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Dynamic behaviors of a delay differential equation model of plankton allelopathy

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Abstract

In this paper, we consider a modified delay differential equation model of the growth of n-species of plankton having competitive and allelopathic effects on each other. We first obtain the sufficient conditions which guarantee the permanence of the system. As a corollary, for periodic case, we obtain a set of delay-dependent condition which ensures the existence of at least one positive periodic solution of the system. After that, by means of a suitable Lyapunov functional, sufficient conditions are derived for the global attractivity of the system. For the two-dimensional case, under some suitable assumptions, we prove that one of the components will be driven to extinction while the other will stabilize at a certain solution of a logistic equation. Examples show the feasibility of the main results.

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1. Introduction

Traditional Lotka–Volterra competitive system can be expressed as follows:

$$\dot{x}_i(t) = x_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right], \quad i = 1, 2, \ldots, n. \tag{1.1}$$

The model has been studied extensively. Many excellent results concerned with permanence, extinction and the existence of a globally attractive positive periodic solution (positive almost periodic solution) of system (1.1) are obtained (see [1,2,5–7,9–11,13–23,26,28,33,35–42] and the references cited therein).

On the other hand, as was pointed out by Chattopadhyay [4], the effects of toxic substances on ecological communities is an important problem from an environmental point of view. In [4], he had proposed the following two species

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competitive system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)[K_1 - \alpha_1 x_1(t) - \beta_{12} x_2(t) - \gamma_1 x_1(t) x_2(t)],\\ \dot{x}_2(t) &= x_2(t)[K_2 - \alpha_2 x_2(t) - \beta_{21} x_2(t) - \gamma_2 x_1(t) x_2(t)],\end{aligned} \tag{1.2}$$

where $x_1(t)$ and $x_2(t)$ denote the population density of two competing species at time t for a common pool of resources. The terms $\gamma_1 x_1(t)x_2(t)$ and $\gamma_2 x_1(t)x_2(t)$ denote the effect of toxic substances. Here Chattopadhyay made the assumption that each species produces a substance toxic to the other, only when the other is present.

Noticing that the production of the toxic substance allelopathic to the competing species will not be instantaneous, but delayed by different discrete time lags required for the maturity of both species, thus, Mukhopadhyay et al. [32] modified system (1.2) to the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)[K_1 - \alpha_1 x_1(t) - \beta_{12} x_2(t) - \gamma_1 x_1(t) x_2(t-\tau_2)],\\ \dot{x}_2(t) &= x_2(t)[K_2 - \alpha_2 x_2(t) - \beta_{21} x_2(t) - \gamma_2 x_1(t-\tau_1) x_2(t)],\end{aligned} \tag{1.3}$$

where $\tau_i > 0,\ i = 1, 2$ are the time required for the maturity of the first species and second species, respectively.

Recently, Jin and Ma [24] argued that the environmental fluctuation is important in an ecosystem, and more realistic models require the inclusion of the effect of environmental changes, especially environmental parameters which are time-dependent and periodically changing (e.g., seasonal changes, food supplies, etc.). They also thought that the distributed delay is more realistic, and proposed the following two-species competition model:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[r_1(t) - \sum_{j=1}^{2} a_{1j}(t)\int_{-T_{1j}}^{0} K_{1j}(s)x_j(t+s)\,\mathrm{d}s - b_1(t)x_1(t)\int_{-\tau_2}^{0} f_2(s)x_2(t+s)\,\mathrm{d}s\right],\\ \dot{x}_2(t) &= x_2(t)\left[r_2(t) - \sum_{j=1}^{2} a_{2j}(t)\int_{-T_{2j}}^{0} K_{1j}(s)x_j(t+s)\,\mathrm{d}s - b_2(t)x_2(t)\int_{-\tau_1}^{0} f_1(s)x_1(t+s)\,\mathrm{d}s\right],\end{aligned} \tag{1.4}$$

where $r_i(t), a_{ij}(t) > 0, b_i(t) > 0$ $(i, j = 1, 2)$ are continuous ω-periodic functions, T_{ij}, τ_i are positive constants, $K_{ij} \in C([-T_{ij}, 0], (0, +\infty))$ and $\int_{-T_{ij}}^{0} K_{ij}(s)\,\mathrm{d}s = 1$, $f_i \in C([-\tau_i, 0], (0, +\infty))$ and $\int_{-\tau_i}^{0} f_i(s)\,\mathrm{d}s = 1$ $(i, j = 1, 2)$. By applying the coincidence degree theory, sufficient conditions which guarantee the existence of at least one positive periodic solutions of system (1.4) are obtained. For additional works related to this topic, see [27,29–31,34].

