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EXTENSIONS AND RESTRICTIONS IN PRODUCTS OF METRIC SPACES

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The structure of covers on subsets of products of metric spaces is investigated. Some applications to extensions of continuous maps and some well-known corollaries are given.

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When solving the Isbell problem concerning locally fine and subfine uniform spaces, the second author found a method of constructing nice covers of products of metric spaces. Using a slight modification of this procedure we are able to use the method to get rather a strong result on the existence of certain nice families in products of metric spaces (see Theorem). Its strength follows from many applications which generalize known results in various directions:

Ščepin's and Klebanov's results [9, 4, 5] when certain subsets are zero sets; Pol's and Morita's results [8, 6] on dimension of certain subspaces of products; the Borsuk-Dugundji theorem [1, 2] on extension of continuous mappings into Banach spaces; Ulmer's and Tkačenko's results [11, 10] concerning C-embedded subspaces. One would expect that one of the consequences of our Theorem should also be the Gulko result that Σ-products of metric spaces are normal. Unfortunately, we were not able to obtain it without imitating the main part of the original proof.

Suppose that $\{X_i: i\in I\}$ is an infinite family of nonvoid topological spaces. For $A=\prod_{i\in I} A_i\subset\prod_{i\in I} X_i$ we put $R(A)=\{i\in I: A_i\neq X_i\}$. We say that $U\subset\prod_I X_i$ depends on $J\subset I$ if $U=\mathrm{pr}_J^{-1}(\mathrm{pr}_J U)$ where pr_J is the projection $\prod_I X_i\to\prod_J X_i$. A basic open (regularly open) set in $\prod_I X_i$ is a set of the form $\prod_I U_i$ where all U_i's are (regularly) open sets and $R(\prod_I U_i)$ is finite.

If $\mathscr{A}$ and $\mathscr{B}$ are collections of subsets of X then $\mathscr{A}\wedge\mathscr{B}=\{A\cap B: A\in\mathscr{A}, B\in\mathscr{B}\}$ and $\mathscr{A}$ refines $\mathscr{B}$ means that each member of $\mathscr{A}$ is contained in some member of

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$\mathscr{B}$ and $\bigcup \mathscr{A} = \bigcup \mathscr{B}$ (notation $\mathscr{A} < \mathscr{B}$). $\mathscr{A}$ is said to be locally finite if for each $x \in \bigcup \mathscr{A}$ there is a neighbourhood of x meeting only finitely many members of $\mathscr{A}$.

A G_δ-closure of $A \subset X$ is the set $\{x \in X$: for every sequence $\{G_n\}$ of neighbourhoods of x, $\bigcap G_n \cap A \neq \emptyset\}$. A is said to be G_δ-dense in X if its G_δ-closure equals to X.

Regularly closed sets are those which are closures of open sets. Zero sets (sometimes called functionally closed) are pre-images of closed sets by continuous real-valued mappings.

In the next, let X be a subset of $\prod_I X_i$ and κ be an *infinate* cardinal. Although we shall consider products of metric spaces, the next lemma is valid for products of arbitrary topological spaces. Also, we shall state and prove it in a more general form than it is needed for our purposes.

A collection $\mathscr{A}$ is said to be weakly locally-($<\kappa$) in X if there is a π-base in X such that each of its elements meets less than κ members of $\mathscr{A}$. By $(\kappa)\prod_I X_i$ we denote the κ-modification of the usual (categorical) product $\prod_I X_i$. Hence the basic κ-neighbourhoods of $x \in \prod_I X_i$ in $(\kappa)\prod_I X_i$ are the intersections of less than κ basic open neighbourhoods in $\prod_I X_i$, i.e. basic $G_{<\kappa}$-sets, or basic κ-open sets. If $\mathscr{U}$ is a collection of κ-open sets in X then $\mathscr{B}^\kappa_{\mathscr{U}} = \{B$: B is a basic κ-open set and $B \cap X \subset U$ for some $U \in \mathscr{U}\}$. The closure of Y in $(\kappa)\prod_I X_i$ will be denoted by $\bar{Y}^\kappa$. If $\kappa = \omega$ then indices κ will be omitted. If κ is not specified then $\kappa = \omega$. If X_i are 1st countable spaces, then $(\omega_1)\prod X_i$ is the ω-box product of discrete spaces.

