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Continued fractions and the polygamma functions

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Abstract: In a previous study we have shown that the polygamma functions (derivatives of the logarithm of the gamma function) relate to Stieltjes transforms in the square of the argument. These transforms in turn may be converted to Stieltjes continued fractions; in the background is a determined Stieltjes moment problem.

In the present study we use the Hamburger form of the Stieltjes integral to produce a set of real monotonic increasing and monotonic decreasing approximants to each of the real and imaginary parts of a polygamma function when the argument is complex. The approximants involve rational fractions which appear to be new.

Special attention is given to $\ln \Gamma(z)$ and the psi function.

Keywords: Generalized continued functions, monotonic sequences, Stieltjes integral.

1. Introduction

It was shown by us [3] that the polygamma function

$$\psi_m(z) = \frac{d^{m+1}}{dz^{m+1}} \ln \Gamma(z), \quad m = 0, 1, 2, \ldots, \tag{1}$$

($\psi(z) = \psi_0(z)$ being the psi or digamma function, $\psi_1(z)$ the trigamma function, etc.) satisfies the relation

$$(-1)^{m+1}\psi_m(z) = \frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \left(\frac{2\pi}{z}\right)^m g_m(z), \quad \operatorname{Re}(z) > 0, \tag{2}$$

$m = 1, 2, 3, \ldots$, where

$$g_m(z) = \int_0^\infty \frac{x^{m/2}\theta_m(y)\,dx}{(x+z^2)(y-1)^{m+1}}, \quad y = e^{2\pi\sqrt{x}},$$

$$\theta_m(y) = y(1-y)\frac{d\theta_{m-1}}{dy} + my\theta_{m-1}, \quad \theta_0 = 1, m = 0, 1, 2, \ldots.$$

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(For $m=0$ the expression $(m-1)!/z^m$ is replaced by $-\ln z$, and 0! by unity). For example,

$$\begin{aligned}\psi(z)&=\ln z-\frac{1}{2z}-\int_0^\infty\frac{\mathrm{d}x}{(x+z^2)(y-1)},\\ \psi_1(z)&=\frac{1}{z}+\frac{1}{2z^2}+\frac{2\pi}{z}\int_0^\infty\frac{y\sqrt{x}\,\mathrm{d}x}{(x+z^2)(y-1)^2},\\ \psi_2(z)&=-\frac{1}{z^2}-\frac{1}{z^3}-\left(\frac{2\pi}{z}\right)^2\int_0^\infty\frac{(y+y^2)x\mathrm{d}x}{(x+z^2)(y-1)^3},\\ \psi_3(z)&=\frac{2}{z^3}+\frac{3}{z^4}+\left(\frac{2\pi}{z}\right)^3\int_0^\infty\frac{(y+4y^2+y^3)\sqrt{x^3}\,\mathrm{d}x}{(x+z^2)(y-1)^4},\end{aligned}\tag{3}$$

showing the polygamma functions in terms of Stieltjes integral transforms with parameter z^2.

Our object here is to derive from (2) new approximation sequences (monotonic increasing, and monotonic decreasing) for the real and imaginary parts of $\psi_m(z)$. These sequences are quite distinct from those which could be derived by separating out the real and imaginary parts from the Stieltjes type continued fractions (c.fs.) derived in Shenton and Bowman [3]; see expressions (19)–(24).

2. The polynomials $\theta_m(y)$ and the integral $g_m(z)$

From (3) the symmetry of the polynomials is obvious. A general proof is to consider the function

$$\phi_m(y)=y^{-(m+1)/2}\theta_m(y),\quad m\geqslant 1,\tag{4}$$

which satisfies the difference-differential equation

$$\phi_m(y)=\tfrac{1}{2}m\left(\sqrt{y}+1/\sqrt{y}\right)\phi_{m-1}(y)+\sqrt{y}(1-y)\phi'_{m-1}(y).$$

It is also readily shown that

$$\phi_m(1/x)=\tfrac{1}{2}m\left(\sqrt{x}+1/\sqrt{x}\right)\phi_{m-1}(1/x)+(1-1/x)\left(1/\sqrt{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\phi_{m-1}(1/x).$$

But $\phi_m(y)=\phi_m(1/y)$, $m=1, 2, 3$.