Here, as far as system (1.4) is concerned, several issues are proposed:

(1) Is it possible to obtain a set of sufficient conditions which guarantee the permanence of the system?
(2) Is it possible to obtain a set of sufficient conditions which guarantee the global attractivity of the positive solution of system (1.4)?
(3) As far as the two species Lotka–Volterra competitive system is concerned, the principle of exclusion is well known (see [1]). However, seldom did scholars consider the final extinction of some of the species in the nonautonomous system (1.2) (we call such a case the partial extinction, see [27,30]). To this day, to the best of our knowledge, no scholar has investigated the partial extinction of system (1.4). Is it possible for us to obtain a set of sufficient conditions which ensure one of the components in system (1.4) will be driven to extinction?

The aim of this paper is to give an affirmative answer to the above three issues. Also, since few things in nature are truly periodic, unlike the consideration of [24], we feel that it is natural to consider the general nonautonomous, nonperiodic system (1.4). Here we propose the following nonautonomous n-species competition system:

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)x_j(t+s)\,\mathrm{d}s - \sum_{j=1, j\neq i}^{n} b_{ij}(t)x_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)x_j(t+s)\,\mathrm{d}s\right], \tag{1.5}$$

where $i = 1, 2, \ldots, n$ and $x_i(t)$ $(1 \leqslant i \leqslant n)$ is the density of ith species at time t.

Throughout this paper, it is assumed that:

(H$_1$) $r_i(t), a_{ij}(t), b_{ij}(t)$ $(i \neq j)$, $i, j=1, 2, \ldots, n$ are continuous and bounded above and below by positive constants on $[0, +\infty)$;

(H$_2$) T_{ij}, τ_{ij} are positive constants, $K_{ij} \in C([-T_{ij}, 0], (0, +\infty))$ and $\int_{-T_{ij}}^{0} K_{ij}(s)\,\mathrm{d}s=1$, $f_{ij} \in C([-\tau_{ij}, 0], (0, +\infty))$ $(i \neq j)$ and $\int_{-\tau_{ij}}^{0} f_{ij}(s)\,\mathrm{d}s = 1$ $(i, j = 1, 2, \ldots, n)$.

We consider (1.5) together with the initial conditions

$$x_i(\theta) = \phi_i(\theta) \geqslant 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0, \tag{1.6}$$

where $\tau = \max_{i,j}\{T_{ij}, \tau_{ij}\}$, ϕ_i are continuous on $[-\tau, 0]$. It is not difficult to see that solutions of (1.5)–(1.6) are well defined for all $t \geqslant 0$ and satisfy

$$x_i(t) > 0 \quad \text{for } t \geqslant 0, \quad i = 1, 2, \ldots, n.$$

Throughout, we shall use the following notations:

- $g^l = \min_{t\geqslant 0} g(t), \quad g^u = \max_{t\geqslant 0} g(t),$
  where g is a continuous bounded function defined on $[0, +\infty)$;
- $J_i = \{1, \ldots, i-1, i+1, \ldots, n\}$.

We say a positive solution of system (1.5) is globally attractive if it attracts all other positive solution of the system.

