Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 157 (2009) 211–218
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Global defensive k-alliances in graphs

J.A. Rodríguez-Velázquez[a,*], J.M. Sigarreta[b]

[a] *Department of Computer Engineering and Mathematics, Rovira i Virgili University of Tarragona, Av. Països Catalans 26, 43007 Tarragona, Spain*
[b] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés (Madrid), Spain*

Received 20 November 2006; received in revised form 5 November 2007; accepted 12 February 2008
Available online 24 March 2008

Abstract

Let $\Gamma = (V, E)$ be a simple graph. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $\delta_X(v)$ denotes the number of neighbors v has in X. A nonempty set $S \subseteq V$ is a *defensive k-alliance* in $\Gamma = (V, E)$ if $\delta_S(v) \geq \delta_{\bar{S}}(v) + k$, $\forall v \in S$. A defensive k-alliance S is called *global* if it forms a dominating set. The *global defensive k-alliance number* of Γ, denoted by $\gamma_k^a(\Gamma)$, is the minimum cardinality of a defensive k-alliance in Γ. We study the mathematical properties of $\gamma_k^a(\Gamma)$.

Keywords: Defensive alliances; Alliances in graphs; Domination

1. Introduction

Since (defensive, offensive and dual) alliances were first introduced by Kristiansen, Hedetniemi and Hedetniemi [12], several authors have studied their mathematical properties [2,4,3,6,9,13,14,16,18,20,22] as well as the complexity of computing minimum cardinality of alliances [1,7,10,11]. The minimum cardinality of a defensive (respectively, offensive or dual) alliance in a graph Γ is called the defensive (respectively, offensive or dual) alliance number of Γ. The mathematical properties of defensive alliances were first studied in [12] where several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliances was investigated in [9] where several bounds on the global (strong) defensive alliance number were obtained. The dual alliances were introduced as powerful alliances in [2,3]. In [14] there were obtained several tight bounds on the defensive (offensive and dual) alliance number. In particular, there was investigated the relationship between the alliance numbers of a graph and its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. Moreover, the study of global defensive (offensive and dual) alliances in a planar graph was initiated in [16] and the study of defensive alliances in the line graph of a simple graph was initiated in [22]. The particular case of global alliances in trees has been investigated in [4]. For many properties of offensive alliances, the readers may refer to [6,13,15,23].

A generalization of (defensive and offensive) alliances called k-alliances was presented by Shafique and Dutton [18,19] where was initiated the study of k-alliance free sets and k-alliance cover sets. The aim of this work is to

* Corresponding author.
E-mail addresses: juanalberto.rodriguez@urv.cat (J.A. Rodríguez-Velázquez), josemaria.sigarreta@uc3m.es (J.M. Sigarreta).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2008.02.006

study mathematical properties of defensive k-alliances. We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v_i \in V$ by $\delta(v_i)$ (or by d_i for short) and the degree sequence of Γ by $d_1 \geq d_2 \geq \cdots \geq d_n$. The subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$ and the complement of the set S in V will be denoted by $\bar{S}$.

A nonempty set $S \subseteq V$ is a *defensive k-alliance* in $\Gamma = (V, E)$, $k \in \{-d_1, \ldots, d_1\}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + k. \tag{1}$$

A vertex $v \in S$ is said to be *k-satisfied* by the set S if (1) holds. Notice that (1) is equivalent to

$$\delta(v) \geq 2\delta_{\bar{S}}(v) + k. \tag{2}$$

A defensive (-1)-alliance is a *defensive alliance* and a defensive 0-alliance is a *strong defensive alliance* as defined in [12]. A defensive 0-alliance is also known as a *cohesive set* [21].