Lemma. *Let $\omega \leqslant \lambda \leqslant \kappa$, $\mathscr{U}$ be a weakly locally ($<\kappa$) κ-open family in $X \subset (\kappa)\prod_I X_i$, $J \in [I]^{<\kappa}$ and H be a basic κ-open set with $R(H) \subset J$. If $X \subset \overline{(\operatorname{int}_\kappa \bar{X}^k)}^\lambda$ then either $X \cap H \subset \bar{U}^\lambda$ for some $U \in \mathscr{U}$ or there is $A \in [I]^{<\kappa}$ such that $R(B) \cap A - J \neq \emptyset$ for each $B \in \mathscr{B}^\kappa_{\mathscr{U}}$ with $\operatorname{pr}_J B \supset \operatorname{pr}_J H$.*

Proof. We may assume $\kappa \leqslant |I|$. Denote $\tilde{\mathscr{B}} = \{B \in \mathscr{B}^\kappa_{\mathscr{U}}$: $\operatorname{pr}_J B \supset \operatorname{pr}_J H\}$. We may assume that $X \cap H$ is nonempty and that $R(B) \subset J$ for no $B \in \tilde{\mathscr{B}}$ (otherwise, clearly, $X \cap H \subset U$ for some $U \in \mathscr{U}$). Suppose that the second part of our assertion is not valid. Then there is a family $\{B_\alpha$: $\alpha \in \kappa\} \subset \tilde{\mathscr{B}}$ such that $R(B_\alpha) \cap R(B_\beta) \subset J$ for $\alpha \neq \beta$. For $\alpha \in \kappa$, choose $U_\alpha \in \mathscr{U}$ with $U_\alpha \supset B_\alpha \cap X$ and assume at first that all U_α's are different. Since $X \cap H \neq \emptyset$, there is a nonvoid basic open set G in $(\kappa)\prod_I X_i$ meeting less than κ members of $\mathscr{U}$, such that $G \subset H \cap \bar{X}^\kappa$. The last property entails the existence of $\alpha \in \kappa$ such that $R(G) \cap R(B_\alpha) \subset J$, $G \cap U_\alpha = \emptyset$; since $\operatorname{pr}_J G \subset \operatorname{pr}_J H \subset \operatorname{pr}_J B$ and $R(G) \cap R(B_\alpha) \subset J$, $G \cap B_\alpha$ must be nonempty, hence also $G \cap B_\alpha \cap X \neq \emptyset$ (because $G \subset \bar{X}^\kappa$), which contradicts $G \cap U_\alpha = \emptyset$. Consequently, we may suppose that U_α coincide with some U for all $\alpha \in \kappa$. Take $x \in X \cap H$ and its basic λ-neighbourhood V. Again there is a nonempty basic κ-open set $G \subset V \cap H \cap \bar{X}^\kappa$ and $\alpha \in \kappa$ with $R(G) \cap R(B_\alpha) \subset J$, hence $G \cap B_\alpha \neq \emptyset$ and $V \cap U \neq \emptyset$ as above, which proves $X \cap H \subset \bar{U}^\lambda$. □

Remarks. (1) If $\lambda = \kappa$ then the condition for X in our Lemma means exactly that $\bar{X}^\kappa$ is regularly closed in $(\kappa)\prod X_i$, which occurs e.g. if X is open or dense in $\prod X_i$

(for $\kappa=\omega$) or if X is a G_δ*-set* in $\prod X_i$ (for $\kappa=\omega_1$) or if $\bar{X}$ is $G_{<\kappa}$-set and X is κ-dense in $\bar{X}$.