The organization of this paper is as follows: in the next section, by using the differential inequality theorem, sufficient conditions are obtained for the permanence of system (1.5). As a corollary, for the periodic case, we obtain a set of delay-dependent conditions which ensure the existence of positive periodic solutions of system (1.5). In Section 3, by constructing a suitable Lyapunov functional, a set of easily verified sufficient conditions are obtained for the global attractivity of positive solutions of system (1.5) with the initial conditions (1.6). In Section 4, we consider a two-dimensional case of system (1.5), by further developing the analysis and technique of Li and Chen [27] and Montes De Oca and Vivas [31], we obtain a set of sufficient conditions which ensure that one of the components will be driven to extinction. We also compare our results with some previously known results. Some interesting relationship among the results of [27,30] and our paper are discovered. In Section 5, some suitable examples are presented, which show the feasibility of main results. For the works on the general nonautonomous ecosystem and the construction of Lyapunov functional, one could refer to [12,8,25,39] and the references cited therein. For the works concerned with partial extinction of the species in the ecosystem, one could refer to [1,27,30,31,36].

2. Permanence

This section is concerned with permanence of system (1.5).

Lemma 2.1. *If* $a > 0, b > 0$ *and* $\dot{x}(t) \leqslant (\geqslant) x(t)(b - ax(t))$, $x(t_0) > 0$, *we have*

$$\limsup_{t\to+\infty} x(t) \leqslant \frac{b}{a} \left(\liminf_{t\to+\infty} x(t) \geqslant \frac{b}{a}\right).$$

Proof. From [8, Lemma 2.2], it follows that

$$x(t) \leqslant (\geqslant) \frac{b}{a}\left[1 + \left(\frac{b}{ax(t_0)} - 1\right)\mathrm{e}^{-b(t-t_0)}\right]^{-1}. \tag{2.1}$$

Letting $t \to +\infty$ in the above inequality, it follows that

$$\limsup_{t\to+\infty} x(t) \leqslant \frac{b}{a} \left(\liminf_{t\to+\infty} x(t) \geqslant \frac{b}{a}\right). \qquad \square$$

Theorem 2.1. *Assume* (H_1)–(H_2) *hold, let* $x(t) = (x_1(t), \ldots, x_n(t))^T$ *be any solution of system* (1.5)–(1.6). *Then*

$$\limsup_{t\to+\infty} x_i(t) \leqslant M_i, \quad i = 1, 2, \ldots, n, \tag{2.2}$$

where

$$M_i = \frac{r_i^u}{a_{ii}^l} \exp\{r_i^u T_{ii}\}, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

Proof. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be any solution of system (1.5)–(1.6). It follows from the positivity of the solution and the ith equation of system (1.5) that

$$\dot{x}_i(t) \leqslant r_i^u x_i(t). \tag{2.4}$$

For $-T_{ii} \leqslant s \leqslant 0$ and $t \geqslant 0$, integrating the above differential inequality on the interval $[t + s, t]$ leads to

$$x_i(t + s) \geqslant x_i(t) \exp\{r_i^u s\} \geqslant x_i(t) \exp\{-r_i^u T_{ii}\}, \quad 0 \geqslant s \geqslant -T_{ii}. \tag{2.5}$$

Owing to (2.5) and $\int_{-T_{ii}}^0 K_{ii}(s)\,\mathrm{d}s = 1$, again, from the positivity of the solution and ith equation of system (1.5) it follows that

$$\begin{aligned}\dot{x}_i(t) &\leqslant x_i(t)\left[r_i^u - a_{ii}^l \int_{-T_{ii}}^0 K_{ii}(s) x_i(t+s)\,\mathrm{d}s\right]\\ &\leqslant x_i(t)[r_i^u - a_{ii}^l \exp\{-r_i^u T_{ii}\} x_i(t)].\end{aligned}$$

By applying Lemma 2.1, it immediately follows that

$$\limsup_{t\to+\infty} x_i(t) \leqslant M_i, \quad i = 1, 2, \ldots, n.$$

This ends the proof of Theorem 2.1. □

Theorem 2.2. *Assume* (H_1)–(H_2) *hold. Assume further that*

(H_3)

$$r_i^l - \sum_{j\in J_i} a_{ij}^u M_j - \sum_{j\in J_i} b_{ij}^u M_i M_j > 0, \quad i = 1, 2, \ldots, n$$

holds; *let* $x(t) = (x_1(t), \ldots, x_n(t))^T$ *be any solution of system* (1.5)–(1.6). *Then*

