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Renormalisation Group, Function Iterations and Computer Algebra

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The set of perturbative solutions of the renormalisation group equation relative to the coupling constant known in quantum field theory is studied. It is shown that it is generated by the continuous iteration of a particular solution. The iteration parameter, which may be complex, is the parameter of an abelian group whose generator is proportional to β. Two new lemmas are derived. They are useful to construct a simple algorithm to find the continuous iteration of various functions and to deduce trivially, from the already known iteration of a function, the continuous iteration of a whole family of functions. It is pointed out how one can find iterations of functions which do not satisfy the perturbative boundary condition. The algorithm has been implemented in REDUCE and several examples are discussed.

1. Introduction

The renormalisation group (RG) equations have been extensively used for many years in quantum field theory (Stueckelberg & Peterman, 1953; Gell-Mann & Low, 1954; Bogoliubov & Shirkov, 1959; Callan, 1970; Symanzik, 1970).

One of these equations expresses the invariance of physical quantities with respect to a redefinition of the coupling constant. It reads

$$\beta(f(\alpha)) = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}\,\beta(\alpha). \tag{1}$$

$\beta(\alpha)$ is the so-called renormalisation group function while α and $f(\alpha)$ are, respectively, the coupling constant and the redefined coupling constants. Equation (1) is a functional equation which was already known to the mathematician Hadamard and which has been considered in several other domains of science. Not willing to be exhaustive, let us mention that it is used in the study of branching processes (Harris, 1963) and it has been studied by computer scientists (Knuth, 1969; Brent & Traub, 1980) essentially from the point of view of complexity theory.

In quantum field theory, the current way to exploit this equation is to find f (also called the "running coupling constant") for a given renormalisation group function β. Moreover, the equation is considered in a *perturbative* framework where β and f are expressed as asymptotic series.

We present here a further investigation of this equation. In section 2, we describe the content of Eq. (1) within the "perturbative" boundary condition. We relate our considerations to the existing literature. We point out that, introducing the operation of "iterative exponentiation" or continuous iteration, the set of solutions generates all the

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elements of an abelian group whose generator is proportional to β. Next, we prove two new lemmas. The first gives a new expression of the solutions f in terms of the iteration parameter, the second expresses relations between solutions corresponding to different βs. This second lemma makes possible the construction of solutions which do not obey the perturbative boundary condition.

In section 3, we explain how to calculate the iterations in a systematic and simple way and we discuss several interesting examples. Section 4 gives a short account of an application of iterations (Hans, 1985) and discusses other closely related conditions imposed on $\beta(\alpha)$ and $f(\alpha)$ which have been considered in the literature.

2. The Solutions of the RG Equation and Their Properties

The *formal* solutions of Eq. (1) are given by

$$f(\alpha, C) = K_\beta^{-1}(K_\beta(\alpha)+C), \tag{2}$$

where

$$K_\beta(\alpha) = \int^\alpha \frac{d\alpha'}{\beta(\alpha')}, \tag{3}$$

and C is an arbitrary constant. Equation (2) is formal to the extent that we do not know K_β and we cannot tell whether K_β^{-1} exists or not. Equation (2) is known as the ABEL equation.

$f(\alpha, C)$ has the two properties

$$f(\alpha, 0) = \alpha, \tag{4a}$$

and

$$\beta \to \lambda\beta \Rightarrow f(\alpha, C) \to f(\alpha, \lambda C). \tag{4b}$$

EXAMPLES

(i) If

$$\beta(\alpha) = \beta\alpha, \tag{5}$$

then

$$f(\alpha) = \alpha e^{\beta C}; \tag{6}$$

(ii) if

$$\beta(\alpha) = \alpha^2, \tag{7}$$

then

$$f(\alpha) = \frac{\alpha}{1-C\alpha}. \tag{8}$$

If $f(\alpha, \xi_1)$ and $f(\alpha, \xi_2)$ are two solutions of Eq. (2) corresponding to $C=\xi_1$ and ξ_2, respectively, then

$$\begin{aligned} f(f(\alpha, \xi_2), \xi_1) &= K_\beta^{-1}(K_\beta(f\alpha, \xi_2)+\xi_1) \\ &= K_\beta^{-1}(K_\beta(\alpha)+\xi_1+\xi_2) \\ &= f(\alpha, \xi_1+\xi_2), \end{aligned} \tag{9}$$

is a solution corresponding to $C=\xi_1+\xi_2$.

