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On $SL(2, R)$ symmetry in nonlinear electrodynamics theories

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Abstract

Recently, it has been observed that the Noether–Gaillard–Zumino (NGZ) identity holds order by order in α' expansion in nonlinear electrodynamics theories as Born–Infeld (BI) and Bossard–Nicolai (BN). The nonlinear electrodynamics theory that couples to an axion field is invariant under the $SL(2, R)$ duality in all orders of α' expansion in the Einstein frame. In this paper we show that there are the $SL(2, R)$ invariant forms of the energy momentum tensors of axion-nonlinear electrodynamics theories in the Einstein frame. These $SL(2, R)$ invariant structures appear in the energy momentum tensors of BI and BN theories at all orders of α' expansion. The $SL(2, R)$ symmetry appears in the BI and BN Lagrangians as a multiplication of Maxwell Lagrangian and a series of $SL(2, R)$ invariant structures.

1. Introduction

Duality transformations in nonlinear electrodynamics has been studied in [1]. Classical electromagnetism is the most familiar duality-invariant theory. The Maxwell's Hamiltonian and the equations of motion are invariant under rotations. Note that the Lagrangian, however, is not

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invariant. This brings that nonlinear deformations of Lagrangian will require modifications which are also non-invariant.

Commonly, duality transformations may be found out in the path integral as a Legendre transform. Given some Lagrangian $L(F)$ depending only on the field strength of a vector field, one can construct [2]

$$\mathcal{L}(\tilde{F}, G) = \mathcal{L}(F) - \frac{1}{2}\epsilon^{abcd} F_{ab}\partial_c \tilde{A}_d \tag{1}$$

in which F is treated as a fundamental field. The classical equations of motion for F require that $G_{ab} = \partial_a \tilde{A}_b - \partial_b \tilde{A}_a$ is related to F by

$$G_{ab} = -2\frac{\partial \mathcal{L}(F)}{\partial F^{ab}} \tag{2}$$

through $G_{ab} = -\frac{1}{2}\epsilon_{abcd}\tilde{G}^{cd}$ and $\tilde{G}^{ab} = \frac{1}{2}\epsilon^{abcd}G_{cd}$.

Consistency of the duality constraint can be expressed as a requirement in which the Lagrangian must transform under duality in a particular way, defined by the Noether–Gaillard–Zumino (NGZ) identity [3].

We consider $\mathcal{L}_{inv}$ that is invariant under the following transformation:

$$\delta\begin{pmatrix}\tilde{F}\\ G\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\tilde{F}\\ G\end{pmatrix} \tag{3}$$

where $A^T = -D$, $B^T = B$ and $C^T = C$ are the infinitesimal parameters of the transformations [2]. By applying the duality symmetry, the NGZ identity could be written as following [4]:

$$\mathcal{L} = \mathcal{L}_{inv} - \frac{1}{4}G_{ab}F^{ab}. \tag{4}$$

When the theory only has linear duality (e.g. only F^2 terms in the action) $\delta L_{inv}/\delta F$ vanishes. So it could be found that any higher order dependence (F^4, $F^6, \ldots$) must be part of $\mathcal{L}_{inv}$. The NGZ identity, along with (2) can be solved to find $G(F)$ and various Lagrangians which provide a duality symmetry between equations of motion and Bianchi identities. We will discuss two cases of nonlinear deformations of the Maxwell theory that depend only on F's without derivatives.

The NGZ identity has a significant consequence for the duality rotations properties of the energy-momentum tensor. The energy-momentum tensor, which can be obtained as the variational derivative of the Lagrangian with respect to the gravitational field, is invariant under duality transformation [5]. It has been found as [6]

$$T_{ab} = g_{ab}\mathcal{L} + G_a{}^c F_{bc} \tag{5}$$

Considering this relation and (4), it could be found that $\mathcal{L}_{inv} = 1/4T_a{}^a$.

The Born–Infeld Lagrangian density can be written intelligently in term of the square root of a determinant [7]:

$$\mathcal{L}_{BI} = \sqrt{-\det(\eta_{ab} + F_{ab})} - 1 \tag{6}$$

where the fundamental $(scale)^2$ $(= T^{-1} = 2\pi\alpha'$ in the string theory context) has been set equal to 1, has many considerable aspects, including electro-magnetic duality symmetry [8]. Since in four dimensions:

$$-\det(\eta_{ab} + F_{ab}) = \frac{1}{2}F_{ab}F^{ab} - \frac{1}{16}(F_{ab}\tilde{F}^{ab})^2, \qquad \tilde{F}^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}, \tag{7}$$

L_{BI} interpolates between the Maxwell Lagrangian $\frac{1}{4}F_{ab}F^{ab}$ for small F and the total derivative (topological density) $\frac{i}{4}(F_{ab}\tilde{F}^{ab})$ for large F. One can find an appropriate form in Euclidean signature as:

$$\mathcal{L}_{BI} = \sqrt{(1+F_2)^2 + 2F_4} - 1 = F_2 + F_4[1 + O(F^2)], \tag{8}$$

where $F_2 = \frac{1}{4}F_{ab}F^{ab}$ and $F_4 = \frac{-1}{8}[F_{ab}F^{bc}F_{cd}F^{da} - \frac{1}{4}(F_{ab}F^{ab})^2]$.

