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Row versus Column Operations

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0. INTRODUCTION AND STATEMENT OF RESULTS

On which matrices can a sequence of row operations always be replaced by a sequence of column operations?

This simple question, which could (should?) be asked by any freshman student of linear algebra, admits an easy analysis when the matrix takes entries in a field. We study here what happens when the entries are taken to lie in a ring, always (except for Proposition 1.4 below) commutative and with identity 1, and later (Sect. 4 onwards) assumed to be an integral domain. When a square matrix A has the property that each sequence of row operations on A is equivalent to a sequence of column operations, and vice versa, we call the matrix *bireducible*.

The first step in the study of row and column operations taken by researchers in K-theory is to observe that, for matrices of determinant 1, only one kind of operation needs be considered, namely, those that add a multiple of one row or column to another. Such an operation is obtained by left (in the case of rows) or right (for columns) multiplication by an *elementary* matrix. Elementary $n \times n$ matrices look like the identity matrix except for a single off-diagonal entry; they generate the subgroup $E_n(R)$ of the special linear group $SL_n(R)$.

One is thus led to consider the equality of the left and right cosets of $E_n(R)$ with representative A. For invertible A, the problem reduces to the normality of $E_n(R)$ in the general linear group $GL_n(R)$. Because $E_{n+1}(R)$ is normalized by $GL_n(R)$ in $GL_{n+1}(R)$ (embedded by direct sum with the identity 1×1 matrix), the problem disappears in the larger dimension (see (1.4) below). It is thus strictly a question in unstable K_1-theory. To ensure the normality of $E_n(R)$ in $GL_n(R)$, certain conditions are commonly imposed.

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We shall refer to the following as the *stock conditions* on R and n. Note that the condition $E_2(R) = SL_2(R)$ which appears below is satisfied by all rings of stable rank 1, or, more generally, by all GE_2-rings (such as Euclidean domains). Alternatively, if R is the ring of all continuous real or complex functions on a topological space, then $E_2(R)$ is the commutator subgroup of $SL_2(R)$. Condition (b)(ii) is satisfied by all Banach algebras.

Stock Conditions. (a) $n \geq 3$ or

(b) $n = 2$ and at least one of the following holds:

(i) $E_2(R) = SL_2(R)$ or a characteristic subgroup of $SL_2(R)$ (such as its commutator subgroup);

(ii) R is an $\mathbb{R}$-algebra such that for all $r \in R$, there exists $\epsilon > 0$ in $\mathbb{R}$ for which $1 + xr$ is a unit in R whenever $|x| < \epsilon$;

(iii) $E_2(R) = SL_2(R) \cap E_4(R)$.

Under these conditions, it follows from [8] (or [9] for (b)(ii)) that $E_n(R)$ is a normal subgroup of $GL_n(R)$. Moreover, within $M_n(R)$ scalar matrices are central. We therefore have, as a guide to our research, the following result.

PROPOSITION 0.1. *Under the stock conditions, every product of a scalar by an invertible matrix is bireducible.*

Our aim is thus to investigate when the converse statement holds, and also what can be said in the more delicate case when $n = 2$ but Condition (b) does not apply. We highlight some of our findings here. We remark that in the next two theorems the stock conditions are in fact used only for the argument in one direction.

THEOREM 0.2. *Under the stock conditions, an $n \times n$ matrix over a domain is bireducible if and only if after each localization at a maximal ideal it becomes the product of a scalar and an invertible matrix.*

This result is proved at the end of Section 4. For computational purposes, the following is perhaps the most useful characterization of bireducibility, at least for relatively small matrices over domains. The equivalence of (i) and (ii) is shown in Section 3, while (iii) is handled in Section 5.

THEOREM 0.3. *Let A be an $n \times n$ matrix A over a domain satisfying the stock conditions. Then the following are equivalent.*

(i) *A is bireducible;*

(ii) *either $A = 0$ or A has a nonzero determinant which divides every product of an entry of A with a cofactor of A;*

(iii) *the determinant of A divides every product of n entries of A.*

Generalization of this result to nondomains is also discussed in Section 3 below. A consequence of the impartiality of Conditions (ii) and (iii) above with respect to left and right multiplication is as follows.

COROLLARY 0.4. *Under the stock conditions on a domain, the property that any sequence of row operations is replaceable by a sequence of column operations is equivalent to the property that any sequence of column operations is replaceable by a sequence of row operations.*

Further analysis, in Section 5, leads to the following result. It is remarkable that a stable K_0 group should emerge here, from a discussion of unstable K_1 phenomena.