So, f also satisfies the translation functional equation discussed by Aczel (1966). This equation can also directly be deduced from Eq. (1) because of the properties of the derivative operator if we confine ourselves to solutions which are differentiable. As long as we consider solutions which admit a well-defined derivative with respect to α, there is a complete equivalence between Eq. (9) and Eq. (1), and the set of solutions of Eq. (1) has the same properties as the corresponding subset of solutions of Eq. (9).

We consider $\xi = 1$ and write

$$f(\alpha, 1) \equiv f(\alpha). \tag{10}$$

From Eq. (9), it follows that the nth iterate of $f(\alpha)$ is also a solution of Eq. (1). We define the iterative exponentiation operation for all n by

$$f(f(\ldots f(\alpha)\ldots)) \equiv f \circ f \circ \ldots \circ f \equiv f \circ\circ\, n; \tag{11}$$

n is the iterative exponentiation exponent.

The functional Eq. (9) allows us to extend this concept for all real or even complex values of n. We can indeed rewrite it.

$$\begin{aligned} f(\alpha, \xi_1) \circ f(\alpha, \xi_2) &= (f(\alpha) \circ\circ\, \xi_1) \circ (f(\alpha) \circ\circ\, \xi_2) \\ &= f(\alpha) \circ\circ\, (\xi_1 + \xi_2). \end{aligned} \tag{12}$$

So, the set of solutions of Eq. (1) is entirely given by the "continuous iteration" of a particular solution (here defined as $f(\alpha, 1)$.

This concept of continuous iteration is not new; it was first introduced by Cayley (1860) but also considered by many other people (Aczel, 1966). Here, we define it through Eqs. (11) and (12).

In perturbative quantum field theory, $f(\alpha)$ must satisfy

$$f(\alpha) = \alpha + f_1\alpha^2 + f_2\alpha^3 + \ldots, \tag{13}$$

and $\beta(\alpha)$ must satisfy

$$\beta(\alpha) = \beta^{(2)}\alpha^2 + \beta^{(3)}\alpha^3 + \ldots, \tag{14}$$

in some positive neighbourhood of $\alpha = 0$.

These are asymptotic expansions, so they are not necessarily convergent. What is crucial is that, in Eq. (13), the linear term is always present while, in Eq. (14), the lowest power is always bigger than one.

If we substitute these expansions in Eq. (1) we obtain the system of equations

$$\begin{gathered} \beta^{(3)}f_1 + \beta^{(2)}(-f_2 + f_1^2) = 0, \\ 2\beta^{(4)}f_1 + 3\beta^{(3)}f_1^2 + \beta^{(2)}(-2f_3 + 2f_1f_2) = 0, \\ \vdots \\ (k-2)\beta^{(k)}f_1 + \{(k-4)f_2 + \mathcal{O}(f_1)\}\beta^{(k-1)} + \ldots \\ + \beta^{(2)}\{(-k+2)f_{k-1} + \mathcal{O}(f_{k-2}, \ldots, f_1)\} = 0. \end{gathered} \tag{15}$$

The interesting fact is that this system can always be solved to find the $\beta^{(i)}$ coefficients when the f_js are known or vice versa (Knuth, 1969). When the $\beta^{(i)}$ are known the solution is determined up to one parameter which is f_1 when $\beta^{(2)}$ is different from zero and f_J when $\beta^{(J+1)}$ is the first non-zero β-coefficient. We call ξ this parameter, so that

$$\{f(\alpha, \xi)\ \forall\, \xi \in [-\infty, +\infty]\} \tag{16}$$

represents the set of all solutions. By construction, $f(\alpha, \xi)$ is expressed as a (formal) power serie in α and ξ. We can write it as

$$f(\alpha, \xi) = \alpha + \xi F(\alpha) + \mathcal{O}(\xi^2). \tag{17}$$

The composition law Eq. (10), necessarily satisfied by it, allows us to write

$$f(\alpha,\xi)=\left(f\left(\alpha,\frac{\xi}{n}\right)\right)\circ\circ\, n, \tag{18}$$

where

$$f\left(\alpha,\frac{\xi}{n}\right)=\alpha+\frac{\xi}{n}F(\alpha)+\mathcal{O}\left(\frac{\xi^2}{n^2}\right). \tag{19}$$

This shows that Eq. (16) forms an *abelian* group whose *generator* is $F(\alpha)$. This is fully discussed in Caprasse & Hans (1987).