It is now evident that under the mapping $x=t^2$ in (2), we may write

$$g_m(z)=\int_{-\infty}^\infty\frac{t^{m+1}\theta_m(\mathrm{e}^{2\pi t})\mathrm{d}t}{(t^2+z^2)(\mathrm{e}^{2\pi t}-1)^{m+1}}=\int_{-\infty}^\infty\frac{t^{m+1}\phi_m(\mathrm{e}^{2\pi t})\mathrm{d}t}{(t^2+z^2)(\mathrm{e}^{\pi t}-\mathrm{e}^{-\pi t})^{m+1}}\tag{5}$$

in which

$$\phi_m(\mathrm{e}^{2\pi t})=\phi_m(\mathrm{e}^{-2\pi t}).$$

Hence the integrand in (5) is an even function and so

$$g_m(z)=\frac{\mathrm{i}}{z}\int_{-\infty}^\infty\frac{\mathrm{d}\sigma(t)}{t+\mathrm{i}z}\quad(\mathrm{Re}(z)>0)\tag{6}$$

where $\sigma(t)$ is a distribution function on $(-\infty,\infty)$; i.e.

$$(-1)^{m+1}\psi_m(z)=\frac{(m-1)!}{z^m}+\frac{m!}{2z^{m+1}}+\frac{\mathrm{i}}{z^{m+1}}\int_{-\infty}^\infty\frac{\Theta_m(t)\mathrm{d}t}{t+\mathrm{i}z}$$

where $\Theta_m(-t)=\Theta_m(t)$ and $m\geqslant 1$.

But there is the basic c.f.–integral relation, namely

$$\mathrm{i}\int_{-\infty}^\infty\frac{\Theta_m(t)\mathrm{d}t}{t+\mathrm{i}z}=\frac{c_0^{(m)}}{z+}\,\frac{c_1^{(m)}}{z+\cdots}.$$

from expression (24a) of Shenton and Bowman (1971). Defining $w = \mathrm{i}z$, we therefore have

$$\int_{-\infty}^{\infty} \frac{\Theta_m(t)\mathrm{d}t}{t+w} = \frac{c_0^{(m)}}{w-}\,\frac{c_1^{(m)}}{w-\cdots}, \quad \mathrm{Im}\, w > 0,\ \mathrm{Re}\, w \neq 0. \tag{7}$$

Note we are interested in $\psi_m(z)$ when z is complex; i.e. $\mathrm{Im}(z) \neq 0$.

3. Real and imaginary parts of $\psi_m(z)$

Let $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ with $-\pi/2 < \theta < \pi/2$, $\theta \neq 0$, and write $(2\pi/z)^m g_m(z)$ in the form

$$\Phi_m(z) = (-1)^{m+1}\psi_m(z) - \frac{(m-1)!}{z^m} - \frac{m!}{2z^{m+1}}. \tag{8}$$

Then from (2) and (7), for $m \geqslant 1$,

$$\mathrm{Re}\{\Phi_m(z)\} = \frac{1}{r^{m+1}} \int_{-\infty}^{\infty} \frac{\{t\sin(m+1)\theta + r\cos(m+2)\theta\}\Theta_m(t)\mathrm{d}t}{t^2 - 2tr\sin\theta + r^2} \tag{9a}$$

and

$$\mathrm{Im}\{\Phi_m(z)\} = \frac{1}{r^{m+1}} \int_{-\infty}^{\infty} \frac{\{t\cos(m+1)\theta - r\sin(m+2)\theta\}\Theta_m(t)\mathrm{d}t}{t^2 - 2tr\sin\theta + r^2}. \tag{9b}$$

We point out in the earlier paper on the subject that we missed the important symmetry properties of the integrand occurring in $\psi_m(z)$. This property focuses attention on the Hamburger aspect of the integral transforms – a simplification is now induced into the expression for the sequence approximants to $\mathrm{Re}(\psi_m) + \mathrm{Im}(\psi_m)$ when we appeal to our own work on second-order continued fractions [2].

4. Link with second order continued fraction

To set up increasing and decreasing sequences to (9a) and (9b), there are three important formal steps.

A. To formulate rational fraction approximants to definite integrals of the form

$$\int_{-\infty}^{\infty} \frac{A(x)B(x)\mathrm{d}\phi(x)}{C(x)}$$

where, A, B, C are polynomials.

B. To describe the basis for the derivation of monotonic sequence approximants.

C. To formulate details of the computational schemes.

The scheme is described in detail in [2]. We recall the main points.