The *defensive k-alliance number* of Γ, denoted by $a_k(\Gamma)$, is defined as the minimum cardinality of a defensive k-alliance in Γ. Notice that

$$a_{k+1}(\Gamma) \geq a_k(\Gamma). \tag{3}$$

The defensive (-1)-alliance number of Γ is known as the *alliance number* of Γ and the defensive 0-alliance number is known as the *strong alliance number*, [12,8,9]. For instance, in the case of the 3-cube graph, $\Gamma = Q_3$, every set composed by two adjacent vertices is a defensive alliance of minimum cardinality and every set composed by four vertices whose induced subgraph is isomorphic to the cycle C_4 is a strong defensive alliance of minimum cardinality. Thus, $a_{-1}(Q_3) = 2$ and $a_0(Q_3) = 4$.

For some graphs, there are some values of $k \in \{-d_1, \ldots, d_1\}$, such that defensive k-alliances do not exist. For instance, for $k \geq 2$ in the case of the star graph S_n, defensive k-alliances do not exist. By (2) we conclude that, in any graph, there are defensive k-alliances for $k \in \{-d_1, \ldots, d_n\}$. For instance, a defensive (d_n)-alliance in $\Gamma = (V, E)$ is V. Moreover, if $v \in V$ is a vertex of minimum degree, $\delta(v) = d_n$, then $S = \{v\}$ is a defensive k-alliance for every $k \leq -d_n$. Therefore, $a_k(\Gamma) = 1$, for $k \leq -d_n$. For the study of the mathematical properties of $a_k(\Gamma)$, $k \in \{d_n, \ldots, d_1\}$, we cite [17].

A set $S \subset V$ is a *dominating set* in $\Gamma = (V, E)$ if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in S). The *domination number* of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ.

A defensive k-alliance S is called *global* if it forms a dominating set. The *global defensive k-alliance number* of Γ, denoted by $\gamma_k^a(\Gamma)$, is the minimum cardinality of a defensive k-alliance in Γ. Clearly,

$$\gamma_{k+1}^a(\Gamma) \geq \gamma_k^a(\Gamma) \geq \gamma(\Gamma) \quad \text{and} \quad \gamma_k^a(\Gamma) \geq a_k(\Gamma). \tag{4}$$

The global defensive (-1)-alliance number of Γ is known as the *global alliance number* of Γ and the global defensive 0-alliance number is known as the *global strong alliance number* [9]. For instance, in the case of the 3-cube graph, $\Gamma = Q_3$, every set composed by four vertices whose induced subgraph is isomorphic to the cycle C_4 is a global (strong) defensive alliance of minimum cardinality. Thus, $\gamma_{-1}^a(Q_3) = \gamma_0^a(Q_3) = 4$.

For some graphs, there are some values of $k \in \{-d_1, \ldots, d_1\}$, such that global defensive k-alliances do not exist. For instance, for $k = d_1$ in the case of nonregular graphs, defensive k-alliances do not exist. Therefore, the bounds showed in this paper on $\gamma_k^a(\Gamma)$, for $k \leq d_1$, are obtained by supposing that the graph Γ contains defensive k-alliances. Notice that for any graph Γ, every dominating set is a global defensive $(-d_1)$-alliance. Hence, $\gamma_{-d_1}^a(\Gamma) = \gamma(\Gamma)$. Moreover, for any d_1-regular graph of order n, $\gamma_{d_1-1}^a(\Gamma) = \gamma_{d_1}^a(\Gamma) = n$.

2. Global defensive k-alliance number

Theorem 1. *Let S be a global defensive k-alliance of minimum cardinality in Γ. If $W \subset S$ is a dominating set in Γ, then for every $r \in \mathbb{Z}$ such that $0 \leq r \leq \gamma_k^a(\Gamma) - |W|$,*

$$\gamma_{k-2r}^a(\Gamma) + r \leq \gamma_k^a(\Gamma).$$

Proof. We can take $X \subset S$ such that $|X| = r$. Hence, for every $v \in Y = S - X$,

$$\begin{aligned}\delta_Y(v) &= \delta_S(v) - \delta_X(v)\\ &\geq \delta_{\bar{S}}(v) + k - \delta_X(v)\\ &= \delta_{\bar{Y}}(v) + k - 2\delta_X(v)\\ &\geq \delta_{\bar{Y}}(v) + k - 2r.\end{aligned}$$