(2) If the members of $\mathcal{U}$ are regularly λ-open in X, then the inclusion $X\cap H\subset\bar{U}^\lambda$ in Lemma is equivalent to $X\cap H\subset U$. The same assertion holds if e.g. H is open in $\prod X_i$ and int $\bar{U}^\lambda\subset U$.

The next simple assertion concerning finite products of metric spaces seems to be well-known. Since we were not able to find a reference, we put it here with the proof.

Proposition 1. *Let $\mathcal{R}$ be a family of open sets in a finite product $\prod_{i=1}^n X_i$ of metrizable spaces X_i. There is a σ-discrete in $\prod_1^n X_i$ locally finite regularly open refinement $\mathcal{A}$ of $\mathcal{R}$ composed of basic open sets.*

Proof. For $i\leqslant n$, let $d_i\leqslant 1$ be a metric inducing the topology of X_i; define the metric d on $\prod_1^n X_i$ by $d(x,y)=\max\{d_i(\mathrm{pr}_i x,\mathrm{pr}_i y): i\leqslant n\}$. For $x\in\bigcup\mathcal{R}$ define $hx=\sup\{r$: the d-ball in $\prod_1^n X_i$ with the centre x and radius r is contained in some $R\in\mathcal{R}\}$. Clearly, $h:\bigcup\mathcal{R}\to\,]0,1]$ is a Lipschitz mapping with the constant 1 (indeed, if $hx>hy$, $d(x,y)<hx$ then $hy\geqslant hx-d(x,y)$). For $k\in\omega$, $i\leqslant n$, take a locally finite and σ-discrete regularly open cover $\mathcal{A}_k^i$ of X_i such that d_i-diameters of members $\mathcal{A}_k^i$ are less than 2^{-k-1}. Define $\mathcal{A}_k=\{\prod_1^n A^i: A^i\in\mathcal{A}_k^i$, $\prod_1^n A^i\subset R$ for some $R\in\mathcal{R}$, $\prod_1^n A^i\subset h^{-1}(]2^{-k-1},2^{-k-3}[)\}$, $\mathcal{A}=\bigcup_{k\in\omega}\mathcal{A}_k$. Then $\mathcal{A}$ is the requested collection.

It is clear that $\mathcal{A}$ is σ-discrete in $\prod X_i$ and that each member of $\mathcal{A}$ is a basic regularly open set and it is contained in some member of $\mathcal{R}$. It remains to prove that $\bigcup\mathcal{A}=\bigcup\mathcal{R}$ and that $\mathcal{A}$ is locally finite in $\bigcup\mathcal{R}$. For $x\in\bigcup\mathcal{R}$ there is $k\in\omega$ such that $hx\in\,]2^{-k},2^{-k+2}[$, and for $i\leqslant n$, take any $A^i\in\mathcal{A}_k^i$ containing $\mathrm{pr}_i x$. Then $\prod_i^n A^i\in\mathcal{A}_k$ since $\prod_1^n A^i$ is contained in the ball around x with radius 2^{-k-1} and by definition of hx, it is a part of some $R\in\mathcal{R}$ and if $y\in\prod_1^n A^i$ then $hy\geqslant hx-d(x,y)\geqslant 2^{-k}-2^{-k-1}=2^{-k-1}$, $hy\leqslant hx+d(x,y)\leqslant 2^{-k+2}+2^{-k-1}\leqslant 2^{-k+3}$. There is a neighbourhood of x meeting members of $\mathcal{A}_k$ for finitely many indices k and each $\mathcal{A}_k$ is locally finite in $\bigcup\mathcal{R}$-consequently, $\mathcal{A}$ is locally finite in $\bigcup\mathcal{R}$.

Remark. If Ind $X_i=0$ for each $i\leqslant n$ one can get stronger results, namely that the refinement $\mathcal{A}$ is composed of disjoint basic open sets. The proof is an easy modification of the above proof, where one takes disjoint families $\mathcal{A}_k^i$ such that $\mathcal{A}_{k+1}^i$ refines $\mathcal{A}_k^i$, and in $\mathcal{A}$ one uses only those $\prod A^i$ not contained in other such sets.