An nth order c.f. is associated with

$$F(z_1, z_2, \ldots, z_n) = \int_0^{\infty} \frac{A(x)B(x)\mathrm{d}\phi(x)}{C_n(x)} \tag{10}$$

where A, B are real polynomials and

$$C_n(x) = \prod_{\lambda=1}^{n} (x + z_\lambda) \quad \text{for } x \geqslant 0,\ z_\lambda \text{ distinct}, \tag{11}$$

and linked to the c.f. (Stieltjes transform)

$$F(z) = \int_0^{\infty} \frac{\mathrm{d}\phi(x)}{x+z} = \frac{a_0}{z+c_1} - \frac{a_1}{z+c_2} - \cdots \quad \left\{ = \frac{P_s(z)}{Q_s(z)} \quad \text{as } s \to \infty \right\}. \tag{12}$$

In the paper we were interested in the general case and in particular, functions $\phi(x)$ as a solution of the

Stieltjes moment problem – the formal change to moment problems for which $\phi(x)$ is a constant for $x < a$, and $x > b(> a)$ presents no problem but care is needed over validity considerations. Again our present study concentrates on the case $n = 2$, involving second-order (generalized) c.fs. In (10) we need only to consider polynomials $A(\cdot)$, $B(\cdot)$ such that the product degree does not exceed two. Sequences of approximants appear involving the parameters in the denominator C_n, which in the present case $(n = 2)$ takes the form

$$C_2(x) = x^2 + p_1 x + p_2. \tag{13}$$

We need to define the following functions:

(i) $$U_s^{(0)} = \frac{1}{\Delta}\begin{vmatrix} z_1 & z_2 \\ Q_s(z_1) & Q_s(z_2) \end{vmatrix}, \quad U_s^{(1)} = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 \\ Q_s(z_1) & Q_s(z_2) \end{vmatrix} \quad (\Delta = z_2 - z_1), \tag{14}$$

(ii) $$(A, B) = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{vmatrix}$$

where

$$\beta_s = B(-z_s)\int_0^\infty \frac{\{A(x) - A(-z_s)\}}{x + z_s}\,\mathrm{d}\phi(x), \qquad s = 1, 2.$$

(iii) $W_s(B)$ is similar to (A, B) except that the last row of the numerator determinant is replaced by y_{1s}, y_{2s} where $y_{rs} = P_s(z_r)B(-z_r)$. As is readily shown,

$$W_s(B) = \sum_{r=1}^{2} (-1)^{2-r} b_r V_s^{(2-r)}$$

where $B(x) = \sum_{r=1}^{2} b_r x^{2-r}$ and $V_s^{(\lambda)}$, $\lambda = 0, 1$, is similar to $U_s^{(1)}$ except that the last row of the numerator determinant is $z_1^\lambda P_s(z_1)$, $z_2^\lambda P_s(z_2)$.

Then an approximating sequence to

$$F(z_1, z_2) = \int_0^\infty \frac{A(x)B(x)\mathrm{d}\phi(x)}{(x + z_1)(x + z_2)}, \tag{15}$$

assuming convergence of the integral, is

$$P_s(A, B)/Q_s(z_1, z_2) \quad (s = 1, 2, \ldots) \tag{16}$$

where

$$P_s(A, B) = \begin{vmatrix} -(A, B) & W_s(B) & W_{s+1}(B) \\ U^{(1)}(A) & U_s^{(1)} & U_{s+1}^{(1)} \\ U^{(0)}(A) & U_s^{(0)} & U_{s+1}^{(0)} \end{vmatrix}, \qquad Q_s(z_1, z_2) = \begin{vmatrix} U_s^{(1)} & U_{s+1}^{(1)} \\ U_s^{(0)} & U_{s+1}^{(0)} \end{vmatrix}.$$

Various choices of the polynomials A, B lead to different forms, but it is fairly obvious that

$$P_s(A, B) = P_s(B, A).$$

For our present purposes we need the result for $A(x) = B(x) = Cx + D$. We find formally the sequence of approximants to

$$\int_0^\infty \frac{(Cx + D)^2 \mathrm{d}\phi(x)}{x^2 + p_1 x + p_2}, \tag{17}$$

namely

$$T_s(A, A; C_2) = C^2 a_0 + \frac{P_s^*(A, A)}{Q_s(z_1, z_2)}, \tag{17a}$$

where

$$P_s^*(A, A) = \begin{vmatrix} DV_s^{(0)} - CV_s^{(1)} & DV_{s+1}^{(0)} - CV_{s+1}^{(1)} \\ CU_s^{(0)} - DU_s^{(1)} & CU_{s+1}^{(0)} - DU_{s+1}^{(1)} \end{vmatrix}.$$

(Note that $P_s^*(\cdot, \cdot)$ is a quadratic form in C, and D with variable coefficients.)

5. Monotonic sequences

5.1. Returning to the function $\Phi_m(z)$ involving the polygamma function $\psi_m(z)$, the real and imaginary parts ((9a), (9b)) will not necessarily, using (16), be approximated by monotone sequences. But for real polynomials there is always a solution to

$$t \sin(m+1)\theta + r\cos(m+2)\theta = (C_1 t + D_1)^2 - C_1^2(t^2 - 2tr\sin\theta + r^2), \tag{18a}$$

namely

$$C_1 = \sqrt{\left\{\frac{1-\cos(m+1)\theta}{2r\cos\theta}\right\}}, \qquad D_1 = \left\{\frac{\sin(m+1)\theta - \sin\theta}{2C_1\cos\theta}\right\}. \tag{18b}$$

Recall that $m = 1, 2, \ldots,$ and $-\pi/2 < \theta < \pi/2$, $\theta \neq 0$.

