A Note on AFLs and Bounded Erasing*

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In this note, properties of the "f-bounded erasing" operator $H_f$ are studied. The operator $H_f$ maps a family $\mathcal{L}$ of languages onto another by means of homomorphisms in which the amount of erasing is bounded by the function $f$. It is shown that under appropriate conditions, this operator composes in a contravariant manner. As a consequence of this result, it is shown that particular families of languages defined by tape-bounded Turing acceptors are not closed under certain classes of bounded erasing.

INTRODUCTION

The studies of AFLs (abstract families of languages) and computational complexity have each received considerable attention during the last few years. The former considers families generated by certain closure operations, while the latter, when considering languages, treats families defined by various measures of "work" required to recognize their elements. Recently, there has been some examination of the relationship between these two apparently unconnected formalisms for defining language families (Book and Greibach, 1970; Book, Greibach, and Wegbreit, 1970; Ginsburg and Hopcroft, 1969). In this note, we continue to explore this relationship by investigating notions defined in (Book, Greibach, and Wegbreit, 1970) and (Ginsburg and Hopcroft, 1969).

In (Book, Greibach, and Wegbreit, 1970), it was shown that the family of languages accepted by nondeterministic Turing acceptors operating within a time bound specified by a function $f$ can be characterized as the image of the quasi-realtime languages (Book and Greibach, 1970) under homomorphisms $H_f$.

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for which the amount of erasing is bounded by \( f \). Similar results were established for the families of languages accepted by nondeterministic (deterministic) Turing acceptors operating within a tape bound and the family of context-sensitive (respectively, deterministic linear bounded automata) languages (Book, Greibach, and Wegbreit, 1970). Hence, in certain cases, the notion of “amount of resource” in complexity theory has an analog in the amount of erasing applied to some AFL.

The purpose of this note is to study properties of the “\( f \)-bounded erasing” operator which maps one family of languages onto another. We prove that under appropriate conditions, this operator composes in a contravariant manner. We also study the relation between this operator and one which allows a family defined by “bounded erasing” to be an AFL. The results are then applied to show that certain families of languages defined by tape-bounded Turing acceptors, both nondeterministic and deterministic, are not closed under certain classes of bounded erasing. In particular, the family of languages defined by deterministic linear-bounded automata is not closed under erasing which is “greater than linear” and the family of context-sensitive languages is not closed under erasing which is bounded below by \( n^{1+\epsilon} \).

In Section 1, definitions and basic results of (Book, Greibach, and Wegbreit, 1970) are given. The main results are developed in Section 2, and Section 3 is devoted to applications.

1. Preliminaries

In this section, we define the families of languages and operators studied in this paper. The definitions are taken from (Book, Greibach, and Wegbreit, 1970). We also state without proof some of the relationships established in (Book, Greibach, and Wegbreit, 1970) and (Ginsburg and Hopcroft, 1969) between these families and operators.

The functions used as bounds in this paper are integer-valued functions of a single real variable with the property that for sufficiently large \( x \), \( f(x) \geq x \).

We assume the reader is familiar with the elementary properties of multitape Turing acceptors and formal languages as described in (Hopcroft and Ullman, 1969).
Define $\text{TAPe}(f) = \{L(M) \mid M \text{ is a nondeterministic multitape Turing acceptor which operates within tape bound } f\}$ and $\det\text{TAPe}(f) = \{L(M) \mid M \text{ is a deterministic multitape Turing acceptor which operates within tape bound } f\}$.

**Definition 1.2.** A multitape Turing acceptor $M$ operates within time bound $f$ if for each input string $w$ accepted by $M$, every accepting computation of $M$ on $w$ has no more than $\max(|w|, f(|w|))$ steps. Define $\text{TIME}(f) = \{L(M) \mid M \text{ is a nondeterministic multitape Turing acceptor which operates within time bound } f\}$.

For any function $f$ and any constant $k > 0$, $\text{TAPe}(kf) = \text{TAPe}(f)$ and $\det\text{TAPe}(kf) = \det\text{TAPe}(f)$. The context-sensitive languages form the family $\text{CS} = \text{TAPe}(i)$ and the family of languages accepted by deterministic linear bounded automata is the family $\det\text{LBA} = \det\text{TAPe}(i)$, where $i$ is the identity function, $i(x) = x$. For any function $f$ and any constant $k > 0$, $\text{TIME}(kf) = \text{TIME}(f)$. The languages accepted in quasi-realtime by nondeterministic multitape Turing acceptors form the family $Q = \text{TIME}(i)$ (Book and Greibach, 1970).