THEOREM 0.5. *The entries of an $n \times n$ bireducible matrix A over a domain R generate an ideal whose class in* $\mathrm{Pic}(R)$ *has order dividing n. If this class is trivial, then A is the product of a scalar and an invertible matrix.*

From a number of interesting examples, we are able to discern the following, in Section 5.

EXAMPLE 0.6. Over the polynomial ring $\mathbb{Z}[\sqrt{-5}][x]$:

not every 2×2 invertible matrix is bireducible;

for even $n \geq 4$, not every bireducible $n \times n$ matrix is the product of a scalar and an invertible matrix;

for odd n, an $n \times n$ matrix is bireducible if and only if it is the product of a scalar and an invertible matrix.

1. PRELIMINARIES

Let R be a commutative ring with unity 1 and A an arbitrary $n \times n$ matrix over R. We always take $n \geq 2$. We denote by $M_n(R)$ the ring of all square matrices of size n, by $GL_n(R)$ the general linear group of invertible $n \times n$ matrices, and by $SL_n(R)$ the special linear group of determinant 1 matrices. We use the usual notation $E_n(R)$ for the subgroup of $SL_n(R)$ generated by the elementary matrices (that is, the matrices $E_{ij}(x) = I_n + xE_{ij}$, where $x \in R$, $i \neq j$, and E_{ij} is the standard matrix unit).

DEFINITION 1.1. Let X be a nonempty subset of R, with $X = \{1\}$ written as 1. We define $E_n(X)$ to be the multiplicative group generated by $n \times n$ elementary matrices over X. We set

$$LR_n(R, X) = \{A \in M_n(R) | E_n(X) A \subseteq AE_n(R)\}$$

and

$$RL_n(R, X) = \{A \in M_n(R) | AE_n(X) \subseteq E_n(R) A\}.$$

In case $X = R$, a matrix in $LR_n(R, R) \cap RL_n(R, R)$ is called a *bireducible* matrix over R.

For any matrix $A \in M_n(RA)$, we define $R\langle A \rangle$ to be the ideal of R generated by the n^2 entries of A.

Observe that $LR_n(R, X) = \{A^t | A \in RL_n(R, X)\}$ where A^t denotes the transpose of A. Therefore, it is sufficient to work with either $LR_n(R, X)$ or $RL_n(R, X)$. It is easy to see that for any result concerning $LR_n(R, X)$, there is a corresponding one for $RL_n(R, X)$. From now on, we shall mainly be concerned with $LR_n(R, X)$. Clearly, $LR_n(R, X)$ is nonempty as it contains the zero matrix. We first list some easy results concerning $LR_n(R, X)$. Their proofs are straightforward, so we leave them to the reader.

LEMMA 1.2. *For any subsets X, Y of R with $X \subseteq Y$, $LR_n(R, Y) \subseteq LR_n(R, X)$. In particular, $LR_n(R, R) \subseteq LR_n(R, 1)$.*

With respect to matrix multiplication, the following are obvious.

LEMMA 1.3. (i) *$A \in LR_n(R, R)$ if A is in the normalizer of $E_n(R)$ in $GL_n(R)$.*

(ii) *$LR_n(R, R)$ is closed under matrix multiplication.*

(iii) *Let $d \in R$. If $A \in LR_n(R, X)$, then $dA \in LR_n(R, X)$. The converse is also true if d is not a zero divisor in R.*

Note that the converse of Lemma 1.3(iii) is false in general. For example, we take $R = \mathbb{Z}/6\mathbb{Z}$ and $A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z}/6\mathbb{Z})$. It is easy to see that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot A \notin A \cdot E_2(R)$, whereas the zero matrix $2A$ is in $LR_n(R, 1)$.

We can apply (1.3) to a result which, although well known, seems difficult to locate in the literature. We embed $GL_{n-1}(R)$ in $GL_n(R)$ via the map $A \mapsto A \oplus 1$.

PROPOSITION 1.4. *For any ring R (not necessarily commutative),*

$$GL_{n-1}(R) \subseteq LR_n(R, R).$$

Proof. For $A = (a_{ij}) \in GL_{n-1}(R)$, and $b \in R$, we use the relation

$$A^{-1}E_{nj}(b)A = \prod_{h=1}^{n-1} E_{nh}(ba_{jh}),$$

together with its counterpart for $E_{in}(b)$. For $n = 2$, there is no other possibility to consider. These relations also suffice for the cases $n \geq 3$, because there we have

$$E_{ij}(b) = E_{in}(1)E_{nj}(b)\big(E_{in}(1)\big)^{-1}\big(E_{nj}(b)\big)^{-1}$$

when $i, j \leq n-1$.