Therefore, Y is a defensive $(k-2r)$-alliance in Γ. Moreover, as $W \subset Y$, Y is a dominating set and, as a consequence, $\gamma^a_{k-2r}(\Gamma) \leq \gamma^a_k(\Gamma) - r$. □

Notice that if every vertex of Γ has even degree and k is odd, $k = 2l-1$, then every defensive $(2l-1)$-alliance in Γ is a defensive $(2l)$-alliance. Hence, in such a case, $a_{2l-1}(\Gamma) = a_{2l}(\Gamma)$ and $\gamma^a_{2l-1}(\Gamma) = \gamma^a_{2l}(\Gamma)$. Analogously, if every vertex of Γ has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in Γ is a defensive $(2l+1)$-alliance. Hence, in such a case, $a_{2l}(\Gamma) = a_{2l+1}(\Gamma)$ and $\gamma^a_{2l}(\Gamma) = \gamma^a_{2l+1}(\Gamma)$. For instance, for the complete graph of order n we have

$$\begin{aligned}n &= \gamma^a_{n-1}(K_n) = \gamma^a_{n-2}(K_n)\\ &\geq \gamma^a_{n-3}(K_n) = \gamma^a_{n-4}(K_n) = n-1\\ &\quad \cdots\\ &\geq \gamma^a_{2-n}(K_n) = \gamma^a_{3-n}(K_n) = 2\\ &\geq \gamma^a_{1-n}(K_n) = 1.\end{aligned}$$

Therefore, for every $k \in \{1-n, \ldots, n-1\}$, and for every $r \in \{0, \ldots, \frac{k+n-1}{2}\}$,

$$\gamma^a_{k-2r}(K_n) + r = \gamma^a_k(K_n). \tag{5}$$

Moreover, notice that for every $k \in \{1-n, \ldots, n-1\}$, $\gamma^a_k(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$.

It was shown in [9] that

$$\frac{\sqrt{4n+1}-1}{2} \leq \gamma^a_{-1}(\Gamma) \leq n - \left\lceil \frac{d_n}{2} \right\rceil \tag{6}$$

and

$$\sqrt{n} \leq \gamma^a_0(\Gamma) \leq n - \left\lfloor \frac{d_n}{2} \right\rfloor. \tag{7}$$

Here we generalize the previous results to defensive k-alliances.

Theorem 2. *For any graph Γ, $\frac{\sqrt{4n+k^2}+k}{2} \leq \gamma^a_k(\Gamma) \leq n - \left\lfloor \frac{d_n-k}{2} \right\rfloor$.*

Proof. If $d_n \leq k$, then $\gamma^a_k(\Gamma) \leq n \leq n - \left\lfloor \frac{d_n-k}{2} \right\rfloor$. Otherwise, consider $u \in V$ such that $\delta(u) \geq \left\lfloor \frac{d_n+d_1}{2} \right\rfloor$. Let $X \subset V$ be the set of neighbors u has in Γ, $X = \{w \in V : w \sim u\}$. Let $Y \subset X$ be a vertex set such that $|Y| = \left\lfloor \frac{d_n-k}{2} \right\rfloor$. In such a case, the set $V - Y$ is a global defensive k-alliance in Γ. That is, $V - Y$ is a dominating set and for every $v \in V - Y$ we have $\frac{\delta(v)-k}{2} \geq \left\lfloor \frac{d_n-k}{2} \right\rfloor \geq \delta_Y(v)$. Therefore, $\gamma^a_k(\Gamma) \leq n - \left\lfloor \frac{d_n-k}{2} \right\rfloor$.