**Definition 1.3.** If $h : \Gamma^* \rightarrow \Delta^*$ is a homomorphism, $L \subseteq \Gamma^*$, and $f$ is a function such that for some $k > 0$ and all $w \in L$, $|w| \leq kf(|h(w)|)$, then $h$ is $f$-bounded on $L$. For any family $\mathcal{L}$ of languages and any function $f$, define $H_f[\mathcal{L}] = \{h(L) \mid L \in \mathcal{L} \text{ and } h \text{ is a homomorphism which is } f\text{-bounded on } L\}$.

Theorems 1.7 and 1.9 of (Book, Greibach, and Wegbreit, 1970) yield certain relationships between some of the families defined above. These are summarized in the following proposition.

**Proposition 1.4.** For any function $f$,

(i) $\text{TAPe}(f) = H_f[\text{CS}]$;

(ii) $\det\text{TAPe}(f) = H_f[\text{detLBA}]$;

(iii) $\text{TIME}(f) = H_f[Q]$.

**Definition 1.5.** (Ginsburg and Greibach, 1969). An abstract family of languages (AFL) is a family of languages containing at least one nonempty set and closed under nonerasing homomorphism, inverse homomorphism, intersection with regular sets, union, concatenation, and Kleene closure.

In (Book, Greibach, and Wegbreit, 1970) the families $\text{TAPe}(f)$, etc., were

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1 For a string $w$, $|w|$ is the length of $w$. 

investigated in order to find sufficient conditions for \( f \) in order that the families in question be AFLs and be principal AFLs. It was shown that two particular properties of the functions play a key role.

**Definition 1.6.** A function \( f \) is superadditive if for every \( x, y \geq 0 \),
\[
f(x) + f(y) \leq f(x + y).
\]
A function \( f \) is semihomogeneous if for every \( k_1 > 0 \) there is a \( k_2 > 0 \) such that for all \( x \geq 0 \), \( f(k_1 x) \leq k_2 f(x) \).

**Notation 1.7.** For any function \( f \) and any integer \( k > 0 \), \( f_k \) is the function given by \( f_k(x) = f(kx) \).

**Definition 1.8.** For any function \( f \) and any family \( \mathcal{L} \) of languages define
\[
\mathcal{S}(H_f[\mathcal{L}]) = \bigcup_k H_{f_k}[\mathcal{L}].
\]
For any function \( f \), define
\[
\mathcal{S}(\text{TAPE}(f)) = \bigcup_k \text{TAPE}(f_k),
\]
\[
\mathcal{S}(\text{detTAPE}(f)) = \bigcup_k \text{detTAPE}(f_k),
\]
and
\[
\mathcal{S}(\text{TIME}(f)) = \bigcup_k \text{TIME}(f_k).
\]

**Corollary 1.9** (of Proposition 1.4). For any function \( f \),
\[
\mathcal{S}(\text{TAPE}(f)) = \mathcal{S}(H_f[\text{CS}]), \quad \mathcal{S}(\text{detTAPE}(f)) = \mathcal{S}(H_f[\text{detLBA}]),
\]
and
\[
\mathcal{S}(\text{TIME}(f)) = \mathcal{S}(H_f[\text{Q}]).
\]

We can now state the results on AFLs as shown by Corollary 2.2, Corollary 2.5, and Theorem 2.7 of (Book, Greibach, and Wegbreit, 1970).

**Proposition 1.10.** For any superadditive \( f \) and any family \( \mathcal{L} \) of languages,
(i) the smallest AFL containing \( H_f[\mathcal{L}] \) is the smallest AFL containing \( \mathcal{S}(H_f[\mathcal{L}]) \),
(ii) if \( f \) is semihomogeneous, then \( \mathcal{S}(H_f[\mathcal{L}]) = H_f[\mathcal{L}] \), and (iii) \( \mathcal{S}(H_f[\mathcal{L}]) \) is an AFL if \( \mathcal{L} \) is an AFL.

2. The Main Results

In this section we study the composition of the operators \( H_f \) and \( \mathcal{S} \). In particular, we give sufficient conditions on the functions \( f \) and \( g \) and on the
AFL $\mathcal{L}$ such that $\mathcal{S}(H_1[\mathcal{S}(H_1[\mathcal{L}])]) = \mathcal{S}(H_{f_0}[\mathcal{L}]).$ Before doing this, it is desirable to point out certain properties of superadditive and semihomogeneous functions.