That $\mathcal{L}_{BI} = F_2 + F_4 + O(F^6)$ is true in all dimensions, however it is only in $D = 4$ that all higher order terms are proportional to F_4.

Adding axion and dilaton fields to the Born–Infeld electrodynamics, the electric–magnetic duality invariance may be extended to $SL(2, R)$ S-duality, relevant to string theory, which insinuates a strong-weak coupling duality of such theories [9]. This action is not invariant under the S-duality, however, its equations of motion and energy-momentum tensor are invariant under the S-duality as it was shown for the electric–magnetic duality [10]. The $SL(2, R)$ S-duality transformation holds order by order in α' and is nonperturbative in the string loop expansion [11].

We will investigate the S-dual structure of energy momentum tensor of nonlinear electrodynamics theory that couples to dilaton and axion fields in the Einstein frame. In fact, we are going to study the energy momentum tensor order by order in α'. We want to show that the energy momentum tensor constructed of $SL(2, R)$ structures at any order in α'. We will find the form of energy momentum tensor that is manifestly S-dual. The action of the theory that appears as a part of energy momentum tensor, can be written in term of a production of Maxwell action and a set of S-dual structures.

As was noted in [3] that the invariance of the energy momentum tensor and thus of the corresponding Lagransian should imply the invariance of the S-matrix. So one expects to find the S-matrix elements in term of the $SL(2, R)$ invariant structures that appear in the energy momentum tensors. On the other hand, it has been found that the tree-level S-matrix elements of some scattering amplitudes including the loops and the nonperturbative effects become symmetric under the $SL(2, Z)$ transformation. Using the Ward identity corresponding to the global S-duality transformations, this point can be extended to all S-matrix as well [15]. It could be expected, considering the $SL(2, R)$ invariant structures of the energy momentum tensors and the Ward identity corresponding to the global S-duality transformations, have been used as guiding principles to find the S-matrix elements [17]. The action of linearized $SL(2, R)$ invariance on S-matrix of scalars and vectors for D_3-brane in tree-level open/closed string theory was studied in [10,15].

The theory that comes from nontrivial nonlinear deformation of classical electrodynamics, which is consistent with NGZ identity, is Bossard–Nicolai theory. This theory is produced in which the NGZ condition work order by order in α'. This theory is equal to Born–Infeld theory up to $O(F^6)$. Indeed, the Bossard–Nicolai theory differs from BI theory starting at $O(F^8)$ [2]. It will be expected that the action and the energy momentum tensor of BN theory appear in term of $SL(2, R)$ structure. In fact, we extend our results from the BI theory to the BN theory.

The outline of the paper is as follows: We begin in section 2 by reviewing the $SL(2, R)$ transformations of bosonic fields. We investigate the behavior of BI energy momentum tensor under the $SL(2, R)$ transformations. In section 3, we extend our calculation in previous section to all orders of α'. We use these considerations to find the BN theory in term of relevant $SL(2, R)$ invariant structure.

2. $SL(2,R)$ invariant structure

In this section we are going to construct relevant structures which are manifestly invariant under $SL(2,R)$ transformation. The nonlinear transformation of the gauge field and the axion-dilaton, $\tau = C_0 + ie^{-\phi_0}$ are given by [6]

$$F_{ab} \to sF_{ab} + r\tilde{G}_{ab}$$
$$G_{ab} \to pG_{ab} - q\tilde{F}_{ab} \quad ; \quad \tau \to \frac{p\tau + q}{r\tau + s} \tag{9}$$

where the antisymmetric tensor G_{ab} is the one appears in (2). It could be written the transformation of gauge field as

$$\mathcal{F}_{ab} \equiv \begin{pmatrix} \tilde{F}_{ab} \\ G_{ab} \end{pmatrix} \to (\Lambda^{-1})^T \begin{pmatrix} \tilde{F}_{ab} \\ G_{ab} \end{pmatrix} \quad ; \quad \Lambda = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2,R) \tag{10}$$

The S-duality transformation on the background fields ϕ_0 and C is

$$\mathcal{M} \to \Lambda \mathcal{M} \Lambda^T \quad ; \quad \mathcal{M} = e^{\phi_0} \begin{pmatrix} |\tau|^2 & C_0 \\ C_0 & 1 \end{pmatrix} \tag{11}$$

Using the above transformations, one can find that the structure $\mathcal{F}^T\mathcal{M}\mathcal{F}$ is a $SL(2,R)$ invariant structure.

It could be useful to consider the Lagrangian at the presence of axion and dilaton couplings in the following form which the contribution of axion coupling is separated.

$$\mathcal{L} = \mathcal{L}' + C_0 z \tag{12}$$

where the Lagrangian was constructed of the following two possible Lorentz invariants ($\mathcal{L}(t,z)$):

$$t = \frac{1}{4}F^{ab}F_{ab} \quad , \quad z = \frac{1}{4}F^{ab}\tilde{F}_{ab} \tag{13}$$

The antisymmetric tensor G_{ab} then separates as $G_{ab} = G'_{ab} - C_0\tilde{F}_{ab}$ where G'_{ab} could be an arbitrary nonlinear function of F which is $G'_{ab} = -2\frac{\partial \mathcal{L}'}{\partial F^{ab}}$.