The substitution of Eq. (19) in Eq. (1) shows that

$$F(\alpha)=\kappa\beta(\alpha), \tag{20}$$

where κ is a constant. It is fixed by a suitable normalisation which tells which function of the family (16) we associate to $\xi=1$. We take

$$\kappa=-1. \tag{21}$$

The fact that $\beta(\alpha)$ is the generator of the abelian group formed by the solutions will play an important role in the construction of the algorithm described in section 3. Furthermore, we are aware that the conditions (13) and (14) are *not* the most general which guarantee that a solution of Eq. (1) can be constructed. This point will be further discussed in the last section. For the time being we stick to these conditions and proceed to prove two new lemmas.

LEMMA 1. *If $\beta(\alpha)$ is infinitely differentiable in some non-empty open set of α, then all solutions of Eq.* (1) *can be written*

$$f(\alpha)\circ\circ\,\xi=(e^{-\xi\beta(\alpha)d/d\alpha})\alpha \tag{22}$$

if Eqs. (13) *and* (14) *are satisfied.*

Indeed, since $f(\alpha,\xi)$ is an element of an abelian group, it is analytic in ξ and we may write

$$f(\alpha,\xi)=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}\left.\frac{d^nf(\alpha,\xi)}{d\xi^n}\right|_{\xi=0}. \tag{23}$$

We express the ξ-derivatives in terms of the α-derivatives. From Eq. (10), we get

$$\left.\frac{df(\alpha,\xi+h)}{d\xi}\right|_{h=0}=\left.\frac{df(\alpha,\xi+h)}{dh}\right|_{h=0}=\left.\frac{df(f(\alpha,\xi),h)}{dh}\right|_{h=0}. \tag{24}$$

From Eqs. (17), (19), (20) and (21) we know that

$$f(f(\alpha,\xi),h)=f(\alpha,\xi)-h\beta(f(\alpha,\xi))+\mathcal{O}(h^2). \tag{25}$$

Applying Eq. (1) we find

$$\frac{df(\alpha,\xi)}{d\xi}=-\left(\beta(\alpha)\frac{d}{d\alpha}\right)f(\alpha,\xi) \tag{26}$$

and

$$\frac{d^nf(\alpha,\xi)}{d\xi^n}=-\left(\beta(\alpha)\frac{d}{d\alpha}\right)^nf(\alpha,\xi). \tag{27}$$

Substitution of Eqs. (26) and (27) into Eq. (23) gives Eq. (22). In particular, if

$$\beta(\alpha_0) = 0 \tag{28}$$

for some $\alpha_0 \neq 0$, we see that

$$f(\alpha) \circ \circ \xi|_{\alpha=\alpha_0} = \alpha_0, \tag{29}$$

for all ξ.

EXAMPLE

$$\beta(\alpha) = \alpha^2, \quad f(\alpha) = \frac{\alpha}{1+\alpha}. \tag{30}$$

We find

$$f(\alpha, \xi) = (e^{-\xi\alpha^2 d/d\alpha})\alpha = \frac{\alpha}{1+\xi\alpha}. \tag{31}$$

REMARK. The conditions (13) and (14) are *sufficient* for the validity of (22) but not necessary. If we take

$$\beta(\alpha) = \beta\alpha \tag{32}$$

and write (22) we get

$$f(\alpha) \circ \circ \xi = (e^{-\xi\beta\alpha d/d\alpha})\alpha = \alpha\, e^{-\beta\xi}, \tag{33}$$

which is indeed the correct solution of Eq. (1).