$$\liminf_{t\to+\infty} x_i(t) \geqslant m_i, \quad i = 1, 2, \ldots, n, \tag{2.6}$$

where

$$m_i = \frac{r_i^l - \sum_{j\in J_i} a_{ij}^u M_j - \sum_{j\in J_i} b_{ij}^u M_i M_j}{a_{ii}^u \Delta_i}, \quad i = 1, 2, \ldots, n, \tag{2.7}$$

and

$$\Delta_i = \exp\left\{\left(r_i^l + \sum_{j=1}^n a_{ij}^u M_j + \sum_{j\in J_i} b_{ij}^u M_i M_j\right) T_{ii}\right\}. \tag{2.8}$$

Proof. Let $x(t) = (x_1(t), \ldots, x_n(t))^T$ be any solution of system (1.5)–(1.6), for any positive constant $\varepsilon > 0$, it follows from Theorem 2.1 that there exists a $T_1 > 0$ such that

$$x_i(t) < M_i + \varepsilon \stackrel{\text{def}}{=} M_i^\varepsilon \quad \text{as } t \geqslant T_1.$$

Here, without loss of generality, from condition (H_3) we may choose ε small enough such that

$$r_i^l - \sum_{j\in J_i} a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon > 0, \quad i = 1, 2, \ldots, n.$$

Thus, for $t \geqslant T_1 + \tau$,

$$\dot{x}_i(t) \geqslant \left(r_i^l - \sum_{j=1}^n a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon \right) x_i(t). \tag{2.9}$$

For $-T_{ii} \leqslant s \leqslant 0$ and $t \geqslant T_1 + \tau$, integrating above differential inequality on the interval $[t+s, t]$ leads to

$$x_i(t+s) \leqslant x_i(t) \exp\left\{ \left(r_i^l - \sum_{j=1}^n a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon \right) s \right\} \leqslant x_i(t) \Delta_i^\varepsilon, \quad -T_{ii} \leqslant s \leqslant 0, \tag{2.10}$$

where

$$\Delta_i^\varepsilon = \exp\left\{ \left(r_i^l - \sum_{j=1}^n a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon \right) T_{ii} \right\}.$$

Owing to (2.10) and $\int_{-T_{ii}}^0 K_{ii}(s)\,\mathrm{d}s = 1$, from the positivity of the solution and ith equation of system (1.5) it follows that

$$\begin{aligned}\dot{x}_i(t) &\geqslant x_i(t) \left[r_i^l - \sum_{j\in J_i} a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon - a_{ii}^u \int_{-T_{ii}}^0 K_{ii}(s) x_i(t+s)\,\mathrm{d}s \right] \\ &\geqslant x_i(t) \left[r_i^l - \sum_{j\in J_i} a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon - a_{ii}^u \Delta_i^\varepsilon x_i(t) \right] \quad \text{for } t \geqslant T_1 + \tau.\end{aligned}$$

By applying Lemma 2.1, it immediately follows that

$$\liminf_{t\to+\infty} x_i(t) \geqslant \frac{r_i^l - \sum_{j\in J_i} a_{ij}^u M_j^\varepsilon - \sum_{j\in J_i} b_{ij}^u M_i^\varepsilon M_j^\varepsilon}{a_{ii}^u \Delta_i^\varepsilon}, \quad i = 1, 2, \ldots, n.$$

Setting $\varepsilon \to 0$, it follows that

$$\liminf_{t\to+\infty} x_i(t) \geqslant m_i, \quad i = 1, 2, \ldots, n.$$

This ends the proof of Theorem 2.2. □

Noting that m_i and M_i in Theorems 2.1 and 2.2 are independent of the solution of system (1.5)–(1.6), thus, as a direct corollary of Theorems 2.1 and 2.2 we have:

Theorem 2.3. *Assume that* (H_1)–(H_3) *hold*, *then system* (1.5) *with initial conditions* (1.6) *is permanent*.

Now let us further assume that

(H_4) $r_i(t), a_{ij}(t), b_{ij}(t)$ $(i \neq j)$ are all continuous positive ω-periodic functions.