Similarly, a solution to

$$t \sin(m+1)\theta + r\cos(m+2)\theta = C_2^2(t^2 - 2tr\sin\theta + r^2) - (C_2 t + D_2)^2$$

is

$$C_2 = \sqrt{\left\{\frac{1+\cos(m+1)\theta}{2r\cos\theta}\right\}}, \qquad D_2 = -\left\{\frac{\sin(m+2)\theta + \sin\theta}{2C_2\cos\theta}\right\}. \tag{18c}$$

We now have for $m = 1, 2, \ldots,$ $-\pi/2 < \theta < \pi/2$, $\theta \neq 0$,

$$\operatorname{Re}\{\Phi_m(z)\} = \frac{1}{r^{m+1}}\{\text{l.i.s. } P_t^{(m)}(r;\theta)\} \tag{19a}$$

$$= \frac{1}{r^{m+1}}\{\text{l.d.s. } Q_t^{(m)}(r;\theta)\} \tag{19b}$$

(limit of the increasing sequence = l.i.s, and limit of the decreasing sequence = l.d.s.) where

$$\text{(i)} \quad P_t^{(m)} = \frac{1}{\omega_t(z)} \begin{vmatrix} D_1V_t^{(0)} - C_1V_t^{(1)} & D_1V_{t+1}^{(0)} - C_1V_{t+1}^{(1)} \\ C_1U_t^{(0)} - D_1U_t^{(1)} & C_1U_{t+1}^{(0)} - D_1U_{t+1}^{(1)} \end{vmatrix} \tag{19c}$$

(ii) $Q_t^{(m)}$ is derived from $P_t^{(m)}$ by replacing D_1 by D_2, C_1 by C_2, and finally changing the sign.
(iii) $\omega_t(z) = U_{t+1}^{(0)}U_t^{(1)} - U_{t+1}^{(1)}U_t^{(0)}$.

5.2. For $\operatorname{Im}\{\Phi_m(z)\}$, C_1, D_1 and C_2, D_2 are replaced by

$$C_1^* = \sqrt{\left\{\frac{1+\sin(m+1)\theta}{2r\cos\theta}\right\}}, \qquad D_1^* = \left\{\frac{\cos(m+2)\theta - \sin\theta}{2C_1^*\cos\theta}\right\}; \tag{20a}$$

$$C_2^* = \sqrt{\left\{\frac{1-\sin(m+1)\theta}{2r\cos\theta}\right\}}, \qquad D_2^* = -\left\{\frac{\cos(m+2)\theta + \sin\theta}{2C_2^*\cos\theta}\right\}. \tag{20b}$$

It is possible for one of the parameters C_1, C_2, C_1^*, C_2^* to be zero resulting in an infinite value for one of the D's. This situation is resolved as follows; there are two cases:

(a) $\sin(m+1)\theta = 0$.

Here for the real part of $\Phi_2(z)$ use the sequence

$$\frac{\cos(m+2)\theta}{r^m}\{\text{l.i.s. } P_t^{(m)}(r;\,\theta)\} \tag{20c}$$

where

$$P_t^{(m)}(r;\,\theta)=\frac{1}{\omega_t(z)}\begin{vmatrix} U_t^{(1)} & U_{t+1}^{(1)} \\ V_t^{(0)} & V_{t+1}^{(0)} \end{vmatrix},$$

and for the complementary decreasing sequence

$$\frac{\cos(m+2)\theta}{r^{m+2}\cos^2\theta}\{\text{l.d.s. } Q_t^{(m)}(r;\,\theta)\} \tag{20d}$$

where

$$Q_t^{(m)}(r;\,\theta)=\frac{1}{\omega_t(z)}\begin{vmatrix} r\sin\theta\, V_t^{(0)}+V_t^{(1)} & r\sin\theta\, V_{t+1}^{(0)}+V_{t+1}^{(1)} \\ U_t^{(0)}+r\sin\theta\, U_t^{(1)} & U_{t+1}^{(0)}+r\sin\theta\, U_{t+1}^{(1)} \end{vmatrix}.$$

(b) $\cos(m+1)\theta=0$.

Use (20c) with $\cos(m+2)\theta/r^m$ replaced by $-\sin(m+2)\theta/r^m$, and (20d) with the factor outside the braces replaced by $-\sin(m+2)\theta/(r^{m+2}\cos^2\theta)$.