Thus in all cases $GL_{n-1}(R)$ normalizes $E_n(R)$, and so Lemma 1.3(i) applies. ∎

Notwithstanding this result, bireducibility does not in general behave well with respect to the usual stabilization; this may be seen from the integral matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Evidently, the first is bireducible, but the second is not. For a variant of stabilization which does work, see Corollary 3.5 below.

Our final preliminary lemma reveals the point of introducing $LR_n(R, X)$, namely, that it defines a functor. We shall often apply it to the situation where R is a domain and f is the embedding of R in its field of quotients.

LEMMA 1.5. *Let S be a ring and $f: R \to S$ a ring homomorphism. If $f_*: M_n(R) \to M_n(S)$ is the ring homomorphism induced by f, then $f_*(LR_n(R, X)) \subseteq LR_n(S, f(X))$. In particular, if R is a subring of S and f is the inclusion, we then have $LR_n(R, 1) \subseteq LR_n(S, 1)$.*

2. MATRICES IN $LR_n(R, 1)$ WITH ZERO DETERMINANT

LEMMA 2.1. *Let $A \in LR_n(R, 1)$ and let $v_1, \dots, v_n$ be the row vectors of A. Suppose that one of the vectors v_i is a linear combination of the others. Then $A = 0$.*

Proof. Suppose that v_i is a linear combination of the other rows. Consider the matrix $E_{ij}(1)A$ with $i \neq j$. By assumption $E_{ij}(1)A = AE$ for some $E \in E_n(R)$. Therefore, $v_t = v_t E$ for $t \neq i$. As v_i is a linear combination of the others, $v_i = v_i E$ also. However, the ith row of $E_{ij}(1)A$ is $v_i + v_j$. It follows that $v_j = 0$. Hence $v_j = 0$ for all $j \neq i$. Again, v_i is a linear combination of the others; we must have $v_i = 0$ also. Therefore $A = 0$. ∎

COROLLARY 2.2. *Let R be a domain. If $A \in LR_n(R, 1)$ and* $\det A = 0$, *then $A = 0$.*

Proof. We first assume that R is a field. Since $\det A = 0$, the row vectors of A are linearly dependent. Therefore, one of the vectors is a linear combination of the others. By Lemma 2.1, $A = 0$. Now, we come back to the case when R is a domain. Let K be the field of quotients of R. By Lemma 1.5, $A \in LR_n(K, 1)$. By the previous argument, $A = 0$. ▮

The following generalization is an easy consequence of Corollary 2.2. Recall that the radical $\operatorname{rad}(I)$ of an ideal I is the intersection of all prime ideals containing I.

COROLLARY 2.3. *Suppose that* $A \in LR_n(R, 1)$. *Then* $R\langle A\rangle \subseteq \operatorname{rad}(R(\det A))$.

Proof. It suffices to show that if P is a prime ideal containing $\det A$, then $R\langle A\rangle \subseteq P$. Consider the domain R/P and epimorphism $f : R \to R/P$. By Lemma 1.5 $f_*(A) \in LR_n(R/P, 1)$ and $\det f_*(A) = 0$. By Corollary 2.2, $R\langle A\rangle \subseteq P$. ▮

As we shall see later, the above result can be improved if R is a domain. Here is an immediate application of Corollary 2.3 to the situation where $\operatorname{rad}(0) = 0$.

COROLLARY 2.4. *Suppose that R is a reduced ring and* $A \in LR_n(R, 1)$ *with* $\det A = 0$. *Then* $A = 0$.

COROLLARY 2.5. *Let R be a domain. If* $A \in LR_n(R, X)$, *then* $\operatorname{adj} A \in LR_n(R, X)$.

Proof. When $\det A = 0$, this is immediate from Corollary 2.2 above. Otherwise, suppose that $E \in E_n(X)$ has $E' \in E_n(R)$ with $EA = AE'$. Pass to the quotient field K, over which A is invertible. There we have, with $d = \det A$,

$$(\operatorname{adj} A)E - E'(\operatorname{adj} A) = A^{-1}(dE - AE'(\operatorname{adj} A)) = 0. \quad \blacksquare$$

3. MATRICES IN $LR_n(R, 1)$ WITH NONZERO DETERMINANT

LEMMA 3.1. *Let* $A \in LR_n(R, 1)$. *Then* $R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle = R(\det A)$.

Proof. Always, $R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle \supseteq R(\det A)$. To show the reverse inclusion, we apply Lemma 1.5 in order to pass to the ring $R/(\det A)$. Equivalently, we assume that $\det A = 0$.