As a direct corollary of Theorem 2 in [35], from Theorem 2.3, we have:

Corollary 2.1. *If* (H_1)–(H_4) *hold*, *then system* (1.5) *admits at least one positive* ω*-periodic solution*.

Remark 2.1. Jin and Ma [24] showed that to ensure the existence of positive periodic solutions of system (1.4), it is enough to make some restriction on $a_{ij}(t)$ and $r_i(t)$, while $b_i(t)$, $i = 1, 2$ and delays T_{ij}, τ_i, $j = 1, 2$ have no influence on the existence of positive periodic solution. Obviously, Corollary 2.1 is different from that of the results of [24], since our conditions are depend on delays and the coefficients $b_{ij}(t)$ of the system. It is in this sense that Corollary 2.1 supplements the main results of [24].

3. Global attractivity

In the following, we will discuss the global attractivity of the positive solutions of system (1.5) by improving the method given in [12,25,38]. Now we state the main results of this section below.

Theorem 3.1. *In addition to* (H_1)–(H_2), *assume further that we have*:

(H_5) *there exist constants* $\lambda_i > 0,\ i = 1, 2, \ldots, n$ *such that*

$$\lim_{t\to+\infty} \inf\{A_i(t) : i = 1, 2, \ldots, n\} > 0, \tag{3.1}$$

where

$$\begin{aligned}
A_i(t) = {} & \lambda_i a_{ii}(t) - \sum_{i\in J_j} \lambda_j \int_{-T_{ji}}^{0} K_{ji}(s) a_{ji}(t-s)\,\mathrm{d}s - \lambda_i \sum_{j\in J_i} b_{ij}(t) M_j \\
& - \sum_{i\in J_j} \lambda_j M_j \int_{-\tau_{ji}}^{0} f_{ji}(s) b_{ji}(t-s)\,\mathrm{d}s - \lambda_i \int_{-T_{ii}}^{0} \int_{t+s}^{t} K_{ii}(s) \\
& \times a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s \left[r_i(t) + \sum_{j=1}^{n} a_{ij}(t) M_j + \sum_{j\in J_i} b_{ij}(t) M_i M_j \right] \\
& - \sum_{j=1}^{n} \lambda_j M_j \int_{-T_{ji}}^{0} \int_{-T_{jj}}^{0} \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) a_{ji}(t-r) K_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \\
& - \sum_{i\in J_j} \lambda_j M_j^2 \int_{-\tau_{ji}}^{0} \int_{-T_{jj}}^{0} \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) b_{ji}(t-r) f_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \\
& - \lambda_i \sum_{j\in J_i} b_{ij}(t) M_i M_j \int_{-T_{ii}}^{0} \int_{t+s}^{t} K_{ii}(s) a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}$$

Then for any two positive solutions $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ *and* $y(t) = (y_1(t), \ldots, y_n(t))^{\mathrm{T}}$ *of system* (1.5), *one has*

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0.$$

As a direct corollary of Theorem 3.1, we have

Corollary 3.1. *In addition to* (H_1)–(H_2), *assume further that* (H_5') *holds, where* (H_5') *there exist constants* $\lambda_i > 0,\ i = 1, 2, \ldots, n$ *such that*

$$\min_i \{A_i^*\} > 0, \tag{3.2}$$

where

$$A_i^* = \lambda_i a_{ii}^l - \sum_{i\in J_j} \lambda_j a_{ji}^u - \lambda_i \sum_{j\in J_i} b_{ij}^u M_j - \sum_{i\in J_j} \lambda_j M_j b_{ji}^u - \lambda_i a_{ii}^u T_{ii}\left[r_i^u + \sum_{j=1}^n a_{ij}^u M_j + \sum_{j\in J_i} b_{ij}^u M_i M_j\right]$$
$$- \sum_{j=1}^n \lambda_j M_j a_{jj}^u a_{ji}^u T_{jj} - \sum_{i\in J_j} \lambda_j M_j^2 a_{jj}^u b_{ji}^u T_{jj} - \lambda_i \sum_{j\in J_i} b_{ij}^u M_i M_j a_{ii}^u T_{ii}.$$