5.3. *The fundamental entities.* The sequences $\{U_t^{(0)}\}$, $\{U_t^{(1)}\}$, $\{V_t^{(0)}\}$ and $\{V_t^{(1)}\}$ are set-up from the recurrences

$$\begin{aligned} U_t^{(0)} &= -C_{t-1}^{(m)}U_{t-2}^{(0)}+r^2U_{t-1}^{(1)}, & U_t^{(1)} &= -2yU_{t-1}^{(1)}-C_{t-1}^{(m)}U_{t-2}^{(1)}-U_{t-1}^{(0)}, \\ V_t^{(0)} &= -C_{t-1}^{(m)}V_{t-2}^{(0)}+V_{t-1}^{(1)}, & V_t^{(1)} &= -2yV_{t-1}^{(1)}-C_{t-1}^{(m)}V_{t-2}^{(1)}-r^2V_{t-t}^{(0)} \end{aligned} \qquad (t\geqslant 2) \tag{21}$$

with initiators

s	$U_s^{(0)}$	$U_s^{(1)}$	$V_s^{(0)}$	$V_s^{(1)}$
0	−1	0	0	0
1	0	1	0	$C_0^{(m)}$

(22)

Note that only the first few partial numerators $C_0^{(m)}$, $C_1^{(m)}$, are known in the general case ([3], (24b)); however, general expressions are known for $\psi_2(x)$ (Stieltjes [5]) and also for $\psi_1(x)$ (Shenton, unpublished). For the latter

$$\psi_1(x)=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{x^2}\left(\frac{\frac{1}{6}}{x+}\,\frac{C_1^{(1)}}{x+\cdots}\right) \tag{23}$$

where $C_{s-1}^{(1)}=s^2(s^2-1)/\{4(4s^2-1)\}$, $s=2,3,\ldots$.

Numerical example. Let $m=1$, $z=1+i\sqrt{3}$, $r=2$, and $\theta=\pi/3$. Then

$$C_1=\tfrac{1}{2}\sqrt{3},\quad D_1=-1,\qquad C_2=\tfrac{1}{2},\quad D_2=-\sqrt{3}, \tag{24}$$

leading to Table 1.

For the *real part* we have for the *coefficients in the determinants* $P_t^{(m)}(r;\,\theta)$:

$\frac{1}{12}\sqrt{3}$	$-0.\dot{3}$	0.503119521	$-0.1\dot{7}$	−1.609082689
1	0.173205081	−4.714285710	13.69145024	−18.0

(change sign of final determinant value), and for the *coefficients in the determinants* $Q_t^{(m)}(r;\,\theta)$:

$\frac{1}{12}$	0	−0.376190476	1.154700536	−1.607792206
$\sqrt{3}$	−3.9	5.691024110	−1.428571490	−19.83985464

Table 1

s	$U_s^{(0)}$	$U_s^{(1)}$	$V_s^{(0)}$	$V_s^{(1)}$
0	−1	0	0	0
1	0	1	0	1/6
2	4.2	$-3.46410\dot{1}\dot{6}$	$0.1\dot{6}$	−0.577350269
3	−13.85640646	7.28571430	−0.577350269	1.247619048
4	25.14285720	−8.08290382	$1.0\dot{8}$	−1.462620685
5	−11.33706004	−8.181818080	−0.587847550	−1.179220764

Table 2a

t	$P_t^{(1)}$	$Q_t^{(1)}$
1	−0.0213294	−0.0193452
2	−0.0213259	−0.0210796
3	−0.0212490	−0.0211917
4	−0.0212129	−0.0211921
	True value	−0.02120402

Table 2b

t	$P_t^{(1)*}$	$Q_t^{(1)*}$
1	−0.001851216	+0.000132911
2	−0.001059332	−0.000813027
3	−0.001057682	−0.001000342
4	−0.001043183	−0.001022385
	True value	−0.00102873

The real part of

$$\Phi_1(1+\mathrm{i}\sqrt{3}) = \mathrm{Re}\left\{-\left(\frac{\mathrm{e}^{-\mathrm{i}\theta}}{r}+\frac{\mathrm{e}^{-2\mathrm{i}\theta}}{2r^2}\right)+\psi_1(z)\right\}$$

follows from Table 2a.