LEMMA 2. *If* f(α) *is a solution of* (1) *and if* h(α) *has an inverse and admits a derivative in some interval* $]\alpha_1, \alpha_2[$, *then the function*

$$f_h(\alpha) = (h^{-1} \circ f \circ h)(\alpha) \tag{34}$$

satisfies Eq. (1) *for*

$$\beta_h(\alpha) = [\beta(h(\alpha))] \Big/ \frac{dh(\alpha)}{d\alpha}. \tag{35}$$

Indeed, we apply the transformation

$$\alpha' = h(\alpha) \tag{36}$$

to Eq. (1). Then

$$\beta((f \circ h)(\alpha)) = \frac{d(f \circ h)(\alpha)}{d\alpha} \left[\beta(h(\alpha)) \Big/ \frac{dh(\alpha)}{d\alpha} \right]. \tag{37}$$

Next, we write

$$f \circ h = h \circ (h^{-1} \circ f \circ h) = h \circ f_h \tag{38}$$

and apply the properties of the derivative to get

$$\beta_h(f_h) = \left(\frac{df_h(\alpha)}{d\alpha}\right) \beta_h(\alpha). \tag{39}$$

Apart from the regularity conditions included in the hypothesis, h is an arbitrary function. This lemma applies whether conditions (13) and (14) are valid or not. We also see immediately that, if

$$h \circ f = f \circ h, \tag{40}$$

then h obeys the *same equation as* f, i.e. there exists ξ_0 such that

$$h(\alpha) = f(\alpha) \circ \circ \xi_0. \tag{41}$$

This lemma is very important since the knowledge of one set of solutions of Eq. (1) allows to deduce directly many others.

The main conclusion of this section is that, when the conditions (13) and (14) are satisfied, we can always solve Eq. (1) and the set of solutions we obtain corresponds to the continuous iteration of a formal power serie in α. When this expansion can be summed it gives the solution of the Abel equation. This possibility opens the way to compute continuous iterations of many functions (algebraic or transcendental) which cannot be found from the direct integration of Eq. (1).

3. The Explicit Calculation of Continuous Iterations

In this section we want to explain the determination of continuous iterations of functions using lemmas 1 and 2 and computer algebra. But, first, it seems worth while to illustrate the use of Eq. (2) for cases where Eqs. (13) and (14) are not valid. We shall call these solutions "non-perturbative solutions".

With

$$\beta(\alpha) = \frac{1}{\cos^2 \alpha}, \tag{42}$$

Eq. (2) gives

$$f(\alpha, \xi) = \operatorname{arctg}(\operatorname{tg} \alpha + \xi). \tag{43}$$

With

$$\beta(\alpha) = -(\ln 2)\alpha \ln \alpha, \tag{44}$$

we find

$$f(\alpha, \xi) = \alpha^2. \tag{45}$$

Both examples violate Eqs. (13) or (and) (14).

Second, lemma 2 applies to all solutions of Eq. (1) whether *it is perturbative or not*. With its help we can start from a perturbative solution and generate non-perturbative ones. Let us give two illustrations. We start from solution (31) and apply lemma 2 with

$$h(\alpha) = \alpha^J, \tag{46}$$

where J is the fixed integer

$$h^{-1}(\alpha) = \alpha^{1/J}; \tag{47}$$

we find [see Eq. (35)]

$$\beta_h = \frac{\beta(h)}{h'} = \frac{\alpha^{J+1}}{J}, \tag{48}$$

and

$$f_h(\alpha, \xi) = (f(\alpha^J, \xi))^{1/J} = \frac{\alpha}{(1+\xi\alpha^J)^{1/J}} \tag{49}$$

It is also a perturbative solution and can be written

$$f_h(\alpha, \xi) = (e^{-\xi(\alpha^{J+1}/J, \mathrm{d}/\mathrm{d}\alpha)})\alpha, \tag{50}$$

from lemma 1.

We take now

$$h(\alpha) = e^{-1/\alpha}. \tag{51}$$

This function has an essential singularity in $\alpha = 0$. But, for all $\alpha > 0$, it is well defined; it admits a derivative and

$$h^{-1}(\alpha) = -\frac{1}{\ln\alpha}. \tag{52}$$

We find

$$f_h = h^{-1}\circ \mathrm{f}\circ h = \frac{\alpha}{1+\alpha\ln(1+\xi\,\mathrm{e}^{-1/\alpha})} \tag{53}$$

and

$$\beta_h = \alpha^2\,\mathrm{e}^{-1/\alpha}. \tag{54}$$

Eqs. (53) and (54) clearly violate (13) and (14). In quantum field theory one would call β_h a "perturbatively invisible" function.