Then for any two positive solutions $x(t) = (x_1(t), \dots, x_n(t))^{\mathrm{T}}$ *and* $y(t) = (y_1(t), \dots, y_n(t))^{\mathrm{T}}$ *of system* (1.5), *one has*

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0.$$

Proof. Noticing that

$$\int_{-T_{ji}}^0 K_{ji}(s) a_{ji}(t-s)\,\mathrm{d}s \leqslant a_{ji}^u \int_{-T_{ji}}^0 K_{ji}(s)\,\mathrm{d}s = a_{ji}^u,$$
$$\int_{-\tau_{ji}}^0 f_{ji}(s) b_{ji}(t-s)\,\mathrm{d}s \leqslant b_{ji}^u.$$

Also, by using mean value theorem, one has

$$\int_{-T_{ii}}^0 \int_{t+s}^t K_{ii}(s) a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s \leqslant a_{ii}^u \int_{-T_{ii}}^0 \int_{t+s}^t K_{ii}(s)\,\mathrm{d}v\,\mathrm{d}s$$
$$= a_{ii}^u \int_{-T_{ii}}^0 K_{ii}(s)(-s)\,\mathrm{d}s = -a_{ii}^u \xi \int_{-T_{ii}}^0 K_{ii}(s)\,\mathrm{d}s \quad (-T_{ii} \leqslant \xi \leqslant 0)$$
$$\leqslant a_{ii}^u T_{ii}.$$

Similarly, we have

$$\int_{-T_{ji}}^0 \int_{-T_{jj}}^0 \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) a_{ji}(t-r) K_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \leqslant a_{jj}^u a_{ji}^u T_{jj}$$

and

$$\int_{-\tau_{ji}}^0 \int_{-T_{jj}}^0 \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) b_{ji}(t-r) f_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \leqslant a_{jj}^u b_{ji}^u T_{jj}.$$

Thus, by simple computation, we have

$$A_i(t) \geqslant A_i^*,$$

and the above inequality together with (3.2) imply that all the conditions of Theorem 3.1 are satisfied. Thus, the conclusion of Corollary 3.1 follows. □

Remark 3.1. Theorem 3.1 and Corollary 3.1 show that in order to ensure that the system to be stable, one should restrict the coefficients $b_{ij}(t)$ to some suitable region, and the delays T_{ii} to be small enough, that is to say, delays and toxicology play important roles on the stability of the system.

Now let us consider the following n-species Lotka–Volterra system with finite time continuous delays

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^n a_{ij}(t) \int_{-T_{ij}}^0 K_{ij}(s) x_j(t+s)\,\mathrm{d}s\right]. \tag{3.3}$$

As a direct corollary of Theorem 3.1, we have

Theorem 3.2. *In addition to* (H_1)–(H_2), *assume further that we have*:

(H_6) *there exist constants* $\lambda_i > 0,\ i = 1, 2, \ldots, n$ *such that*

$$\lim_{t\to+\infty} \inf \{B_i(t) : i = 1, 2, \ldots, n\} > 0, \tag{3.4}$$

where

$$\begin{aligned} B_i(t) = {} & \lambda_i a_{ii}(t) - \sum_{j\in J_i} \lambda_j \int_{-T_{ji}}^{0} K_{ji}(s) a_{ji}(t+s)\,\mathrm{d}s \\ & - \lambda_i \int_{-T_{ii}}^{0} \int_{t+s}^{t} K_{ii}(s) a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s \left(r_i(t) + \sum_{j=1}^{n} a_{ij}(t) M_j \right) \\ & - \sum_{j\in J_i} \lambda_j M_j \int_{-T_{ji}}^{0} \int_{-T_{jj}}^{0} \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) a_{ji}(t-r) K_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r. \end{aligned}$$

Then for any two positive solutions $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ *and* $y(t) = (y_1(t), \ldots, y_n(t))^{\mathrm{T}}$ *of system* (3.3), *one has*

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0.$$

Similarly to the proof of Corollary 3.1, for system (3.3), we can also obtain the following corollary.