For the *imaginary part* we have for the *coefficients in the determinants* $P_t^{(m)*}(r;\,\theta)$,

$$-(2+\sqrt{3})\begin{vmatrix} V_t^{(0)}+\frac{1}{2}V_t^{(1)} & V_{t+1}^{(0)}+\frac{1}{2}V_{t+1}^{(1)} \\ \frac{1}{2}U_t^{(0)}+U_t^{(1)} & \frac{1}{2}U_{t+1}^{(0)}+U_{t+1}^{(1)} \end{vmatrix}:$$

$\frac{1}{12}$	−0.122008468	0.046459255	0.357578546	−1.177457932
1	−1.364101616	0.357511070	4.488524780	−13.85034810

and for the *coefficients in the determinants* $Q_t^{(m)}(r;\,\theta)$,

$$-(2-\sqrt{3})\begin{vmatrix} V_t^{(0)}-\frac{1}{2}V_t^{(1)} & V_{t+1}^{(0)}-\frac{1}{2}V_{t+1}^{(1)} \\ \frac{1}{2}U_t^{(0)}-U_t^{(1)} & \frac{1}{2}U_{t+1}^{(0)}-U_{t+1}^{(1)} \end{vmatrix}:$$

$-\frac{1}{12}$	0.455341801	−1.201159793	1.820199231	0.001762832
1	5.566101616	−14.21391753	20.65433242	2.51328806

The imaginary part of

$$\Phi_1(z) = \mathrm{Im}\left\{\psi_1(z)-\frac{\mathrm{e}^{-\mathrm{i}\theta}}{r}-\frac{\mathrm{e}^{-2\mathrm{i}\theta}}{2r^2}\right\}$$

follows from Table 2b.

6. Remarks on validity

The distribution functions (weight functions $\Theta_m(t)$ given in (7)) for ln $\Gamma(z)$ and the polygamma functions relate to determined Stieltjes moment problems. For example, omitting the first two terms in

$\psi_m(z)$, the series coefficients in the remaining series are the moments of a bounded non-decreasing function with infinitely many points of increase on (0, ∞). But in our earlier study we overlooked the fact (described in Section 1) that for $m \geqslant 1$, the distribution functions relate to the whole axis of reals, so that the corresponding moment problem is the Hamburger; i.e. the basic function (5) relates to

$$I(z;\sigma)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\sigma(t)}{\mathrm{i}z+t}\quad (z=x+\mathrm{i}y). \tag{25}$$

Clearly for validity here, $\mathrm{Re}(z)\neq 0$ at least. Notice that nothing is altered to any extent with the c.f. form

$$\psi_1(z)=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{z}\left\{\frac{1/6}{z^2+}\,\frac{C_1^{(1)}}{1+}\,\frac{C_2^{(1)}}{z^2+\cdots}\right\}$$
$$=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{z^2}\left\{\frac{1/6}{z+}\,\frac{C_1^{(1)}}{z+}\,\frac{C_2^{(1)}}{z+\cdots}\right\},$$

valid for $\mathrm{Re}(z)>0$ (the forms are not valid for $\mathrm{Re}(z)<0$ because of the initial formulation of the relation between $\psi_1(z)$ and its asymptotic series).

We have to consider the validity of generalized c.f. derived from integrals such as (see (10), (15), (17))

$$I(z_1,z_2;\sigma)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\sigma(t)}{(t+z_1)(t+z_2)}.$$

Briefly one approach is to consider the problem of the validity of expansions stemming from

$$\min_{\pi_s}\int_{-\infty}^{\infty}(t+z_1)(t+z_2)\left\{\frac{1}{(t+z_1)(t+z_2)}-\pi_s(t)\right\}^2\mathrm{d}\sigma(t) \tag{26}$$

where we assume $(t+z_1)(t+z_2)>0$ for $t\in(-\infty,\infty)$ and π_s are real polynomials; i.e. the question whether Parseval's formula applies to distribution functions

$$\sigma^*(t)=\int_{-\infty}^{\infty}(x+z_1)(x+z_2)\mathrm{d}\sigma(x) \tag{27}$$

for $\sigma(\cdot)$ a solution of the Hamburger moment problem. But (Shohat & Tamarkin [4]) a necessary and sufficient condition due to M. Riesz [1] for the validity of Parseval's formula for functions $f(\cdot)$, is that $f\in L_\sigma^2$ and the moment problem is determined. Add to these Carleman's criterion for the moments and the validity question is settled sufficient for our requirements. In particular for $\psi_m(z)$, $m\geqslant 1$, Carleman's criterion is satisfied for the moments [see [3], (14) and (15)), appropriate distribution functions exist, and the real and imaginary parts involve positives polynomials $(t+z_1)(t+z_2)$ for all real t.