**Remark 2.1.** (i) The following hold for superadditive functions $f, g$: (a) $f$ is nondecreasing; (b) for every integer $k > 0$ and every $x > 0$, $kf(x) \leq f(kx) = f_k(x)$; (c) for every integer $k > 0$, $f_k$ is superadditive; (d) $f \circ g$ is superadditive. (ii) The following hold for semihomogeneous functions $f, g$: (a) for every integer $k > 0$, $f_k$ is semihomogeneous; (b) $f \circ g$ is semihomogeneous.

In this section and the next we shall consider only those functions $f$ such that for all $x \geq 0, f(x) \geq x$. This is done in order to simplify the arguments.

**Lemma 2.2.** For any functions $f, g$ and any family $\mathcal{L}$ of languages, the following hold:

(i) $H_g[\mathcal{S}(H_f[\mathcal{L}])] \subseteq \mathcal{S}(H_g[\mathcal{S}(H_f[\mathcal{L}])]) \subseteq \mathcal{S}(H_g[\mathcal{S}(H_f[\mathcal{L}])])$;

(ii) $H_g[\mathcal{S}(H_f[\mathcal{L}])] \subseteq \mathcal{H}_g[\mathcal{S}(H_f[\mathcal{L}])] \subseteq \mathcal{S}(H_g[\mathcal{S}(H_f[\mathcal{L}])])$.

**Proof.** These are immediate when we notice that $H_f, H_g,$ and $\mathcal{S}$ are each monotone increasing with respect to inclusion of families.

**Lemma 2.3.** For any nondecreasing function $f$, any superadditive function $g$, and any family $\mathcal{L}$ of languages, $\mathcal{S}(H_g[\mathcal{S}(H_f[\mathcal{L}])]) \subseteq \mathcal{S}(H_{f_0}[\mathcal{L}])$.

**Proof.** If $L \in \mathcal{S}(H_g[\mathcal{S}(H_f[\mathcal{L}])])$, then there exist homomorphisms $h_1$ and $h_2$, a language $L_0 \in \mathcal{L}$, and positive integers $t_1, t_2$ such that $h_1$ is $f_{t_1}$-bounded on $L_0$, $h_2$ is $g_{t_2}$-bounded on $h_1(L_0)$, and $L = h_2(h_1(L_0))$. Thus there exist positive integers $k_1, k_2$ such that for all $w \in L_0$,

\[|w| \leq k_1 f_{t_1}(|h_1(w)|),\]  
\[|h_1(w)| \leq k_2 g_{t_2}(|h_2(h_1(w))|).\]  

Let $h_3$ be the homomorphism determined by composing $h_3$ with $h_1$; so for all $w \in L_0$, $h_3(w) = h_2(h_1(w))$. Since $f$ is nondecreasing, (2) implies for all $w \in L_0$,

\[f_{t_1}(|h_1(w)|) = f(t_1 |h_1(w)|) \leq f(t_1 k_2 g_{t_2}(|h_2(h_1(w))|)).\]

$^2$ For functions $f$ and $g$, $f \circ g$ is the function defined for all $x$ by $(f \circ g)(x) = f(g(x))$. 


Since $g$ is superadditive, for all $w \in L_0$,
\[ t_1k_2g_{t_2}\left(\left| h_3(w)\right|\right) \leq g(t_2t_1k_2 \left| h_3(w)\right|) = g_{t_2t_1k_2}\left(\left| h_3(w)\right|\right), \tag{4} \]
so that $f$ nondecreasing and (3) yield, for all $w \in L_0$,
\[ f_{t_1}\left(\left| h_1(w)\right|\right) \leq f(g_{t_2t_1k_2}\left(\left| h_3(w)\right|\right)) = (f \circ g)_{t_2t_1k_2}\left(\left| h_3(w)\right|\right). \tag{5} \]
Thus by (1) and (5), for all $w \in L_0$,
\[ \left| w \right| \leq k_1f_{t_1}\left(\left| h_1(w)\right|\right) \leq k_1(f \circ g)_{t_2t_1k_2}\left(\left| h_3(w)\right|\right), \]
so that $h_3$ is $(f \circ g)_{t_2t_1k_2}$-bounded on $L_0$. Hence
\[ L = h_3(h_1(L_0)) = h_3(L_0) \subseteq H(f \circ g, L). \]

**Corollary 2.4.** For any nondecreasing function $f$ and any family $\mathcal{L}$ of languages, if $g$ is a linear function, then $H(f \circ g, L) \subseteq H(f, L)$. Thus, $H(f, L)$ is closed under linear erasing.