On the other hand, let $S \subseteq V$ be a dominating set in Γ. Then,

$$n - |S| \leq \sum_{v \in S} \delta_{\bar{S}}(v). \tag{8}$$

Moreover, if S is a defensive k-alliance in Γ,

$$k|S| + \sum_{v \in S} \delta_{\bar{S}}(v) \leq \sum_{v \in S} \delta_S(v) \leq |S|(|S|-1). \tag{9}$$

Hence, solving

$$0 \le |S|^2 - k|S| - n \tag{10}$$

we deduce the lower bound. □

The upper bound is attained, for instance, for the complete graph $\Gamma = K_n$ for every $k \in \{1-n, \ldots, n-1\}$. The lower bound is attained, for instance, for the 3-cube graph $\Gamma = Q_3$, in the following cases: $2 \le \gamma^a_{-3}(Q_3)$ and $4 \le \gamma_1(Q_3) = \gamma_0(Q_3)$.

It was shown in [9] that for any bipartite graph Γ of order n and maximum degree d_1,

$$\gamma^a_{-1}(\Gamma) \ge \left\lceil \frac{2n}{d_1+3} \right\rceil \quad \text{and} \quad \gamma^a_0(\Gamma) \ge \left\lceil \frac{2n}{d_1+2} \right\rceil.$$

Here we generalize the previous bounds to defensive k-alliances. Moreover, we show that the result is not restrictive to the case of bipartite graphs.

Theorem 3. *For any graph Γ,* $\gamma^a_k(\Gamma) \ge \left\lceil \frac{n}{\left\lfloor \frac{d_1-k}{2} \right\rfloor + 1} \right\rceil$.

Proof. If S denotes a defensive k-alliance in Γ, then

$$d_1 \ge \delta(v) \ge 2\delta_{\bar{S}}(v) + k, \quad \forall v \in S.$$

Therefore,

$$\left\lfloor \frac{d_1-k}{2} \right\rfloor \ge \delta_{\bar{S}}(v), \quad \forall v \in S. \tag{11}$$

Hence,

$$|S| \left\lfloor \frac{d_1-k}{2} \right\rfloor \ge \sum_{v \in S} \delta_{\bar{S}}(v). \tag{12}$$

Moreover, if S is a dominating set, S satisfies inequality (8). The result follows by (8) and (12). □

The above bound is tight. For instance, for the Petersen graph the bound is attained for every k: $3 \le \gamma^a_{-3}(\Gamma)$, $4 \le \gamma^a_{-2}(\Gamma) = \gamma^a_{-1}(\Gamma)$, $5 \le \gamma_0(\Gamma) = \gamma_1(\Gamma)$ and $10 \le \gamma_2(\Gamma) = \gamma_3(\Gamma)$. For the 3-cube graph $\Gamma = Q_3$, the above theorem leads to the following exact values of $\gamma^a_k(Q_3)$: $2 \le \gamma^a_{-3}(Q_3)$, $4 \le \gamma_0(Q_3) = \gamma_1(Q_3)$ and $8 \le \gamma_2(Q_3) = \gamma_3(Q_3)$.

Hereafter, we denote by $\mathcal{L}(\Gamma) = (V_l, E_l)$ the line graph of a simple graph Γ. The degree of the vertex $e = \{u, v\} \in V_l$ is $\delta(e) = \delta(u) + \delta(v) - 2$. If the degree sequence of Γ is $d_1 \ge d_2 \ge \cdots \ge d_n$, then the maximum degree of $\mathcal{L}(\Gamma)$, denoted by Δ_l, is bounded by $\Delta_l \le d_1 + d_2 - 2$.

Corollary 4. *For any graph Γ of size m and maximum degrees $d_1 \ge d_2$,*

$$\gamma^a_k(\mathcal{L}(\Gamma)) \ge \left\lceil \frac{m}{\left\lfloor \frac{d_1+d_2-2-k}{2} \right\rfloor + 1} \right\rceil.$$

The above bound is attained for $k \in \{-3, -2, -1, 2, 3\}$ in the case of the complete bipartite graph $\Gamma = K_{1,4}$. Notice that $\mathcal{L}(K_{1,4}) = K_4$ and $\gamma^a_{-3}(K_4) = 1$, $\gamma^a_{-2}(K_4) = \gamma^a_{-1}(K_4) = 2$, $\gamma^a_2(K_4) = \gamma^a_3(K_4) = 4$.