Finally, we assume the validity of Eqs. (13) and (14) and explain how we systematically compute $f(\alpha)\circ\circ\,\xi$.

The starting function $f(\alpha)$ may be transcendental; in that case we first try to expand it around $\alpha = 0$. In all cases we verify that it can be written in the form (13). Then, we solve the system of equations (15) to find the coefficients $\beta^{(i)}$ in the expansion (14). Finally, knowing $\beta(\alpha)$, we calculate the ξ iteration of $f(\alpha)$ directly from Eq. (22).

The first step requires (eventually) performing a Taylor expansion; the second step requires solving a system of an infinite number of linear equations up to some order in α; the third step requires the repeated application of the derivative operator $\mathrm{d}/\mathrm{d}\alpha$. All these calculations are elementary but long and tedious. We have used the computer algebra system REDUCE 3.2 to do them.

The second step is the most delicate. If we want to maintain efficiency we have to take into account that, when some of the coefficients f_i are equal to zero, the system contains trivial equations. Therefore, we have constructed a procedure which adapts the way of solving the system to the form of the input. It is given in Fig. 1.

The third step can also be optimised in order not to remake derivatives of a given order several times. For illustration we give in Fig. 2 the general expression of the coefficient of ξ^9 in Eq. (22). We insist that this third step relies heavily on Eq. (22).

EXAMPLES OF PERTURBATIVE SOLUTIONS

$$f(\alpha) = \ln(1+\alpha) \tag{55}$$

gives

$$\beta(\alpha) = \tfrac{1}{2}(\alpha^2 - \tfrac{1}{6}\alpha^3 + \tfrac{1}{24}\alpha^4 + \tfrac{1}{90}\alpha^5 + \tfrac{11}{4320}\alpha^6 + \mathcal{O}(\alpha^7)), \tag{56}$$

and

$$\ln(1+\alpha)\circ\circ\,\xi = \alpha - \tfrac{1}{2}\xi\alpha^2 + \tfrac{1}{4}\xi(\xi+\tfrac{1}{3})\alpha^3 - \tfrac{1}{8}\xi(\xi^2+\tfrac{5}{6}\xi+\tfrac{1}{6})\alpha^4 + \mathcal{O}(\alpha^5). \tag{57}$$

If we substitute $\xi = -1$ in Eq. (57) we obtain the expansion of

$$\mathrm{e}^{\alpha} - 1, \tag{58}$$

which is indeed $f^{-1}(\alpha)$. We can also see that the coefficient of $-\xi$ is precisely $\beta(\alpha)$ as given by Eq. (56).

An interesting case is the one of a *polynomial.* Let

$$f(\alpha) = \alpha + B\alpha^2 + C\alpha^3 \equiv P(\alpha). \tag{59}$$

Since Eq. (59) is *exact*, this is a case where the computation of $\beta(\alpha)$ must be made up to any order irrespective of the degree of the polynomial. The procedure of Fig. 2 is applied with

```
PROCEDURE BETA(F,X,ORD)$
BEGIN INTEGER F1,FDEG,N,NN SCALAR BETTA,NF,FF,DERF,DERFBET,R,BR,PA,FT$
    DELPROP !ε!εBET$ OPERATOR !ε!εBET$
    FF:=F$
    NF:=NUM FF$
    FDEG:=LOWDEG(NF,X)$
    IF FDEG NEQ 1 THEN LISP
    REDERR(LIST("INPUT FUNCTION MUST BE ",X,"+F1*",X,"**2+ . ."))$
    N:=IF NOT ORD=0 THEN ORD ELSE DEG(NF,X)$
    FF:=FF-X$
    FDEG:=LOWDEG(NUM FF,X)$NN:=N+FDEG$
    X**NN:=0$
    FT:=-POW(FF,X,FDEG)$
    !εBETTA:=FT*X**FDEG+FOR I:=(FDEG+1):N SUM !ε!εBET(I)*X**I$
    DERF:=1+DF(FF,X)$ DERFBET:=DERF*!εBETTA$
    BETTA:=SUB(X=X+FF,!εBETTA)$
    BETTA:=BETTA-DERFBET $  FDEG:=LOWDEG(NUM BETTA,X)$
    PA:=FDEG/2-1$FDEG:=FDEG-1$
L:  FDEG:=FDEG+1$
    R:=POW(BETTA,X,FDEG)$F1:=FDEG-PA$
    BR:=POW(R,!ε!εBET(F1),1)$
    IF BR NEQ 0 THEN !ε!εBET(F1):=!ε!εBET(F1)-R/BR$
    IF NN-FDEG>=0 THEN GO TO L$
    CLEAR X**NN$
    WRITE "PREVIOUS TRUNCATION ON ",X," SHOULD BE REDEFINED"$
    RETURN !εBETTA END$
END$
```