**Proof.** By Lemmas 2.2 and 2.3, $H(f \circ g, L) \subseteq H(f, L)$. Let $t > 0$ be an integer such that for all $x$, $g(x) \leq tx$. Then for all $x$, $(f \circ g)(x) \leq f(tx) = f(x)$ so that $H(f \circ g, L) \subseteq H(f, L)$. 

**Lemma 2.5.** For any nondecreasing semihomogeneous function $f$, any function $g$, and any family $\mathcal{L}$ of languages, $H_f[H_f, L] \subseteq H_{f \circ g}(L)$.

**Proof.** The argument parallels the proof of Lemma 2.3 as follows: Since we deal with $H_f[H_f, L]$ as opposed to $H(f \circ g, L)$, let $t_1 = t_2 = 1$. Then (1) becomes
\[ \left| w \right| \leq k_1f(\left| h_1(w)\right|), \tag{1'} \]
and (3) becomes
\[ f(\left| h_1(w)\right|) \leq f(k_2g(\left| h_3(w)\right|)). \tag{3'} \]
Since $f$ is semihomogeneous, there is a $k_a > 0$ such that for all $x$, $f(k_a x) \leq k_a f(x)$. Hence by (1') and (3'), we have for all $w \in L_0$,
\[ \left| w \right| \leq k_1f(\left| h_1(w)\right|) \leq k_1f(k_2g(\left| h_3(w)\right|)) \leq k_1k_3f(g(\left| h_3(w)\right|)). \]
Hence $h_3$ is $f \circ g$-bounded on $L_0$, so that $L = h_3(L_0) \subseteq H(f \circ g, L)$. 

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We wish to show that $H_{fog}[\mathcal{L}] \subseteq H_g[H_f[\mathcal{L}]]$, so that $\mathcal{P}(H_{f}[\mathcal{P}(H_f[\mathcal{L}])]) = \mathcal{P}(H_{fog}[\mathcal{L}])$. However, we have been unable to do this for arbitrary $f$, $g$, and $\mathcal{L}$, but if attention is restricted to AFLs, $\mathcal{L}$ and functions $f$ and $g$ which are "countable with respect to $\mathcal{L}$," then the desired result is obtained.

**Definition 2.6.** For $w \in \Gamma^*$, $a \in \Gamma$, let $\#_a(w)$ be the number of occurrences of $a$ in $w$. For any function $f$, a set $\{f, \Gamma, \Omega, d\} = \{w \in \Gamma^* | \exists Z \in \Omega, \sum_{a \in \Gamma} \#_a(w) = f(Z)\}$, where $d \in \Omega \subseteq \Gamma$, is said to be an $f$-counting set. If $f$ is a function and $\mathcal{L}$ is a family of languages, $f$ is $\mathcal{L}$-countable if for any integer $k > 0$, any $kf$ counting set $\{kf, \Gamma, \Omega, d\}$, and any $L \in \mathcal{L}$, $L \cap \{kf, \Gamma, \Omega, d\} \in \mathcal{L}$.

It is easy to see that if $f$ is tape constructable in the sense of (Stearns, Hartmanis, and Lewis, 1965), then $f$ is detLBA-countable and also CS-countable. Similarly, if $f$ is the appropriate generalization of the real-time countable functions of (Yamada, 1962), then $f$ is $Q$-countable.

**Lemma 2.7.** For any AFL $\mathcal{L}$ and any functions $f$, $g$ which are both $\mathcal{L}$-countable, $H_{fog}[\mathcal{L}] \subseteq H_g[H_f[\mathcal{L}]]$.

**Proof.** If $L \in H_{fog}[\mathcal{L}]$, then there is a language $L_0 \in \mathcal{L}$ and a homomorphism $h_1 : \Gamma^* \rightarrow \Delta^*$ such that $L_0 \subseteq \Gamma^*$, $L_1(L_0) = L$, and $h_1$ is $f \circ g$-bounded on $L_0$. Let $c$ and $d$ be new symbols not in $\Delta$, let $\Delta_c = \Delta \cup \{c\}$, and $\Delta_{c,d} = \Delta_c \cup \{d\}$. Let $\mu_c : \Delta_{c,d}^* \rightarrow \Delta_c^*$ be the homomorphism determined by defining $\mu_c(c) = e$ and $\mu_c(a) = a$ for $a \in \Delta_c$. Let $\mu_d : \Delta_{d,c}^* \rightarrow \Delta_d^*$ be the homomorphism determined by defining $\mu_d(d) = e$ and $\mu_d(a) = a$ for $a \in \Delta_d$.