The next Proposition may be known, too. It is trivial for 1st-countable spaces, less trivial for their products (in fact, in products one must add a condition on X).

Proposition 2. *Let $\{X_i: i\in I\}$ be a family of* 1st-*countable spaces and $X\subset\prod X_i$ with $X\subset\mathrm{int}_{\omega_1}\bar{X}$. Let $\mathcal{P}$ be a family of open sets in X.*

(i) *The set $M=\{x\in\prod X_i$: $\mathcal{P}$ is locally finite at $x\}$ is G_δ-closed in $\prod X_i$.*

(ii) *If $\mathcal{P}$ is weakly locally finite in X and each $P\in\mathcal{P}$ is regularly open in X, then the set $\bigcup\mathcal{B}_{\mathcal{P}}$ is G_δ-closed in $\prod X_i$.*

Remark. If one assumes $X \subset \overline{\operatorname{int}_{\omega_1} \bar{X}^{\omega_1}}$, then the condition on $P \in \mathcal{P}$ may be weakened to int $\bar{P}^{\omega_1} \subset P$ (the operations are in X).

Proof. For each $i \in I$, let $\{U_n^i(a): n \in \omega\}$ be a countable base of open neighbourhoods at a in X_i.

(i) We will prove that $\bar{M}^{\omega_1} \subset M$. Take $x \in \bar{M}^{\omega_1}$ and suppose that $x \notin M$. By induction on $k \in \omega$, we can define sets $I_k \in [I]^{\omega}$ (say, $I_k = \{i_k^m: m \in \omega\}$) and for each n (and a given k) one can find infinitely many basic open sets $\{B_j(n, k): j \in \omega\}$ such that each $B_j(n, k) \cap X$ is contained in some $P_j(n, k)$, and $P_j(n, k) \neq P_i(n, k)$ for $j \neq i$, $U(n, k) \cap B_j(n, k) \cap X$ contains a point $z_j(n, k) \in \operatorname{int}_{\omega_1} \bar{X}$ (here $U(n, k) = \prod\{U_n^i(\mathrm{pr}_i x): i \in \{i_k^m: m \leq n\}\} \times \prod\{X_i \mid i \in I - \{i_k^m: m \leq n\}\}$). Thus there is $A_j(n, k) \in [I]^{<\omega}$ such that $\{x: \mathrm{pr}_{A_j(n,k)} x = \mathrm{pr}_{A_j(n,k)} z_j(n, k)\} \subset \bar{X}$ and we assume that $I_{k+1} \supset I_k \cup \bigcup\{R(B_j(n, k) \cup A_j(n, k): j \in \omega, n \in \omega\}$. For $J = \bigcup\{I_k: k \in \omega\}$ take $y \in M \cap \mathrm{pr}_J^{-1} \mathrm{pr}_J x$ and its basic neighbourhood U meeting only finite many members of $\mathcal{P}$. There is $k \in \omega$ such that $I_k \supset R(U) \cap J$ and there is $n \in \omega$ such that $\mathrm{pr}_{I_k} U(n, k) \subset \mathrm{pr}_{I_k} U$. For each $j \in \omega$, $z_j(n, k) \in U(n, k) \cap B_j(n, k)$, hence (since $A_j(n, k) \subset J$) $U \cap B_j(n, k) \cap \bar{X} \neq \emptyset$, hence $U \cap \underline{P_j}(n, k) \neq \emptyset$, which is a contradiction.

