{k+1} &= -r_k s_k, & \ldots, & t_{2k-2} &= -r_{3} s_3, & t_{2k-1} &= r_0 s_1 - 1 r_2 s_2, & t_{2k} &= -r_2 s_0 + 1 s_1
\end{align*}
\]

and so setting

\[
\begin{align*}
s_0 &= -\frac{1}{2} r_0^{-1} r, & s_1 &= -\frac{1}{2} r_0^{-1} r_1^{-1} r_2 r, & s_2 &= -\frac{1}{2} r_1^{-1} r, & s_3 &= \cdots = s_k = 0
\end{align*}
\]

gives \( t_1 = r \) and \( t_2 = t_3 = \cdots = t_{2k} = 0 \).

12.2. \( \mathfrak{g} = \Sigma_4 \) in \( F_4 \)

Let \( \gamma_1 = e_{+++}, \gamma_2 = e_{++--}, \gamma_3 = e_2 + e_4 \) and \( \beta_1 = e_2, \beta_2 = e_{++--}, \beta_3 = e_3 + e_4 \). Then

\[
\begin{align*}
t_1 &= -2 r_2 s_2 - 1 r_3 s_3, & t_2 &= 0, & t_3 &= r_3 s_1 + 1 r_1 s_2, & t_4 &= r_2 s_1 + 1 r_1 s_3
\end{align*}
\]

and so setting

\[
\begin{align*}
s_1 &= \frac{1}{3} r_1 r_1^{-1} r, & s_2 &= -\frac{1}{3} r_2^{-1} r, & s_3 &= -\frac{1}{3} r_3^{-1} r
\end{align*}
\]

gives \( t_1 = r \) and \( t_2 = t_3 = t_4 = 0 \).

12.3. \( \mathfrak{g} = \Sigma_4 \) in \( E_6 \)

Let \( \gamma_1 = e_{+++++++}, \gamma_2 = e_{++++--}, \gamma_3 = e_5 + e_7, \gamma_4 = e_4 + e_8, \gamma_5 = e_4 - e_8 \) and \( \beta_1 = e_{+++++++}, \beta_2 = e_6 + e_7, \beta_3 = e_5 + e_8, \beta_4 = e_5 - e_8, \beta_5 = e_4 - e_7 \). Then

\[
\begin{align*}
t_1 &= -r_1 s_2 - r_2 s_4, & t_2 &= -r_1 s_1 - r_3 s_4, & t_3 &= -r_3 s_3 - r_5 s_5, & t_4 &= 0, & t_5 &= r_2 s_1 + r_3 s_2, & t_6 &= r_4 s_3 + r_5 s_4
\end{align*}
\]

and so setting
12.4. $\mathcal{G} = \Sigma_9$ in $E_7$

Let $\gamma_1 = e_+ - - - + - -$, $\gamma_2 = e_+ - - - + + +$, $\gamma_3 = e_+ - - - + - -$, $\gamma_4 = e_3 + e_4$ and $\beta_1 = e_+ + - - - + + -$, $\beta_2 = e_+ - - - + - -$, $\beta_3 = e_3 + e_5$, $\beta_4 = e_+ + - - - + +$. Then

\begin{align*}
t_1 &= -r_2s_3, \quad t_2 = 0, \quad t_3 = -r_1s_1 - r_3s_4, \quad t_4 = 0, \\
t_5 &= -r_2s_1 - r_3s_2, \quad t_6 = -r_4s_3, \quad t_7 = r_1s_2 + r_2s_4
\end{align*}

and so setting

\begin{align*}
s_1 &= -\frac{1}{2}r_1^{-1}r, \quad s_2 = \frac{1}{2}r_1^{-1}r_2r_3^{-1}r, \quad s_3 = 0, \quad s_4 = -\frac{1}{2}r_3r^{-1}r
\end{align*}

gives $t_3 = r$ and $t_1 = t_2 = t_4 = t_5 = t_6 = t_7 = 0$.