By this consideration one gets the $SL(2,R)$ invariant structure in the following form:

$$(\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{bc} = e^{-\phi_0}\,\tilde{F}_a{}^c\,\tilde{F}_{bc} + e^{\phi_0}\,G'_a{}^c\,G'_{bc} \tag{14}$$

It is easy to check that the above structure is invariant under the linear transformations $G' \longrightarrow \tilde{F}$, $\tilde{F} \longrightarrow -G'$ and $e^{-\phi_0} \longrightarrow e^{\phi_0}$.

3. Nonlinear electrodynamics theories

In this section we review the nonlinear electrodynamics theories that couple to axion field and investigate their behavior under the $SL(2,R)$ symmetry. The actions of these nonlinear theories are the generalized form of Maxwell action. In fact, it was demonstrated that the Maxwell action is the leading-order term in the expansion in F of nonlinear electrodynamics theories. These actions satisfy the NGZ identity, however they are not invariant under the S-duality transformation. In following, we consider the S-duality transformations in the axion-Maxwell theory and show the energy momentum tensor of this theory is invariant under the $SL(2,R)$ symmetry. We extend this consideration to nonlinear theories: Axion-Born–Infeld and Axion-Bossard–Nicolai.

3.1. Axion-Maxwell theory

The simplest example of duality invariant theories is Maxwell's electromagnetism. The Maxwell action can be generalized to the case of the presence of background axion field[1] [6]

$$\mathcal{L} = -e^{-\phi_0} t + C_0 z \tag{15}$$

where t and z are the ones appear in (13). According to (5) and (2), the energy momentum tensor of the above action could be found in the following form:

$$T_{ab} = e^{-\phi_0}\left[-\frac{1}{4}F_{cd}F^{cd}g_{ab} + F_a{}^c F_{bc}\right] \tag{16}$$

On the other hand, replacing in the S-duality transformation (10) and (11) the relevant G_{ab} from (2), one finds the following $SL(2, R)$ invariant structure:

$$(\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{bc} = e^{-\phi_0}[\tilde{F}_a{}^c \tilde{F}_{bc} + F_a{}^c F_{bc}] = e^{-\phi_0}[-\frac{1}{2}F_{cd}F^{cd}g_{ab} + 2F_a{}^c F_{bc}] \tag{17}$$

where we use of the identity $\tilde{\tilde{F}}_{ab} = -F_{ab}$ and

$$\epsilon^{abcd}\epsilon^{efgh} = -\begin{vmatrix} \eta^{ae} & \eta^{af} & \eta^{ag} & \eta^{ah} \\ \eta^{be} & \eta^{bf} & \eta^{bg} & \eta^{bh} \\ \eta^{ce} & \eta^{cf} & \eta^{cg} & \eta^{ch} \\ \eta^{de} & \eta^{df} & \eta^{dg} & \eta^{dh} \end{vmatrix}. \tag{18}$$

Comparing the energy momentum tensor (16) and the structure (17), the energy momentum tensor in the form that is manifestly $SL(2, R)$ invariant could be presented as:

$$T_{ab} = \frac{1}{2}(\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{bc} \tag{19}$$

It is obvious that $Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F})$ will thus be zero. (In the next sections, we see that the trace of $SL(2, R)$ invariant structure is nonzero for nonlinear electrodynamics theories.) We will find the energy momentum tensors corresponding to the BI and BN theories in the forms which are manifestly $SL(2, R)$ invariant.

3.2. Axion-Born–Infeld theory

The Born–Infeld theory, perhaps is the simplest nonlinear deformation of Maxwell's theory. The Axion-Born–Infeld Lagrangian, the general nonlinear electromagnetic theory couple to the axion field that the equations of motion are $SL(2, R)$ invariant, in the Einstein frame is:

$$\mathcal{L} = \mathcal{L}_{BI} + C_0 z = g^{-2}\left[1 - \sqrt{1 + 2g^2 e^{-\phi_0} t - g^4 e^{-2\phi_0} z^2}\right] + C_0 z \tag{20}$$

where $g = 2\pi\alpha'$. Regardless the axion term, it is clear that classical electromagnetism corresponds to $g^2 \to 0$. In the following calculation, we set $\alpha' = 1/2\pi$. Using (2), we find the following expression for G:

---

[1] We have also introduced a background dilaton field ϕ_0 with $e^{\phi_0/2}$ playing the role of an effective gauge coupling constant.

$$G_{ab} = \frac{e^{-\phi_0} F_{ab} - e^{-2\phi_0} z \tilde{F}_{ab}}{\sqrt{1 + 2e^{-\phi_0} t - e^{-2\phi_0} z^2}} - C_0 \tilde{F}_{ab} \tag{21}$$

One can expand the Born–Infeld Lagrangian as:

$$\mathcal{L}_{BI} = -te^{-\phi_0} + \frac{1}{2} e^{-2\phi_0} \left(t^2 + z^2\right) - \frac{1}{2} te^{-3\phi_0} \left(t^2 + z^2\right) + \frac{1}{8} e^{-4\phi_0} \left(t^2 + z^2\right) \left(5t^2 + z^2\right) + \ldots \tag{22}$$

Using (18), we can write the field variable z^2 in term of the following trace terms:

$$z^2 = \frac{1}{4} Tr(F \cdot F \cdot F \cdot F) - \frac{1}{8} Tr(F \cdot F)^2 \tag{23}$$