Let us write $A = (a_{rs})$ and $\operatorname{adj} A = (c_{uv})^t$ where c_{uv} denotes the (u, v)-cofactor. For $p \neq q$, we consider the matrix $E_{pq}(1)A$. By assumption, $A \in LR_n(R, 1)$; therefore $E_{pq}(1)A = AE$ for some $E \in E_n(R)$. Left-multiplying both sides by adj A, we obtain that $(\operatorname{adj} A)(I_n + E_{pq})A = (\det A)E$. Consequently, we get $(\operatorname{adj} A)E_{pq}A = (\det A)(E - I_n)$. As the ijth entry of $(\operatorname{adj} A)E_{pq}A$ is $c_{pi}a_{qj}$, we conclude that $c_{pi}a_{qj} = 0$.

Thus for all p, i the matrix $c_{pi}A$ has qth row zero whenever $q \neq p$, and is in $LR_n(R, 1)$ by Lemma 1.3(iii). Hence by Lemma 2.1 it is the zero matrix, as required. ∎

Note that, even over a domain, the converse of Lemma 3.1 as worded is not true in general. When $n \geq 3$ one has to make provision for the possibility that a nonzero matrix has zero adjoint, as happens with, for example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From Corollary 2.2, such a matrix cannot lie in $LR_n(R, 1)$. The next examples show that there are deeper obstacles when $n = 2$.

Example 3.2. (a) Let F be field and $R = F[x, y]$. In [4], Cohn has proved that the matrix

$$A = \begin{pmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{pmatrix} \in SL_2(R) \setminus E_2(R).$$

As $A \in SL_2(R)$, it is clear that $R\langle A\rangle R\langle \operatorname{adj} A\rangle = R(\det A) = R$. On the other hand, Silvester [7, p. 121] has shown that $A^{-1}E_{12}(1)A \notin E_2(R)$. Hence $A \in SL_2(R) \setminus LR_2(R, 1)$.

(b) It follows from [3] that the argument of (a) goes through when the field F is replaced by any Noetherian ring of dimension 0 (for example, $\mathbb{Z}/m\mathbb{Z}$).

(c) Similarly (after [4] again), over the discretely ordered ring $R = \mathbb{Z}[x]$,

$$\begin{pmatrix} 1 - 2x & -x^2 \\ 4 & 1 + 2x \end{pmatrix} \in SL_2(R) \setminus LR_2(R, 1).$$

(d) Again using [3], one can show that, over the polynomial ring $R = \mathbb{Z}[\sqrt{-5}\,][x]$, the matrix

$$\begin{pmatrix} 1 + 3(2 + \sqrt{-5}\,)x & (1 - 4\sqrt{-5}\,)x \\ 9x & 1 - 3(2 + \sqrt{-5}\,)x \end{pmatrix} \in SL_2(R) \setminus LR_2(R, 1).$$

(Actually, the arguments of [3] pertain to $\mathbb{Z}[\sqrt{-5}][x, x^{-1}]$; however, we may apply Lemma 1.5 to reduce to R.)

However, as we show now, the converse to Lemma 3.1 is true with some extra assumption on R. This proves the equivalence of (i) and (ii) in Theorem 0.3.

THEOREM 3.3. *Suppose that $A \in M_n(R)$ satisfies*

$$R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle = R(\det A) \neq 0.$$

Suppose further either that R satisfies Stock Condition (a) *or* (b)(ii) *or else, with N as the annihilator ideal of* $(\det A)^2$, *that R/N satisfies* (b)(i) *or* (b)(iii). *Then* $(\det A)^3 A$ *is bireducible. In particular, if* $\det A$ *is not a zero divisor in R, then A is bireducible.*

Proof. We write $d = \det A$ and pass to $\overline{R} = R/N$ where $A \mapsto \overline{A}$. By assumption, for any matrix B in $M_n(R)$ there are matrices F, F' such that

$$dF = (\operatorname{adj} A)BA, \qquad dF' = AB(\operatorname{adj} A).$$

Although F, F' need not be unique in $M_n(R)$, their images in $M_n(\overline{R})$ are unique, and so determine functions $\Phi_A, \Phi'_A : M_n(\overline{R}) \to M_n(\overline{R})$, respectively. To see that $\Phi'_A \circ \Phi_A$ is the identity mapping, suppose that G in $M_n(R)$ has $dG = AF(\operatorname{adj} A)$ where, as above, $dF = (\operatorname{adj} A)BA$. Then $d^2G = d^2B$, so that, in $M_n(\overline{R})$, $\overline{G} = \overline{B}$, as required. Similarly, $\Phi_A \circ \Phi'_A$ is the identity, so that Φ_A is bijective. We next show that $\Phi_A(\overline{B}_1)\Phi_A(\overline{B}_2) = \Phi_A(\overline{B}_1\overline{B}_2)$. From the equations $dF_i = (\operatorname{adj} A)B_iA$ and $dH = (\operatorname{adj} A)B_1B_2A$ we obtain

$$d^2F_1F_2 = (\operatorname{adj} A)B_1A(\operatorname{adj} A)B_2A = d(\operatorname{adj} A)B_1B_2A = d^2H,$$

which gives the required relation. Hence Φ_A restricts to an automorphism of $GL_n(\overline{R})$.