(ii) Take $x \in \overline{\bigcup \mathcal{B}_{\mathcal{P}}}^{\omega_1}$. If $x \notin \bar{X}$ then clearly $x \in \bigcup \mathcal{B}_{\mathcal{P}}$, so suppose $x \in \bar{X}$. We can define by induction on $\alpha \in \omega_1$, sets $I_\alpha \in [I]^{\omega}$, $B_\alpha \in \mathcal{B}_{\mathcal{P}}$ such that $\mathrm{pr}_{I_\alpha} B_\alpha \supset \mathrm{pr}_{I_\alpha} x$ and $I_\alpha \supset \bigcup\{I_\beta \cup R(B_\alpha): \beta < \alpha\}$. We may suppose that $\{R(B_\alpha): \alpha \in \omega_1\}$ forms a Δ-system (denote by D its kernel) with $D \subset I_0$ and that $U = \bigcap\{\mathrm{pr}_D^{-1} \mathrm{pr}_D B_\alpha: \alpha \in \omega_1\}$ is a neighbourhood of x in $\prod X_i$. We may also assume that there is $P \in \mathcal{P}$ such that $B_\alpha \cap X \subset P$ for all $\alpha \in \omega_1$. Indeed, there is a basic open set $V \subset U$ with $V \cap X \neq \emptyset$ meeting only finitely many members of $\mathcal{P}$, and because of $V \cap \operatorname{int}_{\omega_1} \bar{X} \neq \emptyset$ there is an $A \in [I]^{<\omega}$ and $z \in V$ with $\mathrm{pr}_A^{-1} \mathrm{pr}_A z \subset V \subset \bar{X}$, hence $V \cap B_\alpha \cap X \neq \emptyset$ for uncountably many $\alpha \in \omega_1$. Now, if $y \in U \cap X$ and V is a basic neighbourhood of y, then again $V \cap B_\alpha \cap X \neq \emptyset$ for some α, which entails $U \cap X \subset \bar{P}$ and hence $U \cap X \subset P$. Consequently, $x \in U \subset \bigcup \mathcal{B}_{\mathcal{P}}$. □

The second part of Proposition 2 in the form we need in our Theorem can be proved directly (and a little more easily) using the procedure of the proof of our Theorem. However, the assertion seem to be interesting to be formulated in a more general form as a proposition.

Theorem. *Let $\{X_i \mid i \in I\}$ be a family of metric spaces and $\omega \leq \lambda \leq \kappa \leq \omega_1$. Let $X \subset \prod X_i$ be such that $X \subset \overline{(\operatorname{int}_\kappa \bar{X}^\kappa)}^\lambda$ and $\mathcal{U}$ be a locally finite open family in X with* int $\bar{U}^\lambda \subset U$ *(relative to X) for each $U \in \mathcal{U}$. Then there is a family V composed of basic regularly open sets in $\prod X_i$ such that*

(i) *$\mathcal{V}$ is σ-discrete in $\prod X_i$;*
(ii) *$\mathcal{V}$ is locally finite;*
(iii) *$\mathcal{V} \wedge (X) < \mathcal{U}$;*
(iv) *$\bigcup \mathcal{V}$ is G_δ-closed in $\prod X_i$ (and equals to $\bigcup \mathcal{B}_{\mathcal{U}}$).*

Proof. For technical reasons, define for $i<0$ $\mathcal{V}_i=(\prod X_j)$ and for $W\in\mathcal{V}_{i-1}$ put $\mathcal{V}_{i,W}=\mathcal{V}_i$, $A_{i,W}=\emptyset$. Now, we shall define by induction for all $i\in\omega$ families $\mathcal{V}_i$, $\mathcal{V}_{i,V}$ and sets $A_{i,V}$ (also the relation $<$ being the transitive hull of the relation $\{((i-1,W),(i,V))\mid V\in\mathcal{V}_{i-1,W}\}$).