Let $h_2 : \Gamma^* \rightarrow \Delta_d^*$ be the homomorphism determined by defining $h_2(a) = h_1(a)$ if $h_1(a) \neq e$ and $h_2(a) = d$ if $h_1(a) = e$. Let $L_2 = h_2(L_0)$. Since $h_2$ is nonerasing and $\mathcal{L}$ is an AFL, $L_0 \in \mathcal{L}$ implies $L_2 \in \mathcal{L}$. Now $L = h_1(L_2) = \mu_d(h_2(L_0)) = \mu_d(L_1)$ and $h_2$ is nonerasing, so that $h_1 \circ g$-bounded on $L_0$ implies that $\mu_d$ is $f \circ g$-bounded on $L_1$. Thus it suffices to show that $\mu_d(L_1)$ is in $H_g[H_f[\mathcal{L}]]$.

To show that $L = \mu_d(L_1)$ is in $H_g[H_f[\mathcal{L}]]$, we proceed by constructing $L_2$ from $L_1$ by substituting some $c$'s for $d$'s in words of $L_1$. By intersecting with the appropriate $f$- and $g$-counting sets, we obtain a language $L_3$ such that $\mu_c(\mu_d(L_3)) = L$.

Let $\tau$ be the substitution on $\Delta_d$ determined by defining $\tau(a) = \{a\}$ for $a \in \Delta$.

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3 For all $x$, $(kf)(x) = kf(x)$. 
and \( \tau(d) = \{c^n \mid n \geq 1\} \cup \{d^n \mid n \geq 1\} \). Then \( \tau \) is an \( e \)-free regular substitution so that \( L_2 = \tau(L_1) \) is in \( \mathcal{L} \) since \( \mathcal{L} \) is an AFL (Ginsburg, and Greibach, 1969). Also \( \mu_b(\mu_d(L_2)) = \mu_d(L_3) = L \).

Since \( \mu_d \) is \( f \circ g \)-bounded on \( L_1 \), there is an integer \( k > 0 \) such that for all \( w \in L_1, |w| \leq kf(g(|\mu_d(w)|)) \). Since \( f \) is \( \mathcal{L} \)-countable, \( L_2 \cap [kf, \Delta_{d}, \{d\}, d] \) is in \( \mathcal{L} \), where \([kf, \Delta_{d}, \{d\}, d]\) is an \( f \)-counting set. Since \( g \) is \( \mathcal{L} \)-countable, 

\[
L_3 = (L_2 \cap [kf, \Delta_{d}, \{d\}, d]) \cap [g, \Delta_{c}, \{c, d\}, c]
\]

is in \( \mathcal{L} \), where \([g, \Delta_{c}, \{c, d\}, c]\) is a \( g \)-counting set. Notice that for \( w \in \Delta_{c,d}^*, \sum_{a \in \Delta_{c,d}} \#_a(w) = |w|, \sum_{a \in \Delta_{c}} \#_a(w) = |\mu_d(w)|, \) and \( \sum_{a \in \Delta} \#_a(w) = |\mu_e(\mu_d(w))| \). Thus \( L_3 = \{w \in L_2 \mid |w| = kf(|\mu_d(w)|), \) and \( |\mu_d(w)| = g(|\mu_e(\mu_d(w))|) \), so that \( \mu_d \) is \( f \)-bounded on \( L_0 \) and \( \mu_e \) is \( g \)-bounded on \( \mu_d(L_0) \), and hence \( \mu_e(\mu_d(L_0)) \in H_f[H_f[\mathcal{L}]] \).

It remains to show that \( \mu_e(\mu_d(L_0)) = L \). Since \( L_3 \subseteq L_2, \mu_e(\mu_d(L_3)) \subseteq \mu_e(\mu_d(L_2)) \). Now for each \( w \in L_1, |w| \leq kf(g(|\mu_d(w)|)) \). Thus there is some \( w_1 \in \tau(w) \subseteq \tau(L_1) = L_2 \) such that \( |w_1| = kf(g(|\mu_e(\mu_d(w_1))|)) \) and such that \( |\mu_d(w_1)| = \sum_{a \in \Delta_{d}} \#_a(w_1) = g(\sum_{a \in \Delta_{c}} \#_a(w_1)) = g(|\mu_e(\mu_d(w_1))|) \). This means \( w_1 \in L_3 \). But \( w_1 \in \tau(w) \) implies \( \mu_e(\mu_d(w_1)) = \mu_d(w) \). Hence \( L = \mu_e(\mu_d(L_0)) \).