When Stock Condition (a) or (b)(ii) holds for R, it is easily seen to hold also for $\overline{R}$, so that we may appeal to the fact that $E_n(\overline{R})$ is characteristic in $GL_n(\overline{R})$ [9, Theorem 3] to deduce that $\Phi_A(E_n(\overline{R})) = E_n(\overline{R})$. To consider the situation when Condition (b)(i) or (iii) holds, first observe from the equation $dF = (\operatorname{adj} A)BA$ that because $n = 2$ we have $\det F - \det B \in N$, whence $\det \Phi_A(\overline{B}) = \det \overline{B}$. Thus Φ_A restricts to an automorphism of $SL_2(\overline{R})$. Then again, under Stock Condition (b)(i), we have $\Phi_A(E_n(\overline{R})) = E_n(\overline{R})$. In the case (b)(iii), this follows from Lemma 3.4 below (which makes $GL_4(\overline{R})$ the image of $\Phi_{A\oplus A}$) and the relation

$$\Phi_{A\oplus A}\big(\overline{B} \oplus I_2\big) = \Phi_A(\overline{B}) \oplus I_2.$$

Because $E_4(\bar{R})$ is characteristic in $GL_4(\bar{R})$, when $\bar{B} \in SL_2(\bar{R})$ each expression lies in $SL_2(\bar{R}) \cap E_4(\bar{R})$. So the condition applies. In summary, we always have $\Phi_A(E_n(\bar{R})) \subseteq E_n(\bar{R})$.

In particular, for any E in $E_n(R)$, since $E_n(R) \to E_n(\bar{R})$ is surjective we can find $E' \in E_n(R)$ with $d^2((\operatorname{adj} A)EA - dE') = 0$. This makes $d^3(EA - AE') = 0$, and therefore $d^3A \in LR_n(R, R)$. Likewise, by arguing with Φ'_A, we obtain that $d^3A \in RL_n(R, R)$. Hence d^3A is after all bireducible.

Finally, when d is not a zero divisor, we can apply Lemma 1.3(ii) to conclude that A is itself bireducible. ∎

This theorem allows us to stabilize in the following way. Given $A \in M_n(R)$, we consider its direct sum $A \oplus \cdots \oplus A \in M_{mn}(R)$. The following is easily checked.

LEMMA 3.4. *For $\hat{A} = A \oplus \cdots \oplus A$ over R, if*

$$R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle = R(\det A)$$

then

$$R\langle \hat{A}\rangle \cdot R\langle \operatorname{adj} \hat{A}\rangle = R(\det \hat{A}).$$

The converse holds provided that d is not a zero divisor in R.

The theorem, combined with Lemma 3.1, now has an easy application here.

COROLLARY 3.5. *If R is a domain and $A \in LR_n(R, X)$, then the direct sum of m copies of A has*

$$A \oplus \cdots \oplus A \in LR_{mn}(R, X).$$

The converse is also true under the stock conditions.

The situation for nondomains in the above theorem is illustrated by the following example.

EXAMPLE 3.6. Let F be a field and let $R = F[t]/(t^4 - t^3)$. Then the matrix

$$A = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}$$

satisfies

$$(R\langle A\rangle)^3 = R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle = R(\det A).$$

However, over the residue ring $F[t]/(t^2)$, A has just one row zero. Hence by Lemmas 1.5 and 2.1 $A \notin LR_3(R, 1)$. On the other hand, observe that $(\det A)A = t^3 I_3$ is bireducible.

It is interesting to note that, for $n \geq 3$, we do not have any example of a matrix A that satisfies the equation

$$R\langle A \rangle \cdot R\langle \operatorname{adj} A \rangle = R(\det A)$$

and yet fails to have $(\det A)A \in LR_n(R, R)$. In some cases, for example when R is a reduced ring, this is because the annihilator of $\det A$ coincides with that of $(\det A)^3$. More seriously, our main device for constructing matrices outside $LR_n(R, 1)$ is Lemma 2.1. However, it has the following counterpart.