Take $i\in\omega$, $W\in\mathcal{V}_{i-2}$, $V\in\mathcal{V}_{i-1,W}$ and put in our Lemma $J=A_{i-1,W}$, $H=V$, which implies that either there is $U\in\mathcal{U}$ such that $X\cap V\subset U$ (then we define $\mathcal{V}_{i,V}=(V)$, $A_{i,V}=A_{i-1,W}$) or there is a set $A_{i,V}\in[I]^{<\kappa}$ (which we shall regard also as a sequence) such that $A_{i,V}\supset A_{i-1,W}$ and $R(B)\cap A_{i,V}-A_{i-1,W}\neq\emptyset$ for all $B\in\mathcal{B}_{\mathcal{U}}$ with $\mathrm{pr}_{A_{i-1,W}}B\supset\mathrm{pr}_{A_{i-1,W}}V$. Then we define $\mathcal{V}_{i,V}=(V)\wedge\mathrm{pr}_{A_{i,V}}^{-1}\mathcal{C}$ for a convenient $\mathcal{C}$: $\mathcal{C}$ is a σ-discrete (in $\prod_{A_{i,V}}X_i$) locally finite family composed of basic regularly open sets in $\prod_{A_{i,V}}X_i$ refining $\mathrm{pr}_{A_{i,V}}\mathcal{B}_{\mathcal{U}}$ such that $\operatorname{diam}\mathrm{pr}_jC<2^{-i-1}$ for each $C\in\mathcal{C}$ and each j from the initial segments of $A_{k,S}$ of length $i-k$ for every $(k,S)<(i,V)$.

The existence of $\mathcal{C}$ for $\kappa=\omega$ or finite $A_{i,V}$ follows directly from our Proposition (in that case one may assume $R(C)=A_{i,V}$). For $\kappa=\omega_1$ and infinite $A_{i,V}$ proceed as follows: $\mathrm{pr}_{A_{i,V}}\mathcal{B}_{\mathcal{U}}$ is an open family in the hereditarily paracompact space $\prod_{A_{i,V}}X_j$, so it has a regularly open locally finite refinement $\mathcal{E}$; assuming that our Theorem was proved for $\kappa=\omega$ ($I=A_{i,V}$, $X=\prod_{A_{i,V}}X_j$, $\mathcal{U}$ equals to $\mathcal{E}$), there is a σ-discrete (in $\prod_{A_{i,V}}X_j$) locally finite family $\mathcal{D}$ composed of basic regularly open sets in $\prod_{A_{i,V}}X_j$ refining $\mathcal{E}$; for every $D\in\mathcal{D}$ put A_D to be the union of $R(D)$ and of all the above mentioned initial segments of $A_{k,S}$ and find a σ-discrete locally finite (all in $\prod_{A_D}X_i$) basic regularly open refinement $\mathcal{C}_D$ of $\mathrm{pr}_{A_D}D$ with $\operatorname{diam}\mathrm{pr}_jC<2^{-i-1}$ for each $j\in A_D$—then $\mathcal{C}=\bigcup_{D\in\mathcal{D}}\mathrm{pr}_{A_D}^{-1}\mathcal{C}_D$.

Let $\mathcal{V}=\mathcal{B}_{\mathcal{U}}\cap\bigcup_{i\in\omega}\mathcal{V}_i$. Clearly, $\mathcal{V}$ is a σ-discrete family composed of basic regularly open sets in $\prod X_i$. To prove (ii) and (iii), take $B\in\mathcal{B}_{\mathcal{U}}$, $x\in B$ and such $k>0$ that the ball around pr_ix with diameter 2^{-k} is contained in pr_iB for each $i\in R(B)$. There is an $n>k$ and a basic open neighbourhood G of x meeting only finitely many members of $\mathcal{V}_n$, say V_n^j for $j\leq l$, and such that $\operatorname{diam}\mathrm{pr}_iV_n^j<2^{-k-1}$ for each $i\in R(B)\cap A_{n,V_n^j}$, $j\leq l$. Take $m>n+|R(B)|$, a basic open $W\subset G$ containing x and meeting only finitely many of V's from $\mathcal{V}_m$. Then $V\in\mathcal{V}$ whenever $V\in\mathcal{V}_n$ and $V\cap W\neq\emptyset$ for otherwise $R(B)\cap A_{i+1,V_i}-A_{i,V_{i-1}}\neq\emptyset$ for the corresponding interval of $(i,V_i)\leq(m,V)$ with $i\geq n$ which entails $|R(B)|>m-n$—a contradiction. Therefore, $\bigcup\mathcal{V}=\bigcup\mathcal{B}_{\mathcal{U}}$ and $\mathcal{V}$ is locally finite. Proposition 2(ii)) implies (iv). □