Fig. 1. The above REDUCE PROCEDURE solves the system (15) for the $\beta^{(i)}$ when $\beta(\alpha)$ satisfies Eq. (14). The calculated $\beta(\alpha)$ is contained in the variable !εBETTA. The first argument of the procedure is $f(\alpha)$, the second one is α and the third one if the maximum order up to which the resolution will be done. When ORD = 0, then, this order is automatically determined from the highest degree of the input $f(\alpha)$. This procedure depends on three non-standard functions: DELPROP, LOWDEG and POW. The first one is a variant of the CLEAR function, the second one extracts the *lowest* degree of a polynomial, the third one extracts the coefficient of a given power within a polynomial.

ORD $\neq 0$. We obtain

$$P(\alpha)\circ\circ\xi = \alpha\alpha + \xi B\alpha^2 + \xi(\xi B^2 - B^2 + C)\alpha^3 + B\xi(B^2\xi^2 - \tfrac{5}{2}B^2\xi + \tfrac{5}{2}C\xi + \tfrac{3}{2}B^2 - \tfrac{5}{2}C)\alpha^4 + \mathcal{O}(\alpha^5). \tag{60}$$

The polynomial which multiplies α^4 can be written

$$(\xi - 1)(B^2\xi - \tfrac{3}{2}B^2 + \tfrac{5}{2}C), \tag{61}$$

as it should since $P(\alpha)\circ\circ 1 \equiv P(\alpha)$.

In general, if we write

$$P(\alpha)\circ\circ\xi = \alpha + \sum_{n=2}^{\infty} \xi P_n(\xi)\alpha^n, \tag{62}$$

we know *a priori* that the polynomial $P_n(\xi)$ can be written as

$$P_n(\xi) = (\xi-1)(\xi-2)\dots(\xi-k)p_{n-k}(\xi), \tag{63}$$

when $n \geqslant 3+k$. When $B=0$, i.e. when

$$P(-\alpha) = -P(\alpha), \tag{64}$$

```
AR(7) := -BETA**8*DF(BETA,A,8)+BETA**7*(-29*DF(BETA,A,7)*
DF(BETA,A)
-64*DF(BETA,A,6)*DF(BETA,A,2)-98*DF(BETA,A,5)*DF(BETA,A,3)-56*DF
(BETA,A,4)**2)+3*BETA**6*(-96*DF(BETA,A,6)*DF(BETA,A)**2-346*DF(
BETA,A,5)*DF(BETA,A,2)*DF(BETA,A)-448*DF(BETA,A,4))*DF(BETA,A,3)*DF(
BETA,A)-256*DF(BETA,A,4)*DF(BETA,A,2)**2-285*DF(BETA,A,3)**2*DF(
BETA,A,2))+BETA**5*(-1206*DF(BETA,A,5)*DF(BETA,A)**3-5142*DF(
BETA,A,4)*DF(BETA,A,2)*DF(BETA,A)**2-2829*DF(BETA,A,3)**2*DF(BETA,A)
**2-5946*DF(BETA,A,3)*DF(BETA,A,2)**2*DF(BETA,A)-496*DF(BETA,A,2)
**4)+BETA**4*DF(BETA,A)**2*(-2127*DF(BETA,A,4)*DF(BETA,A)**2-
9204*DF(BETA,A,3)*DF(BETA,A,2)*DF(BETA,A)-4288*DF(BETA,A,2)**3)+3*
BETA**3*DF(BETA,A)**4*(-463*DF(BETA,A,3)*DF(BETA,A)-968*
DF(BETA,A,2)
**2)-247*BETA**2*DF(BETA,A,2)*DF(BETA,A)**6-BETA*DF(BETA,A)**8$
```