We can now establish the desired equality.

**Theorem 2.8.** Let \( \mathcal{L} \) be an AFL and let \( f, g \) be \( \mathcal{L} \)-countable functions. If \( f \) is nondecreasing and \( g \) is superadditive, then

\[
\mathcal{S}(H_{f \circ g}[\mathcal{L}]) = \mathcal{S}(H_f[H_{g}[\mathcal{L}]])) = \mathcal{S}(H_g[H_f[\mathcal{L}]]).
\]

**Proof.** By Lemma 2.7, for any integer \( k > 0, \)

\[
H_{(f \circ g)}[\mathcal{L}] = H_{f \circ g}[\mathcal{L}] \subseteq H_{g}[H_f[\mathcal{L}]],
\]

so that \( \mathcal{S}(H_{f \circ g}[\mathcal{L}]) = \bigcup_k H_{(f \circ g)}[\mathcal{L}] \subseteq \bigcup_k H_g[H_f[\mathcal{L}]] = \mathcal{S}(H_g[H_f[\mathcal{L}]])). \) By Lemma 2.2, \( \mathcal{S}(H_g[H_f[\mathcal{L}]])) \subseteq \mathcal{S}(H_g[H_f[\mathcal{L}]])) \), and by Lemma 2.3, \( \mathcal{S}(H_g[H_f[\mathcal{L}]])) \subseteq \mathcal{S}(H_{f \circ g}[\mathcal{L}]). \)

**Corollary 2.9.** Let \( \mathcal{L} \) be an AFL and let \( f, g \) be \( \mathcal{L} \)-countable functions. If \( f \) is nondecreasing, and \( g \) is superadditive and semihomogeneous, then \( \mathcal{S}(H_{f \circ g}[\mathcal{L}]) = H_g[H_f[\mathcal{L}]]. \)

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4 A function \( \tau : \Gamma_1 \rightarrow \Gamma_2^* \) is a substitution on \( \Gamma_1 \); \( \tau \) is extended to \( \Gamma_1^* \rightarrow \Gamma_2^* \) by defining \( \tau(e) = \{e\} \) and \( \tau(a_1 \cdots a_n) = \tau(a_1) \cdots \tau(a_n) \) for \( n > 1, a_i \in \Gamma_1 \). If \( L \subseteq \Gamma_1^* \), \( \tau(L) = \bigcup_{a \in \Gamma_1} \tau(a) \). A substitution \( \tau \) is \( e \)-free if for every \( a \in \Gamma \), \( e \notin \tau(a) \), and is regular if for every \( a \in \Gamma \), \( \tau(a) \) is a regular set.
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Proof. By Proposition 1.10, \( \mathcal{P}(H_o[H_f[\mathcal{L}]]) = H_o[H_f[\mathcal{L}]] \), so the result follows from the theorem.

Notice that by Lemma 2.5 and 2.7, we have the following result.

**COROLLARY 2.10.** Let \( \mathcal{L} \) be an AFL and let \( f, g \) be \( \mathcal{L} \)-countable functions. If \( f \) is nondecreasing and semihomogeneous, then \( H_o[H_f[\mathcal{L}]] = H_f[\mathcal{L}] \).

The requirement in Lemma 2.7 that \( \mathcal{L} \) be an AFL is not necessary; it is sufficient that \( \mathcal{L} \) be closed under \( e \)-free regular substitution. However, the applications in Section 3 all involve AFL’s, hence the stronger result is not used here.

If 2.7–2.10 are studied by means of AFA (Ginsburg and Greibach, 1969), the requirement that \( f \) be \( \mathcal{L} \)-countable is not necessary. However we conjecture that \( g \) must be \( \mathcal{L} \)-countable for the results to hold.

3. APPLICATIONS TO FAMILIES DEFINED
BY TAPE-BOUNDED TURING ACCEPTORS

In this section we apply the results of Section 2 to the families \( \text{TAPE}(f) \), \( \text{detTAPE}(f) \), and \( \text{TIME}(f) \). In particular, we show that for certain \( f \) and \( g \), \( \text{TAPE}(f) \) and \( \text{detTAPE}(f) \) are not closed under \( g \)-bounded homomorphic mappings.