Remarks. (1) If $\mathcal{U}$ covers X, then $\mathcal{B}_{\mathcal{U}}$ contains $X\cup(\prod X_i-\bar{X})$ and, hence, it contains also G_δ-closure of the last set. A special case of the assertion of Theorem is thus the following:

There exists a σ-discrete (in $\prod X_i$) locally finite (in G_δ-closure of $X\cup(\prod X_i-\bar{X})$) collection $\mathcal{V}$ composed of basic open sets such that $\mathcal{V}\wedge(X)$ refines $\mathcal{U}$.

Therefore, $\mathcal{V}$ is locally finite in $\prod X_i$ provided X is G_δ-dense in $\bar{X}$, which occurs e.g. if X is closed or contains a Σ-product of $\{X_i\}$.

(2) If we use Lemma in its full generality, we obtain:

Every weakly locally finite regularly open cover of X is refined by a locally finite regularly open cover of X.

Another possibility is to notice that, in some situations, our Lemma is valid for point-finite families. Indeed, if all B_n's in the proof of Lemma lie in $\bar{X}$ and $\mathrm{pr}_K \bar{X} \subset \mathrm{pr}_K X$ for all $K \in [I]^\omega$, then $X \cap \bigcap B_n \neq \emptyset$ and again one can assume that all $X \cap B_n$ lie in the same $U \in \mathcal{U}$. Thus we get:

If X is G_δ-dense in $\prod X_i$ then every point-finite regularly open cover of X is refined by a locally finite regularly open cover of X.

(3) Our Theorem does not hold for $\kappa \geq \omega_2$ even if $X = \prod X_i$. The power N^{ω_1} is not normal, so that there is a finite open cover $\mathcal{W}$ of N^{ω_1} which is not uniformizable, i.e. it cannot be refined by a locally finite cozero cover. Since $(\omega_2)N^{\omega_1}$ is discrete, for any $W \subset N^{\omega_1}$ and any uncountable cardinal τ, $\mathrm{pr}_{\omega_1}^{-1} W$ is regularly open in $(\omega_2)N^\tau$. Consequently, $\mathcal{U} = \mathrm{pr}_{\omega_1}^{-1} \mathcal{W}$ satisfies the conditions of Theorem for $\kappa = \omega_2$, $X = N^\tau$, but there is no $\mathcal{V}$ having the required properties (otherwise the trace of $\mathcal{V}$ on a canonical embedding of N^{ω_1} into N^τ would be a locally finite cozero refinement of $\mathcal{W}$).

(4) In the case of strongly 0-dimensional spaces ($\dim X_i = 0$) we may take in the proof of Theorem the refining families $\mathcal{C}$ to be disjoint and the procedure yields disjoint $\mathcal{V}$ (see Corollary 3).

In the next corollaries, we assume that $\{X_i;\ i \in I\}$ is a family of metric spaces.

Corollary 1 (Ščepin [9]). *Every regularly closed set in* $\prod X_i$ *is a zero set.*

We shall prove the following more general result:

Corollary 2 (Klebanov [5]). *Every closure of a union of* G_δ*-sets in* $\prod X_i$ *is a zero set.*

Proof. Suppose that M is the closure of a union P of G_δ-sets. In theorem, put $X = \prod X_i$, $\mathcal{U} = (X - M)$, $\lambda = \kappa = \omega_1$. Then indeed $\mathrm{int}\, \overline{X - M}^{\omega_1} \subset \mathrm{int}(X - P) = X - M$, and as a result, $X - M$ is a union of a σ-discrete family of cozero sets (members of $\mathcal{V}$), thus $X - M$ is a cozero set, too.