12.5. $\mathcal{G} = \Sigma_6$ in $E_8$

Let $\gamma_1 = e_+ - - - + - - - +$, $\gamma_2 = e_+ - - - + + + + -$, $\gamma_3 = e_+ - - - + + + -$, $\gamma_4 = e_4 + e_6$, $\gamma_5 = e_3 + e_7$, $\gamma_6 = e_2 + e_8$, $\gamma_7 = e_2 - e_8$ and $\beta_1 = e_+ - - - + + + - -$, $\beta_2 = e_+ - - - + + + -$, $\beta_3 = e_5 + e_6$, $\beta_4 = e_4 + e_7$, $\beta_5 = e_3 + e_8$, $\beta_6 = e_3 - e_8$, $\beta_7 = e_2 - e_7$. Then

\begin{align*}
t_1 &= -r_1s_3 - r_2s_4 - r_3s_6, \quad t_2 = -r_2s_2 - r_5s_6 - r_6s_7, \\
t_3 &= -r_5s_5 - r_7s_7, \quad t_4 = r_1s_1 + r_4s_4, \\
t_5 &= 0, \quad t_6 = r_2s_1 + r_4s_3, \\
t_7 &= r_3s_2 + r_5s_4, \quad t_8 = r_6s_6 + r_7s_6
\end{align*}

and so setting

\begin{align*}
s_1 &= \frac{2}{5}r_1^{-1}r_2^{-1}r_4r, \quad s_2 = \frac{2}{5}r_2^{-1}r_3^{-1}r_5r, \quad s_3 = -\frac{2}{5}r_1^{-1}r, \quad s_4 = -\frac{2}{5}r_2^{-1}r, \\
s_5 &= \frac{1}{5}r_3^{-1}r_6^{-1}r_7r, \quad s_6 = -\frac{1}{5}r_3^{-1}r, \quad s_7 = -\frac{1}{5}r_5r_6r^{-1}r
\end{align*}

gives $t_1 = r$ and $t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = 0$. 
12.6. $\mathcal{G} = \Sigma_{10}$ in $E_8$

Let $\gamma_1 = e_{++++---}$, $\gamma_2 = e_{++++---}$, $\gamma_3 = e_{++++---}$, $\gamma_4 = e_{++++---}$, $\gamma_5 = e_{++++---}$, $\gamma_6 = e_2 + e_4$ and $\beta_1 = e_{++++---}$, $\beta_2 = e_{++++---}$, $\beta_3 = e_{++++---}$, $\beta_4 = e_{++++---}$, $\beta_5 = e_3 + e_4$, $\beta_6 = e_2 + e_5$. Then

$$
t_1 = -r_4 s_5 - r_5 s_6, \quad t_2 = 0, \quad t_3 = -r_2 s_3 - r_3 s_4,$$
$$
t_4 = r_1 s_1, \quad t_5 = r_2 s_2 + r_5 s_4, \quad t_6 = r_4 s_3 + r_6 s_6,$$
$$
t_7 = r_2 s_1, \quad t_8 = r_3 s_2 + r_5 s_3 + r_6 s_5$$

and so setting

$$
s_1 = 0, \quad s_2 = \frac{1}{3} r_3^{-1} r_4^{-1} r_6 r, \quad s_3 = \frac{1}{3} r_4^{-1} r_5^{-1} r_6 r,$$
$$
s_4 = -\frac{1}{3} r_2 r_3^{-1} r_4^{-1} r_5^{-1} r_6 r, \quad s_5 = -\frac{2}{3} r_4^{-1} r, \quad s_6 = -\frac{1}{3} r_5^{-1} r$$

gives $t_1 = r$ and $t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = 0$.

12.7. $\mathcal{G} = \Sigma_{15}$ in $E_8$

Let $\gamma_1 = e_{++++---}$, $\gamma_2 = e_{++++---}$, $\gamma_3 = e_{++++---}$, $\gamma_4 = e_{++++---}$ and $\beta_1 = e_{++++---}$, $\beta_2 = e_{++++---}$, $\beta_3 = e_{++++---}$, $\beta_4 = e_{++++---}$. Then

$$
t_1 = 0, \quad t_2 = r_2 s_3 + r_3 s_4, \quad t_3 = 0, \quad t_4 = r_1 s_1,$$
$$
t_5 = r_2 s_2 + r_4 s_4, \quad t_6 = 0, \quad t_7 = r_3 + r_4 s_3, \quad t_8 = r_2 s_1$$

and so setting

$$
s_1 = 0, \quad s_2 = -\frac{1}{2} r_2^{-1} r_3^{-1} r_4 r, \quad s_3 = \frac{1}{2} r_2^{-1} r, \quad s_4 = \frac{1}{2} r_3^{-1} r$$

gives $t_2 = r$ and $t_1 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = 0$.

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References