**DEFINITION 3.1.** A function \( f \) is **tape constructible** (deterministic-tape constructible) if there is a multitape Turing machine (deterministic multitape Turing machine) \( M \) such that for any input \( w \) to \( M \) any resulting computation of \( M \) on \( w \) visits precisely \( f(|w|) \) tape squares on at least one of its storage tapes and visits no more than \( f(|w|) \) tape squares on any one of its storage tapes. A function \( f \) is said to be **time constructible** if there is a multitape Turing machine \( M \) such that for any input \( w \) to \( M \), any resulting computation of \( M \) on \( w \) requires precisely \( f(|w|) \) steps.

It is clear that a function is tape constructible (deterministic-tape constructible, time constructible) if and only if it is CS-countable, (detLBA-countable, \( Q \)-countable). Thus the results of Section 2 and Corollary 1.9 yield the following proposition.

**PROPOSITION 3.2.** Let \( f \) and \( g \) be functions such that \( f \) is nondecreasing and \( g \) is superadditive. Then:

(i) if \( f \) and \( g \) are tape constructible, then

\[ \mathcal{P}(\text{TAPE}(f \circ g)) = \mathcal{P}(H_o[\mathcal{P}(\text{TAPE}(f))]) = \mathcal{P}(H_o[\text{TAPE}(f)]) \]
(ii) if \( f \) and \( g \) are deterministic-tape constructible, then
\[
\mathcal{S}(\text{detTAPE}(f \circ g)) = \mathcal{S}(H_g[\mathcal{S}(\text{detTAPE}(f))]) = \mathcal{S}(H_g[\text{detTAPE}(f)]);
\]

(iii) if \( f \) and \( g \) are time constructible, then
\[
\mathcal{S}(\text{TIME}(f \circ g)) = \mathcal{S}(H_g[\mathcal{S}(\text{TIME}(f))]) = \mathcal{S}(H_g[\text{TIME}(f)]).
\]

It should be noted that the analog of Proposition 3.2 applies to any AFA (Ginsburg and Greibach, 1969) when \( f \) and \( g \) are appropriately restricted.

We now apply the results of Section 2 to the families \( \text{TAPE}(f) \) and \( \text{detTAPE}(f) \).

**Notation 3.3.** For any function \( g \), \( g^{(1)} = g \) and \( g^{(i+1)} = g \circ g^{(i)} \). For any function \( g \) and any family \( \mathcal{L} \) of languages, \( H_g^{(1)}[\mathcal{L}] = H_g[\mathcal{L}] \) and \( H_g^{(i+1)}[\mathcal{L}] = H_g[H_g^{(i)}[\mathcal{L}]] \). For any function \( g \) and all \( x \), \( g^2(x) = (g(x))^2 \), and \( 2^g(x) = 2^{g(x)} \).

**Theorem 3.4.** For any deterministic-tape constructible functions \( f, g \) such that \( \lim_{x \to \infty} f(x)g(x) = 0 \), the family \( \text{detTAPE}(f) \) is not closed under \( g \)-bounded homomorphic mappings, i.e., \( H_g[\text{detTAPE}(f)] \nsubseteq \text{detTAPE}(f) \).

**Theorem 3.5.** For any deterministic-tape constructible functions \( f, g \) such that for some \( m > 0 \), \( \lim_{x \to \infty} f(x)^m/(f \circ g(m))(x) = 0 \), the family \( \text{TAPE}(f) \) is not closed under \( g \)-bounded homomorphic images, i.e., \( H_g[\text{TAPE}(f)] \nsubseteq \text{TAPE}(f) \).

To prove Theorems 3.4 and 3.5, we rely on two theorems which we now state in the notation of this paper.

**Proposition 3.6** (Stearns, Hartmanis, and Lewis, 1965). If \( f \) and \( g \) are deterministic tape constructible functions such that \( \lim_{x \to \infty} f(x)/g(x) = 0 \), then \( \text{detTAPE}(f) \nsubseteq \text{detTAPE}(g) \).

**Proposition 3.7** (Savitch, 1970). If \( f \) is a tape constructible function, then \( \text{TAPE}(f) \nsubseteq \text{detTAPE}(f^2) \).

**Proof of Theorem 3.4.** By Proposition 3.6, \( \text{detTAPE}(f) \nsubseteq \text{detTAPE}(f \circ g) \).

Using Lemma 2.7,
\[
\text{detTAPE}(f \circ g) = H_{f \circ g}[\text{detLBA}] \subseteq H_g[H_f[\text{detLBA}]] = H_g[\text{detTAPE}(f)].
\]
Hence, \( H_g[\text{detTAPE}(f)] \nsubseteq \text{detTAPE}(f) \).
Corollary 3.8. The family detLBA is not closed under g-bounded homomorphic mappings for any g such that \( \lim_{x \to \infty} \frac{x}{g(x)} = 0 \).