To simplify, we choose the $SL(2, R)$ matrix Λ in (10) which has been made of the components $p = 0, q = 1, r = -1$ and $s = 0$. This consideration brings the following linear S-duality transformations

$$F_{ab} \to e^{-\phi_0} \tilde{F}_{ab} \quad ; \quad \tilde{F}_{ab} \to -e^{-\phi_0} F_{ab} \quad ; \quad \phi_0 \to -\phi_0 \tag{24}$$

According to (18) and the following trace S-duality transformations

$$\begin{aligned} &e^{-\phi_0} Tr(F \cdot F) \to -e^{-\phi_0} Tr(F \cdot F); \\ &e^{-2\phi_0} Tr(F \cdot F \cdot F \cdot F) \to e^{-2\phi_0} Tr(F \cdot F \cdot F \cdot F) \end{aligned} \tag{25}$$

it will be revealed that $e^{-2\phi_0} z^2$ is invariant under the linear S-duality transformation. Replacing the field variable z^2 from (23) in the Born–Infeld Lagrangian, one finds the expansion of $\mathcal{L}_{BI}$ at the all levels of F^{2n} in term of trace structures $Tr(F \cdot F)$ and $Tr(F \cdot F \cdot F \cdot F)$.

$$\begin{aligned} \mathcal{L}_{BI} = {} & e^{-\phi_0} \left[\frac{1}{4} Tr(F \cdot F) \right] \\ & + e^{-2\phi_0} \left[\frac{1}{8} \text{Tr}(F \cdot F \cdot F \cdot F) - \frac{1}{32} \text{Tr}(F \cdot F)^2 \right] \\ & + e^{-3\phi_0} \left[-\frac{1}{32} \text{Tr}(F \cdot F \cdot F \cdot F) \text{Tr}(F \cdot F) + \frac{1}{128} \text{Tr}(F \cdot F)^3 \right] \\ & + e^{-4\phi_0} \left[\frac{1}{128} \text{Tr}(F \cdot F \cdot F \cdot F)^2 + \frac{1}{256} \text{Tr}(F \cdot F)^2 \text{Tr}(F \cdot F \cdot F \cdot F) \right. \\ & \left. - \frac{3}{2048} \text{Tr}(F \cdot F)^4 \right] \\ & + \mathcal{O}(F^{10}) + \ldots \end{aligned} \tag{26}$$

It is easy to show that the above Lagrangian $\mathcal{L}_{BI}$ in the all levels of F^{2n} is self-dual and anti-self-dual under the linear S-duality transformation when n is even and odd respectively. By expanding the denominator in (21), we can find the G_{ab} in the following expression

$$\begin{aligned} G_{ab} &= G'_{ab} - C_0 \tilde{F}_{ab} \\ &= e^{-\phi_0} F_{ab} - C_0 \tilde{F}_{ab} + e^{-2\phi_0} \left(F_a \cdot F \cdot F_b - \frac{1}{4} Tr(F \cdot F) F_{ab} \right) \\ &+ e^{-3\phi_0} \left(\frac{1}{8} Tr(F \cdot F \cdot F \cdot F) F_{ab} + \frac{1}{4} Tr(F \cdot F) F_a \cdot F \cdot F_b - \frac{3}{32} Tr(F \cdot F)^2 F_{ab} \right) \end{aligned}$$

$$+e^{-4\phi_0}\bigg(\frac{1}{8}Tr(F\cdot F\cdot F\cdot F)F_a\cdot F\cdot F_b+\frac{1}{32}Tr(F\cdot F)Tr(F\cdot F\cdot F\cdot F)F_{ab}$$
$$+\frac{1}{32}Tr(F\cdot F)^2F_a\cdot F\cdot F_b-\frac{3}{128}Tr(F\cdot F)^3F_{ab}\bigg)+\mathcal{O}(F^9)+\dots \quad (27)$$

Replacing the above expansions of $\mathcal{L}_{BI}$ and G_{ab} in (5), the energy momentum tensor becomes

$$T_{ab}=e^{-\phi_0}\left[\frac{1}{4}g_{ab}Tr(F\cdot F)-F_a\cdot F_b\right]$$
$$+e^{-2\phi_0}\bigg[g_{ab}\left(\frac{1}{8}Tr(F\cdot F\cdot F\cdot F)-\frac{1}{32}Tr(F\cdot F)^2\right)-F_a\cdot F\cdot F\cdot F_b$$
$$+\frac{1}{4}Tr(F\cdot F)F_a\cdot F_b\bigg]$$
$$+e^{-3\phi_0}\bigg[g_{ab}\left(\frac{1}{32}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)-\frac{1}{128}Tr(F\cdot F)^3\right)$$
$$-\frac{1}{8}Tr(F\cdot F\cdot F\cdot F)F_a\cdot F_b-\frac{1}{4}Tr(F\cdot F)F_a\cdot F\cdot F\cdot F_b+\frac{3}{32}Tr(F\cdot F)^2F_a\cdot F_b\bigg]$$
$$+e^{-4\phi_0}\bigg[g_{ab}\bigg(\frac{1}{128}Tr(F\cdot F\cdot F\cdot F)^2+\frac{1}{256}Tr(F\cdot F)^2Tr(F\cdot F\cdot F\cdot F)$$
$$-\frac{3}{2048}Tr(F\cdot F)^4\bigg)$$
$$-\frac{1}{8}Tr(F\cdot F\cdot F\cdot F)F_a\cdot F\cdot F\cdot F_b-\frac{1}{32}Tr(F\cdot F)Tr(F\cdot F\cdot F\cdot F)F_a\cdot F_b$$
$$-\frac{1}{32}Tr(F\cdot F)^2F_a\cdot F\cdot F\cdot F_b+\frac{3}{128}Tr(F\cdot F)^3F_a\cdot F_b\bigg]+\mathcal{O}(F^{10})+\dots \quad (28)$$