Fig. 2. The formal expression of the ξ^9 coefficient in Eq. (22) called here AR(7). In this expression, DF is the derivative operator, A is α and BETA is $\beta(\alpha)$ given in many cases as a (formal) power series expansion.

we get a very simple form for $P^{-1}(\alpha)$, i.e.

$$P^{-1}(\alpha) = \alpha - C\alpha^3 + 3C^2\alpha^5 - 12C^3\alpha^7 + 55C^4\alpha^9 + \mathcal{O}(\alpha^{11}), \tag{65}$$

and

$$\beta(-\alpha) = -\beta(\alpha). \tag{66}$$

This is a general property we can deduce from lemma 2.

Indeed, when $f(\alpha)$ is odd, if we take

$$h(\alpha) = -\alpha, \tag{67}$$

we find from Eq. (39)

$$\beta(-\alpha f(-\alpha)) = \frac{\mathrm{d}(-\alpha f(-\alpha))}{\mathrm{d}\alpha} \frac{\beta(-\alpha)}{-1}, \tag{68}$$

i.e.

$$\beta(f(\alpha)) = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} \frac{\beta(-\alpha)}{-1}, \tag{69}$$

$$= \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} \beta(\alpha); \tag{70}$$

$$f(\alpha) = \sin \alpha \tag{71}$$

corresponds to

$$\beta(\alpha) = \tfrac{1}{6}(\alpha^3 + \tfrac{1}{3}\alpha^5 + \tfrac{41}{630}\alpha^7 + \tfrac{8}{315}\alpha^9 + \mathcal{O}(\alpha^{11})), \tag{72}$$

which is indeed odd in α.

We can go further with lemma 2 and deduce without calculation the $\beta(\alpha)$ function which corresponds to

$$f(\alpha) = \mathrm{sh}\, \alpha. \tag{73}$$

Indeed, with

$$h(\alpha) = i\alpha, \tag{74}$$

Eqn (38) gives

$$f_h = -i \sin(i\alpha) = \mathrm{sh}\, \alpha \tag{75}$$

and Eq. (39) gives

$$\beta_h(\mathrm{sh}\, \alpha) = \frac{\mathrm{d}\, \mathrm{sh}\, \alpha}{\mathrm{d}\alpha} \frac{\beta(i\alpha)}{i}. \tag{76}$$

Therefore

$$\beta_h(\alpha) = -i\beta(i\alpha). \tag{77}$$

All above calculations give rise to convergent power series expansions. The last example we take is not of that kind.

$$f(\alpha) = \sum_{n=1}^{\infty} n!\, \alpha^n, \tag{78}$$

has zero radius of convergence. The corresponding β and $f(\alpha) \circ\circ \xi$ are given by

$$\beta(\alpha) = -2(\alpha^2 + \alpha^3 + 3\alpha^4 + \tfrac{41}{3}\alpha^5 + \ldots), \tag{79}$$

$$f(\alpha) \circ\circ \xi = \alpha + 2\xi\alpha^2 + 2\xi(\xi+1)\alpha^3 + 2\xi(4\xi^2 + 5\xi + 3)\alpha^4 + \mathcal{O}(\alpha^5). \tag{80}$$

4. Concluding Remarks

The main aim of our work has been to propose a very simple algorithm to compute formally the set of solutions of Eq. (1) when the conditions (13) and (14) are verified. This algorithm, in its last stage, makes use of the fact that the "generating" function $\beta(\alpha)$ is the generator of the abelian group formed by the solutions.

We have also shown that, from a solution of (1), we can generate (thanks to lemma 2) almost trivially a whole family of solutions which do not necessarily obey Eqs. (13) and (14).