7. Sequences for $\ln\Gamma(z)$ and $\psi(z)$

We have

$$\ln\Gamma(z)=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+J(z),$$

where

$$J(z)=z\int_{-\infty}^{\infty}\frac{\mathrm{d}\sigma(t)}{t+z^2}, \tag{28}$$

with

$$\sigma(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{x}}\{\ln(1-\mathrm{e}^{-2\pi\sqrt{x}})^{-1}\}\mathrm{d}x\quad (\mathrm{Re}(z)>0).$$

Moreover

$$J(z)=\frac{B_0}{z+A_1}\ \frac{B_1}{-\ z+A_2}\ -\ \cdots$$

where

$$B_0=a_0,\quad B_s=a_{2s-1}a_{2s},\quad s\geqslant 1,\qquad A_1=a_1,\quad A_s=a_{2s-2}+a_{2s-1},\quad s\geqslant 2.$$

($a_0=-1/12$, $a_1=1/30$, $a_2=53/210$, $a_3=195/371$, $a_4=22999/22737$, etc.) A slight change of notation shows that

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\sigma(t)}{t+z^2}=\frac{\beta_0}{z^2+A_1}\ \frac{\beta_1}{-\ z^2+A_2}\ -\ \cdots.$$

But

$$\operatorname{Re}\{J(z)\}=x\int_0^{\infty}\frac{(t+x^2+y^2)}{(t+z^2)(t+\bar{z}^2)}\mathrm{d}\sigma(t),\qquad \operatorname{Im}\{J(z)\}=y\int_0^{\infty}\frac{(t-x^2-y^2)}{(t+z^2)(t+\bar{z}^2)}\mathrm{d}\sigma(t),$$

where $z=x+\mathrm{i}y$, $x>0$, and $y\neq 0$. As in Section 5, we now can set up monotonic sequences of approximants. We find

$$\operatorname{Re}\{J(z)\}=x\{\text{l.i.s. } P_m(r,\theta)\}=x\{\text{l.d.s. } Q_m(r,\theta)\} \tag{29a}$$

where

$$P_m(r,\theta)=\frac{1}{\omega_m(z)}\begin{vmatrix} d_1V_m^{(0)}-C_1V_m^{(1)} & d_1V_{m+1}^{(0)}-C_1V_{m+1}^{(1)} \\ C_1U_m^{(0)}-d_1U_m^{(1)} & C_1U_{m+1}^{(0)}-d_1U_{m+1}^{(1)} \end{vmatrix}, \tag{29b}$$

$$Q_m(r,\theta)=\frac{1}{\omega_m(z)}\begin{vmatrix} d_2V_m^{(0)}-C_2V_m^{(1)} & d_2V_{m+1}^{(0)}-C_2V_{m+1}^{(1)} \\ -C_2U_m^{(0)}+d_2U_m^{(1)} & -C_2U_{m+1}^{(0)}+d_2U_{m+1}^{(1)} \end{vmatrix}, \tag{29c}$$

with

$$\begin{aligned} c_1&=\frac{\sqrt{\{|\sin\theta|-\sin^2\theta\}}}{r|\sin(2\theta)|}, & d_1&=\frac{2c_1^2r^2\cos(2\theta)+1}{2c_1}; \\ c_2&=\frac{\sqrt{\{|\sin\theta|+\sin^2\theta\}}}{r|\sin(2\theta)|}, & d_2&=\frac{2c_2^2r^2\cos 2\theta-1}{2c_2}. \end{aligned} \tag{30}$$

For the imaginary part we have similarly

$$y^{-1}\operatorname{Im}\{J(z)\}=\text{l.i.s. } P_m^*(r;\theta)=\text{l.d.s. } Q_m^*(r;\theta),$$

for which c_1, d_1 etc. are replaced by

$$\begin{aligned} c_1^*&=\frac{\sqrt{\{\cos\theta+\cos^2\theta\}}}{r|\sin(2\theta)|}, & d_1^*&=\frac{2c_1^{*2}r^2\cos(2\theta)+1}{2c_1^*}; \\ c_2^*&=\frac{\sqrt{\{\cos\theta-\cos^2\theta\}}}{r|\sin(2\theta)|}, & d_2^*&=\frac{2c_2^{*2}r^2\cos(2\theta)-1}{2c_2^*}. \end{aligned} \tag{31}$$

The fundamental entities are defined in

$$\begin{aligned} U_s^{(0)}&=A_sU_{s-1}^{(0)}-B_{s-1}U_{s-2}^{(0)}+r^4U_{s-1}^{(1)}, \\ U_s^{(1)}&=\{A_s+2r^2\cos(2\theta)\}U_{s-1}^{(1)}-B_{s-1}U_{s-2}^{(1)}-U_{s-1}^{(0)}; \\ V_s^{(0)}&=A_sV_{s-1}^{(0)}-B_{s-1}V_{s-2}^{(0)}+V_{s-1}^{(1)}, \\ V_s^{(1)}&=\{A_s+2r^2\cos(2\theta)\}V_{s-1}^{(1)}-B_{s-1}V_{s-2}^{(1)}-r^4V_{s-1}^{(0)}. \end{aligned} \tag{32}$$