Proof of Theorem 3.5. Suppose TAPE\((f)\) is closed under g-bounded homomorphic mappings. Then for any integer \( k > 0 \), \( H_{g}^{(k)}[TAPE(f)] \subseteq TAPE(f) \). By Lemma 2.7,

\[
H_{f \circ g^{(k)}}[CS] \subseteq H_{g}[H_{f \circ g^{(k-1)}}[CS]] \subseteq \cdots \subseteq H_{g}^{(k)}[H_{f}[CS]] = H_{g}^{(k)}[TAPE(f)] \subseteq TAPE(f),
\]

so that TAPE\((f \circ g^{(k)})\) = \( H_{f \circ g^{(k)}}[CS] \subseteq TAPE(f) \). By Proposition 3.6 and choice of \( g \) and \( m \), detTAPE\((f^{2})\) \( \subseteq \) detTAPE\((f \circ g^{(m)})\). By Definition 1.1,

\[
\text{detTAPE}(f \circ g^{(m)}) \subseteq \text{TAPE}(f \circ g^{(m)}).
\]

By Proposition 3.7, TAPE\((f) \subseteq \text{detTAPE}(f^{2})\). Hence we have

\[
\text{detTAPE}(f^{2}) \subseteq \text{detTAPE}(f \circ g^{(m)}) \subseteq \text{TAPE}(f \circ g^{(m)}) \subseteq \text{TAPE}(f) \subseteq \text{detTAPE}(f^{2}),
\]

a contradiction.

Corollary 3.9. The family CS is not closed under g-bounded homomorphic mappings for any deterministic-tape constructible g such that for some \( \epsilon > 0 \) and all large \( x \), \( g(x) = x^{1+\epsilon} \).

Note that Corollary 3.8 shows that the family detLBA is not closed under mappings which erase more than a linear amount. Corollary 3.9 shows that the family CS is not closed under mappings which erase more than an amount bounded by \( x^{1+\epsilon} \), e.g., CS is not closed under polynomial erasing. It is an open question whether CS is closed under g-bounded mappings for functions such as \( g(x) = x(\log x) \) or \( g(x) = x(\log \log x) \).

As noted in (Book, Greibach, and Wegbreit, 1970), if CS = detLBA, then for any \( f \), TAPE\((f) = \text{detTAPE}(f)\). In (Savitch, 1970), it is shown that if for some deterministic-tape constructible \( f \), TAPE\((f) = \text{detTAPE}(f)\), then for any deterministic-tape constructible \( g \) such that \( g > f \), TAPE\((g) = \text{detTAPE}(g)\). By Theorem 2.8, we have an analog:

(A) If \( f \) is a nondecreasing deterministic tape constructible function such that \( \mathcal{S}(\text{TAPE}(f)) = \mathcal{S}(\text{detTAPE}(f)) \), then for any superadditive deterministic tape constructible function \( g \), \( \mathcal{S}(\text{TAPE}(f \circ g)) = \mathcal{S}(\text{detTAPE}(f \circ g)) \).
By Corollary 1.9 and Proposition 1.10, we can interpret (A) for the AFLs defined by these classes as:

(B) Let \( f \) and \( g \) be superadditive functions and let \( g \) be tape constructible. If \( \mathcal{I}(\text{TAP}(f)) = \mathcal{I}(\text{detTAP}(f)) \), then the smallest AFL containing \( \text{TAP}(f \circ g) \) is the smallest AFL containing \( \text{detTAP}(f \circ g) \).

In (Book, Greibach, and Wegbreit, 1970) it was shown that if \( f \) is a superadditive deterministic-tape constructible function such that for some \( \epsilon > 0 \) and all large \( x \), \( (f(2x)/f(x))^\epsilon \leq (f(x))' \), then \( \mathcal{I}(\text{TAP}(f)) = \mathcal{I}(\text{detTAP}(f)) \).

Hence (A) and (B) are applicable to such \( f \). The function \( f(x) = 2^x \) is an example of a function meeting these conditions. Thus we see that for any \( g \) which is superadditive and tape constructible, the smallest AFL containing \( \text{TAP}(2^g) \) is the smallest AFL containing \( \text{detTAP}(2^g) \).

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References