In the case of cubic graphs[1] $\gamma(\Gamma) = \gamma^a_{-3}(\Gamma) \le \gamma^a_{-2}(\Gamma) = \gamma^a_{-1}(\Gamma) \le \gamma^a_0(\Gamma) = \gamma^a_1(\Gamma) \le \gamma^a_2(\Gamma) = \gamma^a_3(\Gamma) = n$. So, in this case we only study, $\gamma^a_{-1}(\Gamma)$ and $\gamma^a_0(\Gamma)$.

Theorem 5. *For any cubic graph Γ, $\gamma^a_{-1}(\Gamma) \le 2\gamma(\Gamma)$.*

[1] A cubic graph is a 3-regular graph.

Proof. Let S be a dominating set of minimum cardinality in Γ. Let $X \subseteq S$ be the set composed by all $v_i \in S$ such that $\delta_S(v_i) = 0$. For each $v_i \in X$ we take a vertex $u_i \in \bar{S}$ such that $u_i \sim v_i$. Let $Y \subseteq \bar{S}$ be defined as $Y = \cup_{v_i \in X}\{u_i\}$. Then we have $|Y| \leq \gamma(\Gamma)$ and the set $S \cup Y$ is a global defensive (-1)-alliance in Γ. □

The above bound is tight. For instance, in the case of the 3-cube graph we have $\gamma_{-1}^a(Q_3) = 2\gamma(Q_3) = 4$.

A set $S \subset V$ is a total dominating set if every vertex in V has a neighbor in S. The total domination number $\gamma_t(\Gamma)$ is the minimum cardinality of a total dominating set in Γ. Notice that if Γ is a cubic graph, then

$$\gamma_{-1}^a(\Gamma) = \gamma_t(\Gamma). \tag{13}$$

It was shown in [5] that if Γ is a connected graph of order $n \geq 3$, then

$$\gamma_t(\Gamma) \leq \frac{2n}{3}. \tag{14}$$

Moreover, by Theorem 3 we have

$$\frac{n}{3} \leq \gamma_{-1}^a(\Gamma) \quad \text{and} \quad \frac{n}{2} \leq \gamma_0^a(\Gamma). \tag{15}$$

3. Defensive k-alliances in planar graphs

It is well-known that the size of a planar graph Γ of order $n \geq 3$ is bounded by $m \leq 3(n-2)$. Moreover, in the case of triangle-free graphs $m \leq 2(n-2)$. This inequalities allow us to obtain tight bounds for the studied parameters.

Theorem 6. *Let $\Gamma = (V, E)$ be a graph of order n. If Γ has a global defensive k-alliance S such that the subgraph $\langle S \rangle$ is planar.*

(i) *If $n > 2(2-k)$, then $|S| \geq \left\lceil \frac{n+12}{7-k} \right\rceil$.*

(ii) *If $n > 2(2-k)$ and $\langle S \rangle$ is a triangle-free graph, then $|S| \geq \left\lceil \frac{n+8}{5-k} \right\rceil$.*

Proof. (i) If $|S| \leq 2$, for every $v \in S$ we have $\delta_{\bar{S}}(v) \leq 1-k$. Thus, $n \leq 2(2-k)$. Therefore, $n > 2(2-k) \Rightarrow |S| > 2$.

If $\langle S \rangle$ is planar and $|S| > 2$, the size of $\langle S \rangle$ is bounded by

$$\frac{1}{2}\sum_{v \in S} \delta_S(v) \leq 3(|S|-2). \tag{16}$$

If S is a global defensive k-alliance in Γ,

$$k|S| + (n - |S|) \leq k|S| + \sum_{v \in S} \delta_{\bar{S}}(v) \leq \sum_{v \in S} \delta_S(v). \tag{17}$$

By (16) and (17) the result follows.