Table 3

s	$U_s^{(0)}$	$U_s^{(1)}$	$V_s^{(0)}$	$V_s^{(1)}$
0	−1	0	0	0
1	$-\frac{1}{30}$	1	0	$\frac{1}{12}$
2	24.98247978	−5.188679246	$\frac{1}{12}$	−0.435167715
3	−66.51864975	−7.504220150	−0.224417989	−0.617170022
4	−624.8213913	89.79538218	−2.08895459	7.553848138

Table 4a
Real parts

m	P_m	$\ln \Gamma$
1	0.016905756	−1.876093669
2	0.016920588	−1.876078837
3	0.016920591	−1.876078834
m	Q_m	$\ln \Gamma$
1	0.016921553	−1.876077872
2	0.016920979	−1.876078446
3	0.016920658	−1.876078767

Table 4b
Imaginary parts

m	$2P_m^*$	$\ln \Gamma$
1	−0.033376754	0.129635518
2	−0.033366135	0.129646136
3	−0.033366003	0.129646268
m	$2Q_m^*$	$\ln \Gamma$
1	−0.033360957	0.129651314
2	−0.033365744	0.129646527
3	−0.033365936	0.129646334

with **initiators**

s	$U_s^{(0)}$	$U_s^{(1)}$	$V_s^{(0)}$	$V_s^{(1)}$
0	-1	0	0	0
1	$-A_1$	1	0	B_0

Numerical example. Let $z = 1 + 2\mathrm{i}$, $r = \sqrt{5}$, $\sin\theta = 2/\sqrt{5}$, $\sin 2\theta = \frac{4}{5}$ $\cos 2\theta = -\frac{3}{5}$. Then

$$c_1 = \frac{\sqrt{(2\sqrt{5}-4)}}{4}, \quad d_1 = \frac{10-3\sqrt{5}}{8c_1}; \qquad c_2 = \frac{\sqrt{(2\sqrt{5}+4)}}{4}, \quad d_2 = \frac{-10-3\sqrt{5}}{8c_2};$$

$$c_1^* = \frac{\sqrt{(\sqrt{5}+1)}}{4}, \quad d_1^* = \frac{5-3\sqrt{5}}{16c_1^*}; \qquad c_2^* = \frac{\sqrt{(\sqrt{5}-1)}}{4}, \quad d_2^* = \frac{-5-3\sqrt{5}}{16c_2^*},$$

and we find the values of Table 3.

The approximants $\{P_m\}$ etc. to $J(z)$, and $\ln \Gamma(z)$ for $z = 1 + 2\mathrm{i}$ follow from Tables 4a, b. Hence, at this stage

$$-1.87607883 < \mathrm{Re} \ln\Gamma(1+2\mathrm{i}) < -1.87607877,$$

$$0.12964627 < \mathrm{Im} \ln\Gamma(1+2\mathrm{i}) < 0.12964634.$$

8. Further applications

The derivation of the real and imagainary parts of $\psi(z)$ follow similarly. Further, if validity can be established, c.fs. such as

$$f(z)=\frac{c_0}{z+}\frac{c_1}{z+}\ \dots$$

can be used to set up the sequence approximants to the real and imaginary parts of the corresponding function.

Appendix

It can be shown that the formulas (18b)–(20b) also hold for $m=0$, for which

$$\Phi_0(z)=-\psi(z)+\ln z-\frac{1}{2z}. \tag{A1}$$

Similarly we have (corresponding to $m=-1$) from (28) the sequences

$$\operatorname{Re}\{J(z)\}=-x\left\{\text{l.i.s.}\begin{vmatrix}V_t^{(0)} & V_{t+1}^{(0)}\\ U_t^{(1)} & U_{t+1}^{(1)}\end{vmatrix}/\omega_t(z)\right\}, \tag{A2}$$

$$=x^{-1}\left\{\text{l.d.s.}\begin{vmatrix}yV_t^{(0)}+V_t^{(1)} & yV_{t+1}^{(0)}+V_{t+1}^{(1)}\\ U_t^{(0)}+yU_t^{(1)} & U_{t+1}^{(0)}+yu_{t+1}^{(1)}\end{vmatrix}/\omega_t(z)\right\}. \tag{A3}$$

For the imaginary part use (19c) with $C_1=1/\sqrt{(2x)}$, $D_1=(x-y)/\sqrt{(2x)}$ for an increasing sequence, and $C_2=1/\sqrt{(2x)}$, $D_2=-(x+y)/\sqrt{(2x)}$ for a decreasing sequence (changing the sign of the numerator determinant).

In both cases, we assume $\operatorname{Re}(z)>0$.

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