In the next corollaries, X is a subset of $\prod X_i$ satisfying $X \subset \overline{(\mathrm{int}_\kappa \bar{X}^\kappa)}^\lambda$ for some λ, κ with $\omega \leq \lambda \leq \kappa \leq \omega_1$ (for instance, if X is open or dense in $\prod X_i$ or if X is a G_δ-set in $\prod X_i$ or $\bar{X}$ is G_δ-set and X is G_δ-dense in $\bar{X}$).

The next result generalizes Pol's [8] and Morita's [6] theorem on products of zero-dimensional spaces: if X is dense in $\prod X_i$ and $\dim X_i = 0$ for each i, then $\dim X = 0$ (Morita for separable spaces). On the other hand, Pol's result is much more general in the sense that it is stated for higher dimensions.

Corollary 3. *If* $\dim X_i = 0$ *for all* i, *then* $\dim X = 0$.

Proof follows from the Remark 4 to Theorem and the Remark to Proposition 1.

Corollary 4. *The fine uniformity of* X *has for its base traces of locally fine* (*in the* G_δ*-closure of* X), σ*-discrete* (*in* $\prod X_i$) *collections of regularly open basic sets.*

Proof follows from the facts that every uniformizable cover is refined by a locally finite regularly open cover, and that every locally finite cozero cover is uniformizable.

Corollary 5. *If X is G_δ-closed in $\bar{X}$ then the fine uniformity of X is the restriction of the fine uniformity of $\prod X_i$.*

Recall that X is G_δ-closed in $\bar{X}$ if e.g. X is a Σ-product of X_i's or, more generally, if $\mathrm{pr}_J X = \prod_J X_i$ for every $J \in [I]^\omega$. In fact, X is G_δ-closed in $\bar{X}$ iff $\mathrm{pr}_J \bar{X} = \mathrm{pr}_J X$ for every $J \in [I]^\omega$.

It is well-known that "the restriction of the fine uniformity on P to its subspace Q is fine" is equivalent to "every continuous map on Q into a Banach space can be continuously extended onto P" (to prove that, use the Borsuk–Dugundji extension of the Tietze theorem). Thus the last two corollaries can be reformulated.

Corollary 6. *Every continuous mapping on X into a Banach space can be continuously extended to the G_δ-closure of $X \cup (\prod X_i - \bar{X})$, in particular to $\prod X_i$ provided X is G_δ-dense in $\bar{X}$.*

As a special case we get

Corollary 7. *Every closed G_δ-set (in particular, every regularly closed set) in $\prod X_i$ is C-embedded in $\prod X_i$.*

In the case that X is regularly closed we can improve Corollary 6 (one could take even metrizable linear spaces instead of normed ones):

Corollary 8. *Every continuous mapping on a regularly closed set X into a normed linear space can be continuously extended onto $\prod X_i$.*

Proof. Take $f: X \to E$. Denote metric on E by ρ. Consider pseudometric $f^{-1}(\rho)$. It can be extended to $\prod X_i$ by Corollary 5 and now use the Borsuk–Dugundji extension of the Tietze theorem. □

From Corollary 4 we get

Corollary 9. *Any continuous map on X into a topologically complete space (i.e. a space induced by a complete uniformity) admits a continuous extension to G_δ-closure of X.*

Recall that topologically complete spaces are e.g. paracompact spaces or real compact spaces.

Remark. Ulmer proved in [11] that Σ-products of metrizable space X_i (in fact of more general spaces) are C-embedded in $\prod X_i$; Tkačenko in [10] generalized his result to the extent that every G_δ-dense subset of a product of metrizable spaces X_i (in fact of spaces X_i with countable open-tightness) is C-embedded in X_i. Our last results generalize those extension theorems for product of metrizable spaces both in more general ranges and domains.

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