PROPOSITION 3.7. *Let A satisfy the equation*

$$R\langle A \rangle \cdot R\langle \operatorname{adj} A \rangle = R(\det A)$$

and suppose that $(\det A)A$ *has as one of its rows a linear combination of the others. Then* $(\det A)A = 0$.

Proof. First observe that the assertion is equivalent to that obtained by replacing the matrix A by EA, where E is any elementary matrix. By taking a suitable product of elementary matrices, we may therefore pass to the situation where $A = (a_{ij})$ has h-th row annihilated by $\det A$. Then, with $\operatorname{adj} A = (c_{ij})^t$, we have for all s, t

$$(\det A) a_{st} = \sum_j a_{hj} c_{hj} a_{st}.$$

However, the hypotheses force all of the right-hand side terms to be zero because $\det A$ divides each $c_{hj} a_{st}$. ∎

4. DESCRIPTION OF MATRICES IN $LR_n(R, 1)$

For the remainder of this paper, R is a commutative domain with identity 1 and quotient field K. In this section, our objective is to determine matrices in $LR_n(R, 1)$. Recall from Proposition 0.1 that under our stock conditions the set

$$R \cdot GL_n(R) := \{aA | a \in R,\ A \in GL_n(R)\}$$

is a subset of $LR_n(R, 1)$. The following elementary example shows that we may reasonably expect the converse often to be true.

PROPOSITION 4.1. *Let R be a unique factorization domain. Then* $LR_n(R,1) \subseteq R \cdot GL_n(R)$.

Proof. Suppose that $A \in LR_n(R,1)$ with d as the greatest common divisor of the entries of A. Then $A = dA'$, with, by Lemma 1.3, also $A' \in LR_n(R,1)$. Now from Corollary 2.3, any prime divisor of $\det A'$ would divide all the entries of A', and so contradict our choice of d. Hence $\det A'$ is a unit in R, so that indeed $A \in R \cdot GL_n(R)$. ∎

Note that Example 3.2(a) shows that when $n = 2$ this inclusion may be proper. By appealing to Lemma 3.1, we may generalize the proposition to GCD-domains. Recall that a domain R is a *GCD-domain* if for every finitely generated ideal I in R, there exists a smallest principal ideal containing I.

LEMMA 4.2. *Let R be a GCD-domain. Ra is the smallest principal ideal containing $a_1, \dots, a_n$ if and only if Rab is the smallest principal ideal that contains $ba_1, \dots, ba_n$.*

Proof. Suppose that Ra is the smallest principal ideal containing $a_1, \dots, a_n$ and Rc is the smallest principal ideal containing $ba_1, \dots, ba_n$. As it is clear that Rab contains $ba_1, \dots, ba_n$, we have $Rc \subseteq Rab$. Therefore, $c = bc'$. As R is a domain, Rc' contains $a_1, \dots, a_n$. Thus $Rc' \supseteq Ra$. Hence $Rc = Rab$. ∎

PROPOSITION 4.3. *Let R be a GCD-domain. Then* $LR_n(R,1) \subseteq R \cdot GL_n(R)$.

Proof. By assumption, there exists a principal ideal Ra which is the smallest principal ideal containing $R\langle A\rangle$. As in Proposition 4.1, we can write $A = aA'$. Write $A' = (a_{ij})$ and $\operatorname{adj} A' = (c_{pq})^t$. To establish the invertibility of A', it suffices to show that $\det A'$ divides all c_{pq}.

We first fix some p, q. By the lemma, the smallest principal ideal that contains each a_{ij} is R and the smallest principal ideal that contains every $a_{ij}c_{pq}$ is Rc_{pq}. From Lemma 3.1, $(\det A') | (a_{ij}c_{pq})$ for all i, j. Hence $R(\det A')$ contains $a_{ij}c_{pq}$ for all i, j. By the assumption on R, we conclude that $R(\det A') \supseteq Rc_{pq}$. This proves that $\det A'$ divides c_{pq}. ∎

Our strategy is to find a sufficient condition for a matrix $A \in LR_n(R,1)$ to be in $R \cdot GL_n(R)$. As the argument above may suggest, principal ideals figure in this approach. Clearly, if a matrix $A \in R \cdot GL_n(R)$, then $R\langle A\rangle$ is principal. Surprisingly, the latter condition is also sufficient for $A \in R \cdot GL_n(R)$ if $A \in LR_n(R,1)$ (but not in general, as the matrix A_1 given below illustrates).

PROPOSITION 4.4. *Let $A \in LR_n(R,1)$. If $R\langle A\rangle$ is principal, then* $A \in R \cdot GL_n(R)$.