(ii) If $\langle S \rangle$ is a triangle-free graph, then

$$\frac{1}{2}\sum_{v \in S} \delta_S(v) \leq 2(|S|-2). \tag{18}$$

The result follows by (17) and (18). □

Corollary 7. *For any planar graph Γ of order n.*

(a) *If $n > 2(2-k)$, then $\gamma_k^a(\Gamma) \geq \left\lceil \frac{n+12}{7-k} \right\rceil$.*

(b) *If $n > 2(2-k)$ and Γ is a triangle-free graph, then $\gamma_k^a(\Gamma) \geq \left\lceil \frac{n+8}{5-k} \right\rceil$.*

The above bounds are tight. In the case of the graph of Fig. 1, the set $S = \{1, 2, 3\}$ is a global defensive k-alliance for $k = -2$, $k = -1$ and $k = 0$, and Corollary 7(a) leads to $\gamma_k^a(\Gamma) \geq 3$. Moreover, if $\Gamma = Q_3$, the 3-cube graph, Corollary 7(b) leads to the following exact values of $\gamma_k^a(Q_3)$: $2 \leq \gamma_{-3}^a(Q_3)$, $4 \leq \gamma_0^a(Q_3) = \gamma_1^a(Q_3)$ and $8 \leq \gamma_3^a(Q_3)$.

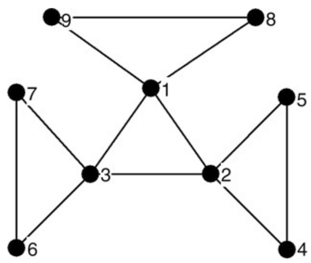

Fig. 1.

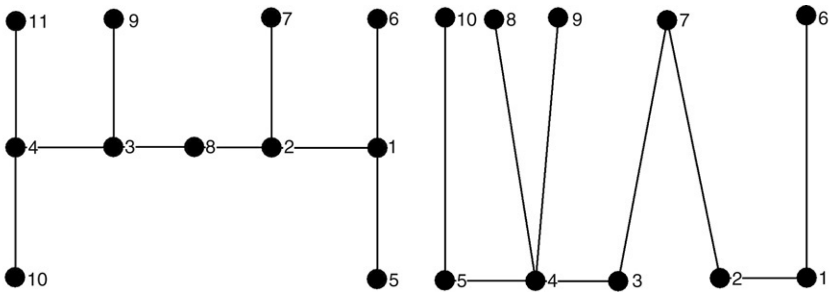

Fig. 2.

Theorem 8. *Let Γ be a graph of order n. If Γ has a global defensive k-alliance S such that the subgraph $\langle S\rangle$ is planar connected with f faces. Then,*

$$|S| \geq \left\lceil \frac{n-2f+4}{3-k} \right\rceil.$$

Proof. By Euler's formula, $\sum_{v\in S}\delta_S(v) = 2(|S|+f-2)$, and (17) we deduce the result. □

In the case of the graph of Fig. 1, the set $S = \{1, 2, 3\}$ is a global defensive k-alliance for $k = -1$, $k = 0$ and $k = 2$. Moreover, $\langle S\rangle$ has two faces. In such a case, Theorem 8 leads to $|S| \geq 3$.

3.1. Defensive *k*-alliances in trees

In this section we study global defensive k-alliances in trees but we impose a condition on the number of connected components of the subgraphs induced by the alliances.