We have not analysed the mathematical properties of the iterations we obtain. When the solution is exactly known as the one given by Eq. (49), it is of course possible to analyse the problem. The mentioned example shows that ξ can indeed be made complex and has a strong bearing on the extent of the convergence radius of the power serie expansion. This is the fact which makes them interesting to use in perturbative quantum field theory (Hans, 1985). There, every physical quantity is written as an expansion with respect to a coupling constant. The choice of this coupling is *not* uniquely defined but the rate of convergence depends on this choice. Suppose that we start from a function $F(\alpha)$ (F is a physical quantity, α a given coupling constant) which admits a Taylor expansion for $|\alpha| < \hat{\alpha}$, where $\hat{\alpha}$ is the position of the nearest singularity of F. We *choose* a function $\beta(\alpha)$ which obeys Eq. (14) and compute $f(\alpha, \xi)$ using lemma 1. We write

$$\alpha(\xi) \equiv f(\alpha, \xi) = f(\alpha) \circ\circ \xi. \tag{81}$$

Therefore

$$\alpha = \alpha(\xi) \circ\circ (-\xi). \tag{82}$$

ξ is a *redundant* parameter in Eq. (82). We next write

$$F(\alpha) = F(\alpha(\xi) \circ\circ (-\xi)) = \bar{F}(\alpha(\xi), \xi). \tag{83}$$

We can write the Taylor expansion of F in terms of $\alpha(\xi)$ (the *new* coupling constant). It is apparent that its convergence radius is different. Indeed, $\hat{\alpha}$ becomes

$$\hat{\alpha}(\xi) = f(\hat{\alpha}) \circ\circ \xi \tag{84}$$

and $\hat{\alpha}(\xi)$ is the location of the nearest singularity of $\bar{F}$. Since the parameter ξ is redundant we can choose it to maximise the rate of convergence of the expansion of $\bar{F}$ in $\alpha(\xi)$.

As we already remarked, solutions of Eq. (1) can exist even if Eqs. (13) and (14) are not satisfied. We can easily generalise our algorithm. Suppose that

$$f(\alpha) = f_0\alpha + f_1\alpha^2 + \ldots. \tag{85}$$

If $f_0^{(n)}$ for a given integer n is equal to 1, then we apply our algorithm for the function

$$g(\alpha) = f_0 \circ \circ n. \tag{86}$$

If such an n does not exist, we solve the equation (Schröder, 1871)

$$S(f(\alpha)) = f_0 S(\alpha), \tag{87}$$

and write

$$\beta(\alpha) = \frac{S(\alpha)}{S'(\alpha)}. \tag{88}$$

From Eq. (88) we see that

$$\beta(\alpha) = \alpha + \beta^{(2)}\alpha^2 + \dots; \tag{89}$$

$f(\alpha)$ and $\beta(\alpha)$, as given by Eqs. (85) and (89), verify Eq. (1). When S^{-1} exists, it is possible to write

$$f(\alpha, \xi) = S^{-1}(f_0^{\xi} S(\alpha)). \tag{90}$$

($f(\alpha, 0) = \alpha$) and *lemma* 1 still applies. It can be used to compute the ξ iteration as before. Equations (32) and (33) are the most trivial illustration of this situation.

The system (15) is evidently no longer valid but can be replaced by a new one, obtained using Eq. (1) and Eqs. (85) and (89). The algorithm of Fig. 1 is to be changed accordingly.

A last interesting case is when

$$f(\alpha) = C + f_0\alpha + f_1\alpha^2 + \dots. \tag{91}$$

If there is a fixed point $\alpha = \alpha_0$, i.e. if

$$f(\alpha_0) = \alpha_0, \tag{92}$$

it is still possible to define the iteration (Knuth, 1961). One writes

$$f(\alpha + \alpha_0) = \alpha_0 + \alpha f'(\alpha)|_{\alpha=\alpha_0} + \frac{\alpha^2}{2!} f''(\alpha)|_{\alpha=\alpha_0} + \dots. \tag{93}$$

For

$$g(\alpha) = f(\alpha + \alpha_0) - \alpha_0, \tag{94}$$

we can write $g(\alpha) \circ \circ \xi$ as in the previous case with

$$f_0 \equiv f'(\alpha)|_{\alpha=\alpha_0}. \tag{95}$$

Then, from Eq. (93), one gets

$$f(\alpha) \circ \circ \xi = [g(\alpha - \alpha_0)] \circ \circ \xi + \alpha_0. \tag{96}$$

It is, of course, obvious that lemma 2 applies in all cases.

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