Proof. By Lemma 2.2, we may assume $\det A \neq 0$. Suppose that $R\langle A\rangle = Ra$ for some $a \in R$. Write $A = aA'$, where $A' \in LR_n(R, 1)$ by Lemma 1.3. Clearly, $R\langle A'\rangle = R$. So, by Lemma 3.1, we have that $R\langle \operatorname{adj} A'\rangle = R(\det A')$. That means $A'^{-1} = (1/\det A')\operatorname{adj} A' \in M_n(R)$. Hence $A' \in GL_n(R)$. ∎

At this point it is useful to insert an immediate deduction from Lemma 3.1.

COROLLARY 4.5. *If $A \in LR_n(R, 1)$ and $A \neq 0$, then the fractional ideal $R\langle A\rangle$ is invertible.*

There is, however, a loss of information in this step. For example, the integral matrix

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

has $R\langle A_1\rangle$ an invertible (indeed, principal) ideal, although evidently $R\langle A_1\rangle \cdot R\langle \operatorname{adj} A_1\rangle \neq R(\det A_1)$.

Next, recall that a commutative ring is *semilocal* if it has only a finite number of maximal ideals.

THEOREM 4.6. *If R is a semilocal domain, then*

$$LR_n(R, 1) = LR_n(R, R) = RL_n(R, 1) = RL_n(R, R) = R \cdot GL_n(R).$$

Proof. As ever (see (0.1) above), $R \cdot GL_n(R) \subseteq LR_n(R, R)$ for $n \geq 3$. The result is also true for $n = 2$, because semilocal rings have stable rank 1, making $E_2(R) = SL_2(R)$ [2, V.3.4]. To see that $LR_n(R, 1) \subseteq R \cdot GL_n(R)$, we again need consider only nonzero $A \in LR_n(R, 1)$. Then by the above corollary, $R\langle A\rangle$ is invertible. A well-known result [5, Theorem 60] guarantees that $R\langle A\rangle$ is principal. So the result follows from Proposition 4.4. ∎

An interesting alternative proof of the theorem for a local domain with $n \geq 3$ derives from Petechuk's characterization of automorphisms of $E_n(R)$ [6]. Thus the automorphism Φ_A of Theorem 3.3 must have the form

$$\Phi_A\big(E_{ij}(1)\big) = CE_{ij}(1)^{\check{k}^{\varepsilon}} C^{-1} \qquad \forall i \neq j$$

for some $C \in GL_n(R)$ and $\varepsilon \in \{0, 1\}$, where the automorphism $\check{k}$ sends an elementary matrix E to $(E^t)^{-1}$.

Suppose that $\varepsilon = 0$. We then have

$$E_{ij}(1)(AC) = (AC)E_{ij}(1) \qquad \forall i \neq j.$$

By a straightforward computation, it is easy to check that AC is a scalar matrix dI_n. Therefore, $A = dC^{-1} \in R \cdot GL_n(R)$.

Suppose that $\varepsilon = 1$. The equation now becomes

$$A^{-1}E_{ij}(1)A = CE_{ij}(1)^{\check{k}}C^{-1} \qquad \forall i \neq j.$$

We thus obtain $E_{ij}(1)(AC) = (AC)E_{ji}(-1)$ for all $i \neq j$. Again by a straightforward computation, we get $AC = 0$. As C is invertible, $A = 0$. Thus A always has the desired form.

We are now ready for the proof of Theorem 0.2. Let A be a nonzero matrix. Then Theorems 3.3 and 4.6 imply the chain of equivalences (only the last of which makes use of the stock conditions):

for any maximal ideal $\mathfrak{m}$, $A \in R_{\mathfrak{m}} \cdot GL_n(R_{\mathfrak{m}})$;
$\Leftrightarrow$ for any maximal ideal $\mathfrak{m}$, $A \in LR_n(R_{\mathfrak{m}}, 1)$;
$\Leftrightarrow$ for any maximal ideal $\mathfrak{m}$, $R_{\mathfrak{m}}\langle A\rangle \cdot R_{\mathfrak{m}}\langle \operatorname{adj} A\rangle = R_{\mathfrak{m}}(\det A)$;
$\Leftrightarrow R\langle A\rangle \cdot R\langle \operatorname{adj} A\rangle = R(\det A)$;
$\Leftrightarrow A \in LR_n(R, R)$.

5. THE PICARD GROUP

The combination of Corollary 4.5 and Proposition 4.4 suggests investigation of when an invertible ideal $R\langle A\rangle$ is principal. This is detected precisely by the class $[R\langle A\rangle]$ of $R\langle A\rangle$ in the Picard group $\operatorname{Pic}(R)$ of R.