In fact, the above BI energy momentum tensor is the generalization of Maxwell energy momentum tensor (16). It is not clear how the above tensor treat under the S-dual transformation. To answer this question, let us try to present the above BI energy momentum tensor in the form that is $SL(2,R)$ invariant. So we calculate appropriate $SL(2,R)$ invariant structure and should be able to write the energy momentum tensor in term of this structure. To this end, we make the nonlinear $SL(2,R)$ invariant structure by replacing in (10) the antisymmetric tensor G_{ab} from (27). This structure becomes:

$$(\mathcal{F}^T)_a{}^c\mathcal{M}_0\mathcal{F}_{cb}=e^{-\phi_0}\left[-\frac{1}{2}g_{ab}Tr(F\cdot F)+2F_a\cdot F_b\right]$$
$$+e^{-2\phi_0}\left[-\frac{1}{2}Tr(F\cdot F)F_a\cdot F_b+2F_a\cdot F\cdot F\cdot F_b\right]$$
$$+e^{-3\phi_0}\bigg[F_a\cdot F\cdot F\cdot F\cdot F\cdot F_b+\frac{1}{4}Tr(F\cdot F\cdot F\cdot F)F_a\cdot F_b$$
$$-\frac{1}{8}Tr(F\cdot F)^2F_a\cdot F_b\bigg]$$
$$+e^{-4\phi_0}\bigg[\frac{1}{2}Tr(F\cdot F)F_a\cdot F\cdot F\cdot F\cdot F\cdot F_b$$

$$
- \frac{1}{4} Tr(F \cdot F)^2 F_a \cdot F \cdot F \cdot F_b
$$
$$
+ \frac{1}{2} Tr(F \cdot F \cdot F \cdot F) F_a \cdot F \cdot F \cdot F_b \Big] + \mathcal{O}(F^{10}) + \ldots \tag{29}
$$

To build the energy momentum tensor (28) in term of the above $SL(2,R)$ invariant structure we need to apply valuable identities at the level of $n \geq 3$. These identities for $n = 3$ are as following:

$$
Tr(F \cdot F \cdot F \cdot F \cdot F \cdot F) - \frac{3}{4} Tr(F \cdot F \cdot F \cdot F) Tr(F \cdot F) + \frac{1}{8} Tr(F \cdot F)^3 = 0
$$
$$
F_a \cdot F \cdot F \cdot F \cdot F \cdot F_b - \frac{1}{2} Tr(F \cdot F) F_a \cdot F \cdot F \cdot F_b
$$
$$
+ \frac{1}{4} F_a \cdot F_b [\frac{1}{2} Tr(F \cdot F)^2 - Tr(F \cdot F \cdot F \cdot F)] = 0 \tag{30}
$$

Using these identities one can write T_{ab} in the form that is manifestly $SL(2,R)$ invariant.

$$
T_{ab} = g_{ab} - \left[g_{ab} + \frac{1}{2} (\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{cb} \right] \sum_{m=0}^{\infty} a_m \, (-1)^m \, 2^{-4m} \, Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F})^m \tag{31}
$$

where

$$
a_0 = 1 \;\; ; a_1 = 1 \;\; ; a_2 = \frac{3}{2} \;\; ; a_3 = \frac{5}{2} \;\; ; a_4 = \frac{35}{8} \;\; ; a_5 = \frac{63}{8} \;\; ; \ldots
$$

One can find the above energy momentum tensor which is expanded in term of the level of coupling factor $e^{-n\phi_0}$, where $n = 2m + 1$. We calculate the expansion factors up to $m = 5$ or up to level of $e^{-11\phi_0}$ ($\mathcal{O}(F^{22})$). It is clear that the Born–Infeld energy momentum tensor leads to the Maxwell energy momentum tensor (19) for $m = 0$.

One may find the manifestly $SL(2,R)$ invariant form of the energy momentum tensor in terms of the different $SL(2,R)$ invariant structures as $Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F} \mathcal{F}^T \mathcal{M}_0 \mathcal{F})$, $Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F} \mathcal{F}^T \mathcal{M}_0 \mathcal{F} \mathcal{F}^T \mathcal{M}_0 \mathcal{F})$ and so on. It could be shown that these structures can be written in terms of powers of $Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F})$ (see appendix). From these considerations, the manifestly $SL(2,R)$ invariant form of the energy momentum tensors in this paper constructed out of the structure $Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F})$ would be unique.

At the end of this section, we are going to find the representation of Born–Infeld Lagrangian that deduces from the relevant $SL(2,R)$ invariant energy momentum tensor (31). According to (5) (and the energy-momentum tensor, which can be obtained as the variational derivative of the Lagrangian with respect to the gravitational field), one can find that only the first term in (29) contributes to the $SL(2,R)$ invariant structure which appears in (31). Considering this, one can find the following representation for Born–Infeld Lagrangian:

$$
\mathcal{L}_{BI} = 1 - \left(1 + e^{-\phi_0} t\right) \sum_{m=0}^{\infty} a_m \, (-1)^m \, 2^{-4m} \, Tr(\mathcal{F}^T \mathcal{M}_0 \mathcal{F})^m \tag{32}
$$

In this form of $\mathcal{L}_{BI}$, it is obvious that the Lagrangian is constructed from two separable parts: invariant and non-invariant $SL(2,R)$ structures. As expected, the $\mathcal{L}_{BI}$ leads to Maxwell Lagrangian for $m = 0$.