Theorem 9. *Let T be a tree of order n. Let S be a global defensive k-alliance in T such that the subgraph $\langle S\rangle$ has c connected components. Then,*

$$|S| \geq \left\lceil \frac{n+2c}{3-k} \right\rceil.$$

Proof. As the subgraph $\langle S\rangle$ is a forest with c connected components,

$$\sum_{v\in S}\delta_S(v) = 2(|S|-c). \tag{19}$$

The bound of $|S|$ follows from (17) and (19). □

The above bound is attained, for instance, for the left-hand-side graph of Fig. 2, where $S = \{1, 2, 3, 4\}$ is a global defensive (-1)-alliance and $\langle S\rangle$ has two connected components. Moreover, the bound is attained in the case of the right-hand-side graph of Fig. 2, where $S = \{1, 2, 3, 4, 5\}$ is a global defensive 0-alliance and $\langle S\rangle$ has two connected components.

Corollary 10. *For any tree T of order n, $\gamma_k^a(T) \geq \left\lceil \frac{n+2}{3-k} \right\rceil$.*

The above bound is attained for $k \in \{-4, -3, -2, 0, 1\}$ in the case of $\Gamma = K_{1,4}$. As a particular case of the above theorem we obtain the bounds obtained in [9]:

$$\gamma_{-1}^a(T) \geq \left\lceil \frac{n+2}{4} \right\rceil \quad \text{and} \quad \gamma_0^a(T) \geq \left\lceil \frac{n+2}{3} \right\rceil.$$

4. Global connected defensive k-alliances

It is clear that a defensive k-alliance of minimum cardinality must induce a connected subgraph. But we can have a global defensive k-alliance of minimum cardinality with nonconnected induced subgraph. We say that a defensive k-alliance S is connected if $\langle S \rangle$ is connected. We denote by $\gamma_k^{ca}(\Gamma)$ the minimum cardinality of a global connected defensive k-alliance in Γ. Obviously, $\gamma_k^{ca}(\Gamma) \geq \gamma_k^a(\Gamma)$. For instance, for the left-hand-side graph of Fig. 2 we have $\gamma_{-1}^{ca}(\Gamma) = 5 > 4 = \gamma_{-1}^a(\Gamma)$ and for the right-hand-side graph of Fig. 2 we have $\gamma_0^{ca}(\Gamma) = 6 > 5 = \gamma_0^a(\Gamma)$.

Theorem 11. *For any connected graph Γ of diameter $D(\Gamma)$,*

(i) $\gamma_k^{ca}(\Gamma) \geq \left\lceil \frac{\sqrt{4(D(\Gamma)+n-1)+(1-k)^2}+(k-1)}{2} \right\rceil$.

(ii) $\gamma_k^{ca}(\Gamma) \geq \left\lceil \frac{n+D(\Gamma)-1}{\left\lfloor \frac{\Delta-k}{2} \right\rfloor+2} \right\rceil$.

Proof. If S is a dominating set in Γ such that $\langle S \rangle$ is connected, then $D(\Gamma) \leq D(\langle S \rangle) + 2$. Hence,

$$D(\Gamma) \leq |S| + 1. \tag{20}$$

Moreover, if S is a global defensive k-alliance in Γ, then $|S|$ satisfies (10). The first result follows by (10) and (20).
As a consequence of (8), (11) and (20) we obtain the second result. □

Both bounds in Theorem 11 are tight. For instance, both bounds are attained for $k \in \{-2, -1, 0\}$ for the graph of Fig. 1. In such a case, both bounds lead to $\gamma_k^{ca}(\Gamma) \geq 3$. Moreover, both bounds lead to the exact values of $\gamma_k^{ca}(K_{3,3})$ in the following cases: $2 \leq \gamma_{-3}^{ca}(K_{3,3}) = \gamma_{-2}^{ca}(K_{3,3}) = \gamma_{-1}^{ca}(K_{3,3})$. Furthermore, notice that bound (ii) leads to the exact values of $\gamma_k^{ca}(Q_3)$ in the cases $4 \leq \gamma_0^{ca}(Q_3) = \gamma_1^{ca}(Q_3)$, while bound (i) only gives $3 \leq \gamma_0^a(Q_3)$ and $3 \leq \gamma_1^{ca}(Q_3)$.
By Theorem 11, and taking into account that $D(\Gamma) - 1 \leq D(\mathcal{L}(\Gamma))$, we obtain the following result on the global connected k-alliance number of the line graph of Γ in terms of some parameters of Γ.