Corollary 3.2. *In addition to* (H_1)–(H_2), *assume further that we have*:

(H_6') *there exist constants* $\lambda_i > 0,\ i = 1, 2, \ldots, n$ *such that*

$$\min_i \{B_i^*\} > 0,$$

where

$$B_i^* = \lambda_i a_{ii}^l - \sum_{j\in J_i} \lambda_j a_{ji}^u - \lambda_i a_{ii}^u T_{ii} \left(r_i^u + \sum_{j=1}^{n} a_{ij}^u M_j \right) - \sum_{j\in J_i} \lambda_j M_j a_{jj}^u a_{ji}^u T_{jj}.$$

Then for any two positive solutions $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ *and* $y(t) = (y_1(t), \ldots, y_n(t))^{\mathrm{T}}$ *of system* (3.3), *one has*

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0.$$

Proof of Theorem 3.1. For two arbitrary nontrivial solutions $x(t)=(x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ and $y(t)=(y_1(t), \ldots, y_n(t))^{\mathrm{T}}$ of system (1.5), from Theorem 2.1, for small enough positive constant ε and M_i $(i=1, 2, \ldots, n)$, there exists a positive constant $T > 0$ such that

$$0 < x_i(t),\ y_i(t) \leqslant M_i + \varepsilon \quad \text{for all } t \geqslant T \text{ and } i = 1, 2, \ldots, n. \tag{3.5}$$

For $i = 1, 2, \ldots, n$, we let

$$V_{i1}(t) = |\ln x_i(t) - \ln y_i(t)|. \tag{3.6}$$

Calculating the upper right derivative of $V_{i1}(t)$ along the solution of system (1.5), by using (3.5), for $t \geqslant T + \tau$, it follows that

$$\begin{aligned}
&D^+V_{i1}(t)\\
&= \left[\frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)}\right]\operatorname{sgn}(x_i(t) - y_i(t))\\
&= \operatorname{sgn}(x_i(t) - y_i(t))\left[-\sum_{j=1}^{n} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s\right.\\
&\quad\left. - \sum_{j\in J_i} b_{ij}(t)\left(x_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)x_j(t+s)\,\mathrm{d}s - y_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)y_j(t+s)\,\mathrm{d}s\right)\right]\\
&= \operatorname{sgn}(x_i(t) - y_i(t))\left[-a_{ii}(t)(x_i(t) - y_i(t)) - \sum_{j\in J_i} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s\right.\\
&\quad - \sum_{j\in J_i} b_{ij}(t)\left(x_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)x_j(t+s)\,\mathrm{d}s - y_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)y_j(t+s)\,\mathrm{d}s\right)\\
&\quad\left. + a_{ii}(t)\int_{-T_{ii}}^{0}\int_{t+s}^{t} K_{ii}(s)(\dot{x}_i(u) - \dot{y}_i(u))\,\mathrm{d}u\,\mathrm{d}s\right]\\
&= \operatorname{sgn}(x_i(t) - y_i(t))\left[-a_{ii}(t)(x_i(t) - y_i(t)) - \sum_{j\in J_i} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s\right.\\
&\quad - \sum_{j\in J_i} b_{ij}(t)\left(x_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s + (x_i(t) - y_i(t))\int_{-\tau_{ij}}^{0} f_{ij}(s)y_j(t+s)\,\mathrm{d}s\right)\\
&\quad + a_{ii}(t)\int_{-T_{ii}}^{0}\int_{t+s}^{t} K_{ii}(s)\left(x_i(u)\left[r_i(u) - \sum_{j=1}^{n} a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)x_j(u+r)\,\mathrm{d}r\right.\right.\\
&\quad\left. - \sum_{j\in J_i} b_{ij}(u)x_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)x_j(u+r)\,\mathrm{d}r\right] - y_i(u)\left[r_i(u) - \sum_{j=1}^{n} a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)y_j(u+r)\,\mathrm{d}r\right.\\
&\quad\left.\left.\left. - \sum_{j\in J_i} b_{ij}(u)y_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r\right]\right)\mathrm{d}u\,\mathrm{d}s\right]\\
&= \operatorname{sgn}(x_i(t) - y_i(t))\left[-a_{ii}(t)(x_i(t) - y_i(t)) - \sum_{j\in J_i} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s\right.\\
&\quad - \sum_{j\in J_i} b_{ij}(t)\left(x_i(t)\int_{-\tau_{ij}}^{0} f_{ij}(s)(x_j(t+s) - y_j(t+s))\,\mathrm{d}s + (x_i(t) - y_i(t))\int_{-\tau_{ij}}^{0} f_{ij}(s)y_j(t+s)\,\mathrm{d}s\right)
\end{aligned}$$