3.3. Graviton-Axion-Born–Infeld theory

It has been shown that the Born–Infeld Lagrangian coupled to gravity $L(F_{ab}, g_{cd})$ that the equations of motion, including the Einstein equations, are invariant under the $SL(2,R)$ duality, has the following form [6]:

$$\mathcal{L} = R + 1 - \sqrt{1 + 2e^{-\phi_0}t - e^{-2\phi_0}z^2} + C_0 z$$

In this section, we want to write the Einstein equation $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$ in the form that is manifestly $SL(2,R)$ invariant.[2] Since the metric is invariant under the $SL(2,R)$ duality in the Einstein frame, one can expect that the scalar curvature and ricci tensor can be constructed in term of $SL(2,R)$ invariant structures. Take the trace of the Einstein equation leads to $R = -T_a{}^a$. By considering (31), one can find the scalar curvature in term of $SL(2,R)$ invariant structure $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})$ as following:

$$R = \frac{1}{4}\sum_{m=0}^{\infty} b_m(-1)^m 2^{-4m} Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^{m+1} \tag{33}$$

$$b_0 = 1 \;\; ; b_1 = \frac{1}{2} \;\; ; b_2 = \frac{1}{2} \;\; ; b_3 = \frac{5}{8} \;\; ; b_4 = \frac{7}{8} \;\; ; b_5 = \frac{63}{4} \dots$$

As we expected for the Maxwell theory, $m = 0$, we get $R = 0$. Replacing in the Einstein equation the above expansion and the expansion (31), the ricci tensor then becomes in term of following $SL(2,R)$ invariant structures:

$$R_{ab} + g_{ab} = \left(g_{ab} + g_{ab}\frac{Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})}{4} - \frac{(\mathcal{F}^T)_a{}^c\mathcal{M}_0\mathcal{F}_{cb}}{2} \right) \times \sum_{m=0}^{\infty} a_m\, (-1)^m 2^{-4m}\, Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^m$$

One can use the above considerations to write Riemann tensor R_{abcd} and any scalar structure as $R_{ab}R^{ab}$ or $R_{abcd}R^{abcd}$ in term of $SL(2,R)$ invariant structures.

3.4. Axion-Bossard–Nicolai theory

Another NGZ-consistent theory is Bossard–Nicolai theory that comes from a distinct nontrivial nonlinear deformation of classical electrodynamics. We are going to show that the Lagrangian and the energy momentum tensor of BN theory appear in term of $SL(2,R)$ structures.

Using the definition of G and $\tilde{G}$ according to (2) and field variables (13), the NGZ consistency could be presented as a differential equation [2,12]. Both Lagrangians $\mathcal{L}_{BI}$ and $\mathcal{L}_{BN}$ are the solution of NGZ differential equation. However unlike the BI Lagrangian, it is not known if the BN Lagrangian has a closed-form expression. Note that this Lagrangian differs from $\mathcal{L}_{BI}$ starting at $\mathcal{O}(F^6)$. In [2], this Lagrangian was calculated up to level of coupling factor $e^{-8\phi_0}$.

$$\mathcal{L}_{BN} = -te^{-\phi_0} + \frac{1}{2}e^{-2\phi_0}\left(t^2 + z^2\right) - \frac{1}{2}te^{-3\phi_0}\left(t^2 + z^2\right) + \frac{1}{4}e^{-4\phi_0}\left(t^2 + z^2\right)\left(3t^2 + z^2\right)$$

[2] We consider the Einstein constant $G = 1/8\pi$.

$$
\begin{aligned}
&-\frac{1}{8}te^{-5\phi_0}\left(t^2+z^2\right)\left(11t^2+7z^2\right)\\
&+\frac{1}{32}e^{-6\phi_0}\left(t^2+z^2\right)\left(91t^4+86t^2z^2+11z^4\right)\\
&-\frac{1}{8}te^{-7\phi_0}\left(t^2+z^2\right)\left(51t^4+64t^2z^2+17z^4\right)\\
&+\frac{1}{64}e^{-8\phi_0}\left(t^2+z^2\right)\left(969t^6+1517t^4z^2+623t^2z^4+43z^6\right)+\ldots
\end{aligned}
\tag{34}
$$

Now using (23), one can expand the above Lagrangian at all orders of α' in term of $Tr(F\cdot F)$ and $Tr(F\cdot F\cdot F\cdot F)$. Similar Axion-Born–Infeld Lagrangian (20), the Axion-Bossard–Nicolai Lagrangian is $\mathcal{L}=\mathcal{L}_{BN}+C_0 z$. Like the antisymmetric tensor (27), we calculate the BN antisymmetric tensor G_{ab} as following:

$$
\begin{aligned}
G_{ab} &= G'_{ab}-C_0\tilde{F}_{ab}\\
&= e^{-\phi_0}F_{ab}-C_0\tilde{F}_{ab}+e^{-2\phi_0}\left(F_a\cdot F\cdot F_b-\frac{1}{4}Tr(F\cdot F)F_{ab}\right)\\
&+e^{-3\phi_0}\left(\frac{1}{8}Tr(F\cdot F\cdot F\cdot F)F_{ab}+\frac{1}{4}Tr(F\cdot F)F_a\cdot F\cdot F_b-\frac{3}{32}Tr(F\cdot F)^2F_{ab}\right)\\
&+e^{-4\phi_0}\left(\frac{1}{4}Tr(F\cdot F\cdot F\cdot F)F_a\cdot F\cdot F_b-\frac{1}{64}Tr(F\cdot F)^3F_{ab}\right)\\
&+e^{-5\phi_0}\left(\frac{15}{2048}Tr(F\cdot F)^4F_{ab}-\frac{5}{128}Tr(F\cdot F)^3F_a\cdot F\cdot F_b\right.\\
&-\frac{15}{256}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2F_{ab}\\
&\left.+\frac{7}{32}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)F_a\cdot F\cdot F_b+\frac{7}{128}Tr(F\cdot F\cdot F\cdot F)^2F_{ab}\right)\\
&+e^{-6\phi_0}\left(\frac{111}{16384}Tr(F\cdot F)^5F_{ab}-\frac{79}{4096}Tr(F\cdot F)^4F_a\cdot F\cdot F_b\right.\\
&-\frac{79}{2048}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^3F_{ab}\\
&+\frac{31}{512}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2F_a\cdot F\cdot F_b\\
&\left.+\frac{31}{1024}Tr(F\cdot F\cdot F\cdot F)^2Tr(F\cdot F)F_{ab}+\frac{33}{256}Tr(F\cdot F\cdot F\cdot F)^2F_a\cdot F\cdot F_b\right)\\
&+e^{-7\phi_0}\left(\frac{63}{32768}Tr(F\cdot F)^6F_{ab}-\frac{5}{4096}Tr(F\cdot F)^5F_a\cdot F\cdot F_b\right.\\
&-\frac{25}{8192}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^4F_{ab}\\
&-\frac{21}{512}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^3F_a\cdot F\cdot F_b\\
&-\frac{63}{2048}Tr(F\cdot F\cdot F\cdot F)^2Tr(F\cdot F)^2F_{ab}\\
&\left.+\frac{51}{256}Tr(F\cdot F\cdot F\cdot F)^2Tr(F\cdot F)F_a\cdot F\cdot F_b+\frac{17}{512}Tr(F\cdot F\cdot F\cdot F)^3F_{ab}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{-8\phi_0}\bigg(-\frac{83}{131072} Tr(F\cdot F)^7 F_{ab} + \frac{271}{65536} Tr(F\cdot F)^6 F_a\cdot F\cdot F_b \\
&+ \frac{813}{65536} Tr(F\cdot F\cdot F\cdot F) Tr(F\cdot F)^5 F_{ab} \\
&- \frac{103}{2048} Tr(F\cdot F\cdot F\cdot F) Tr(F\cdot F)^4 F_a\cdot F\cdot F_b \\
&- \frac{103}{2048} Tr(F\cdot F\cdot F\cdot F)^2 Tr(F\cdot F)^3 F_{ab} \\
&+ \frac{483}{4096} Tr(F\cdot F\cdot F\cdot F)^2 Tr(F\cdot F)^2 F_a\cdot F\cdot F_b \\
&+ \frac{161}{4096} Tr(F\cdot F\cdot F\cdot F)^3 Tr(F\cdot F) F_{ab} + \frac{43}{512} Tr(F\cdot F\cdot F\cdot F)^3 F_a\cdot F\cdot F_b \bigg) \\
&+ \mathcal{O}(F^{17}) + \ldots
\end{aligned}
$$

One can construct the $SL(2,R)$ invariant structures for this theory by replacing in (10) the above antisymmetric tensor. Using (5), the BN energy momentum tensor could be derived. The form of the energy momentum tensor in term of the invariant structure then becomes:

$$
\begin{aligned}
T_{ab} &= -\frac{1}{4}(\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{cb} \\
&\quad - \frac{1}{4}\bigg[(\mathcal{F}^T)_a{}^c \mathcal{M}_0 \mathcal{F}_{cb} - \frac{Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}) g_{ab}}{4}\bigg] \sum_{m=0}^{\infty} (-1)^m 2^{-4m} a_m Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^m \\
&\quad a_0 = 1 \;\; ; a_1 = 2, \;\; ; a_2 = 1 \;\; ; a_3 = \frac{1}{2}; \ldots
\end{aligned}
\tag{35}
$$

where we calculate the coefficients of series up to coupling factor $e^{-8\phi_0}$. Using (5), we find the BN Lagrangian in term of the $SL(2,R)$ invariant structure as following:

$$
\begin{aligned}
\mathcal{L}_{BN} &= -\frac{1}{2} e^{-\phi_0} t - \frac{1}{16}\bigg[8 e^{-\phi_0} t - Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})\bigg] \\
&\quad \times \sum_{m=0}^{\infty} (-1)^m 2^{-4m} a_m Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^m
\end{aligned}
\tag{36}
$$

This ends our illustration of the behavior of nonlinear electrodynamics theories at the presence of dilaton and axion fields under the $SL(2,R)$ symmetry. We could write the actions of these theories in term of $SL(2,R)$ invariant structure. We could explicitly check the invariance of the energy momentum tensor of Axion-dilaton-nonlinear electrodynamics theories under the S-duality transformation and found relevant $SL(2,R)$ invariant structures.