PROPOSITION 5.1. *Suppose that $A \in LR_n(R, 1)$. Then $A \in R \cdot GL_n(R)$ if and only if $[R\langle A\rangle] = 0 \in \operatorname{Pic}(R)$.*

Thus much of our earlier information on GCD-domains and semilocal domains is simply expressed by the statement that there the Picard group vanishes. On the other hand, even when the group is nontrivial, it can still be very useful, by virtue of the next lemma.

LEMMA 5.2. *If $A \in LR_n(R, 1)$, then $R\langle A\rangle^n = R(\det A)$.*

Proof. To prove $R\langle A\rangle^n = R(\det A)$, it suffices to show that for every maximal ideal $\mathfrak{m}$, $R_{\mathfrak{m}}\langle A\rangle^n = R_{\mathfrak{m}}(\det A)$. Since $A \in LR_n(R, 1)$, also $A \in LR_n(R_{\mathfrak{m}}, 1)$ by Lemma 1.5. As $R_{\mathfrak{m}}$ is a local domain, we can apply Theorem 4.6 to see that $A = aA'$ where $a \in R_{\mathfrak{m}}$ and $A' \in GL_n(R_{\mathfrak{m}})$. Obviously, $R_{\mathfrak{m}}\langle A\rangle^n = R_{\mathfrak{m}}(a^n) = R_{\mathfrak{m}}(\det A)$. ∎

These two results immediately yield Theorem 0.5, the equivalence of (i) and (iii) in Theorem 0.3, and the following.

THEOREM 5.3. *If* $\operatorname{Pic}(R)$ *contains no* n*-torsion, then* $LR_n(R,1) \subseteq R \cdot GL_n(R)$.

COROLLARY 5.4. $LR_n(R,1) \subseteq R \cdot GL_n(R)$, *provided that either*

(a) *the (multiplicative) group of units of* $K_0(R)$ *is* n*-torsion free; or*

(b) R *is a Krull domain whose ideal class group* $\operatorname{Cl}(R)$ *is* n*-torsion free; or*

(c) R *is a ring of algebraic integers in a finite field extension of the rationals* $\mathbb{Q}$ *whose class group has order coprime to* n.

Proof. In each case, $\operatorname{Pic}(R)$ embeds in the given group. For (a), see [7, p. 48]; for (b), see [1, p. 187]; (c) is a special case of (b) where the class group is known to be finite. ∎

We give an example to show the necessity of the coprimality condition in the corollary.

Let $R = \mathbb{Z}[\sqrt{-5}]$. Its class group is of order 2, corresponding to the fact that the ideal $I = R(2) + R(1 + \sqrt{-5})$ is not principal, although $I^2 = R(2)$ is principal. This prompts us to take $n = 2$, and consider the matrix

$$A = \begin{pmatrix} 1+\sqrt{-5} & 2 \\ 2 & 1-\sqrt{-5} \end{pmatrix}.$$

Evidently, $R\langle A\rangle = R\langle \operatorname{adj} A\rangle = I$. Thus

$$R\langle A\rangle R\langle \operatorname{adj} A\rangle = R(2) = R(\det A).$$

Unfortunately, the stock conditions are not known to apply here; so we cannot use Theorem 3.3 to deduce that $A \in LR_2(R,1)$. Instead we pass into "stable territory" via Lemma 3.4, to conclude that $A \oplus A \in LR_4(R,1)$. However, since $R\langle A \oplus A\rangle = R\langle A\rangle$ is not a principal ideal, $A \oplus A \notin R \cdot GL_4(R)$. We therefore have $R \cdot GL_4(R)$ as a *proper* subset of $LR_4(R,1)$. A similar statement holds when 4 is replaced by any larger even integer.

Since $\mathbb{Z}[\sqrt{-5}]$ embeds in $\mathbb{Z}[\sqrt{-5}][x]$ as a direct summand, the example also shows (via Lemma 1.5) that the corresponding statement holds over the polynomial ring. This ring was the subject of Example 3.2(d). We may therefore summarize our knowledge of $LR_n(R,1)$ for $R = \mathbb{Z}[\sqrt{-5}][x]$ as follows:

$$
\begin{array}{ll}
n = 2: & LR_2(R, 1) \text{ does not contain } R \cdot GL_2(R). \\
n \text{ even}, n \geq 4: & LR_n(R, 1) \textit{ properly} \text{ contains } R \cdot GL_n(R). \\
n \text{ odd}: & LR_n(R, 1) = R \cdot GL_n(R).
\end{array}
$$

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