Corollary 12. *For any connected graph Γ of size m, diameter $D(\Gamma)$, and maximum degrees $d_1 \geq d_2$,*

(i) $\gamma_k^{ca}(\mathcal{L}(\Gamma)) \geq \left\lceil \frac{\sqrt{4(D(\Gamma)+m-2)+(1-k)^2}-(1-k)}{2} \right\rceil$.

(ii) $\gamma_k^{ca}(\mathcal{L}(\Gamma)) \geq \left\lceil \frac{2(m+D(\Gamma)-2)}{d_1+d_2-k+1} \right\rceil$.

References

[1] A. Cami, H. Balakrishnan, N. Deo, R.D. Dutton, On the complexity of finding optimal global alliances, J. Combin. Math. Combin. Comput. 58 (2006) 23–31.
[2] R.C. Brigham, R.D. Dutton, S.T. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$, Congr. Numer. 167 (2004) 57–63.
[3] R.C. Brigham, R.D. Dutton, T.W. Haynes, S.T. Hedetniemi, Powerful alliances in graphs, Discrete Appl. Math. (in press).
[4] M. Chellali, T. Haynes, Global alliances and independence in trees, Discuss. Math. Graph Theory 27 (1) (2007) 19–27.
[5] E.J. Cockayne, R. Dawes, S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–215.
[6] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar, R.D. Skaggs, Offensive alliances in graphs, Discuss. Math. Graph Theory 24 (2) (2004) 263–275.
[7] H. Fernau, D. Raible, Alliances in graphs: A complexity-theoretic study, SOFSEM 2 (2007) 61–70.

[8] G.H. Fricke, L.M. Lawson, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, A note on defensive alliances in graphs, Bull. Inst. Combin. Appl. 38 (2003) 37–41.
[9] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, Global defensive alliances in graphs, Electron. J. Combin. 10 (2003) 139–146.
[10] L.H. Jamieson, S.T. Hedetniemi, A.A. McRae, The algorithmic complexity of alliances in graphs, J. Combin. Math. Combin. Comput. (2007) (in press).
[11] L.H. Jamieson, Algorithms and complexity for alliances and weighted alliances of different types, Ph.D. Thesis, Clemson University, 2007.
[12] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004) 157–177.
[13] J.A. Rodríguez, J.M. Sigarreta, Offensive alliances in cubic graphs, Internat. Math. Forum 1 (36) (2006) 1773–1782.
[14] J.A. Rodríguez, J.M. Sigarreta, Spectral study of alliances in graphs, Discuss. Math. Graph Theory 27 (1) (2007) 143–157.
[15] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global offensive alliances in graphs, Electron. Notes Discrete Math. 25 (2006) 157–164.
[16] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global alliances in planar graphs, AKCE Int. J. Graphs Comb. 4 (1) (2007) 83–98.
[17] J.A. Rodríguez-Velázquez, I.G. Yero, J.M. Sigarreta, Defensive k-alliances in graphs, Appl. Math. Lett. (in press).
[18] K.H. Shafique, R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congr. Numer. 162 (2003) 139–146.
[19] K.H. Shafique, R. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, J. Combin. Math. Combin. Comput. 56 (2006) 139–145.
[20] K.H. Shafique, Partitioning a graph in alliances and its application to data clustering. Ph.D. Thesis, 2004.
[21] K.H. Shafique, R.D. Dutton, On satisfactory partitioning of graphs, Congr. Numer. 154 (2002) 183–194.
[22] J.M. Sigarreta, J.A. Rodríguez, On defensive alliance and line graphs, Appl. Math. Lett. 19 (12) (2006) 1345–1350.
[23] J.M. Sigarreta, J.A. Rodríguez, On the global offensive alliance number of a graph, Discrete Appl. Math. 157 (2) (2009) 219–226.