The nonlinear electrodynamics theory has been expressed as a leading term in the low-energy effective action of string theory. The BI action may be defined as the action representing the D_3-branes in 4 dimensions [8]. The equations of motion and the energy momentum tensor of the effective action of D_3-branes are invariant under $SL(2,R)$ transformation [13]. It is expected that this symmetry will be held at higher order fields. The effective action of D_3-brane has been presented in [14] in term of two linear S-dual invariant Q-tensors in which $Q = \partial F \partial F + \partial \tilde{F} \partial \tilde{F}$. On the other hand, we have shown in this paper that the $SL(2,R)$ invariant structures appear nonlinearly in the effective action of D_3-brane. So using nonlinear invariant structure as $Q' = \partial G' \partial G' + \partial \tilde{F} \partial \tilde{F}$, it will be expected that one can derive more higher order couplings.

In the approach suggested in [16] just mentioned the whole information about the given duality invariant system is encoded, in a closed form, in some invariant function of tensorial auxiliary fields, while the representation of the relevant Lagrangians as infinite series over powers of the Maxwell field strength arises as a result of elimination of these auxiliary fields by their algebraic equation of motion. It would be interesting to find a closed form of the relevant energy momentum tensor in terms of their $SL(2,R)$ invariants using this formulation.

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Appendix

In this appendix we want to show that the power series appearing in the energy-momentum tensors (or Lagrangians) of the nonlinear electromagnetic theories in this paper, is uniquely determined by the $SL(2,R)$ invariant structure $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})$ and hence, the expression that we have found for the energy-momentum tensors would be unique. In fact, one may use $SL(2,R)$-invariant expressions $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})$, $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})$ and so on to write the energy-momentum tensors. It will be shown that these $SL(2,R)$-invariant expressions can be written in term of the power-series expansion of the $SL(2,R)$ invariant structure $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})$.

Taking the trace of (29), we have

$$\begin{aligned} Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}) = {} & 2e^{-2\phi}Tr(F\cdot F\cdot F\cdot F) \\ & -\frac{1}{2}e^{-2\phi}Tr(F\cdot F)^2 + e^{-3\phi}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F) \\ & -\frac{1}{4}e^{-3\phi}Tr(F\cdot F)^3 + \frac{1}{2}e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)^2 \\ & +\frac{1}{8}e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2 \\ & -\frac{1}{16}e^{-4\phi}Tr(F\cdot F)^4 + \mathcal{O}(F^{10}) + \ldots \end{aligned} \tag{37}$$

where we use (30) to write the $Tr(F^6)$, in terms of the $Tr(F^2)$ and $Tr(F^4)$. It could be found the following expression for the structure $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})$

$$\begin{aligned} Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}) = {} & 4e^{-2\phi}Tr(\cdot F\cdot F\cdot F\cdot F) - e^{-2\phi}Tr(F\cdot F)^2 \\ & + 2e^{-3\phi}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F) \\ & -\frac{1}{2}e^{-3\phi}Tr(F\cdot F)^3 + 3e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)^2 \\ & -\frac{3}{4}e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2 + \mathcal{O}(F^{10}) + \ldots \end{aligned} \tag{38}$$

where we use the following identity to write the $Tr(F^8)$, in terms of the expression including $Tr(F^2)$ and $Tr(F^4)$.

$$Tr(F\cdot F\cdot F\cdot F\cdot F\cdot F\cdot F\cdot F)-\frac{1}{4}Tr(F\cdot F\cdot F\cdot F)^2$$
$$-\frac{1}{4}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2+\frac{1}{16}Tr(F\cdot F)^4=0 \tag{39}$$

One can find this identity (and all similar identities that appear in this paper) by comparing the expansion of (20) with the expansion of the BI action in the Einstein frame and explicitly check with substituting the four dimensional matrix form of the gauge fields in the identity.

Using the identity (39), we find the following expression for $Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})$

$$\begin{aligned}Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})=\;&6e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)^2\\&-3e^{-4\phi}Tr(F\cdot F\cdot F\cdot F)Tr(F\cdot F)^2\\&+\frac{3}{8}e^{-4\phi}Tr(F\cdot F)^4+\mathcal{O}(F^{10})+\ldots\end{aligned} \tag{40}$$

Considering the relation (37), (38) and (40) one can write the last two structures in terms of the first structure as following

$$Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})=2Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})+\frac{1}{2}Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^2+\ldots$$
$$Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F})=\frac{3}{2}Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^2+\frac{1}{4}Tr(\mathcal{F}^T\mathcal{M}_0\mathcal{F})^3+\ldots$$

where we use the following identities to write the $Tr(F^{10})$ and the $Tr(F^{12})$, in terms of the expression including $Tr(F^2)$ and $Tr(F^4)$.

$$Tr(F^{10})+\frac{1}{64}Tr(F\cdot F)^5-\frac{5}{16}Tr(F\cdot F)Tr(F\cdot F\cdot F\cdot F)^2=0$$
$$Tr(F^{12})+\frac{3}{64}Tr(F\cdot F)^4Tr(F\cdot F\cdot F\cdot F)$$
$$-\frac{3}{16}Tr(F\cdot F)^2Tr(F\cdot F\cdot F\cdot F)^2-\frac{1}{16}Tr(F\cdot F\cdot F\cdot F)^3=0.$$

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