$$
\begin{aligned}
&+ a_{ii}(t)\int_{-T_{ii}}^{0}\int_{t+s}^{t} K_{ii}(s)\Bigg((x_i(u)-y_i(u))\Bigg[r_i(u)\\
&-\sum_{j=1}^{n} a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)y_j(u+r)\,\mathrm{d}r-\sum_{j\in J_i} b_{ij}(u)y_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r\Bigg]\\
&+x_i(u)\Bigg[-\sum_{j=1}^{n}a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)(x_j(u+r)-y_j(u+r))\,\mathrm{d}r\\
&-\sum_{j\in J_i} b_{ij}(u)\left(x_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)x_j(u+r)\,\mathrm{d}r-y_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r\right)\Bigg]\Bigg)\,\mathrm{d}u\,\mathrm{d}s\Bigg]\\
\leqslant &-a_{ii}(t)|x_i(t)-y_i(t)|+\sum_{j\in J_i} a_{ij}(t)\int_{-T_{ij}}^{0} K_{ij}(s)|x_j(t+s)-y_j(t+s)|\,\mathrm{d}s\\
&+\sum_{j\in J_i} b_{ij}(t)(M_i+\varepsilon)\int_{-\tau_{ij}}^{0} f_{ij}(s)|x_j(t+s)-y_j(t+s)|\,\mathrm{d}s\\
&+\sum_{j\in J_i} b_{ij}(t)(M_j+\varepsilon)|x_i(t)-y_i(t)|+a_{ii}(t)\int_{-T_{ii}}^{0}\int_{t+s}^{t} K_{ii}(s)|x_i(u)-y_i(u)|\\
&\times\left[r_i(u)+\sum_{j=1}^{n} a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)y_j(u+r)\,\mathrm{d}r+\sum_{j\in J_i} b_{ij}(u)y_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r\right]\mathrm{d}u\,\mathrm{d}s\\
&+a_{ii}(t)\int_{-T_{ii}}^{0}\int_{t+s}^{t} K_{ii}(s)\Bigg[x_i(u)\sum_{j=1}^{n} a_{ij}(u)\int_{-T_{ij}}^{0} K_{ij}(r)|x_j(u+r)-y_j(u+r)|\,\mathrm{d}r\\
&+x_i(u)\sum_{j\in J_i} b_{ij}(u)x_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)|x_j(u+r)-y_j(u+r)|\,\mathrm{d}r\\
&+x_i(u)\sum_{j\in J_i} b_{ij}(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r|x_i(u)-y_i(u)|\Bigg]\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}
$$

Define

$$
\begin{aligned}
V_{i2}(t)=&\sum_{j\in J_i}\int_{-T_{ij}}^{0}\int_{t+s}^{t} K_{ij}(s)a_{ij}(v-s)|x_j(v)-y_j(v)|\,\mathrm{d}v\,\mathrm{d}s\\
&+\sum_{j\in J_i}(M_i+\varepsilon)\int_{-\tau_{ij}}^{0}\int_{t+s}^{t} f_{ij}(s)b_{ij}(v-s)|x_j(v)-y_j(v)|\,\mathrm{d}v\,\mathrm{d}s\\
&+\int_{-T_{ii}}^{0}\int_{t+s}^{t}\int_{v}^{t} K_{ii}(s)a_{ii}(v-s)|x_i(u)-y_i(u)|\Bigg[r_i(u)+\sum_{j=1}^{n} a_{ij}(u)\\
&\times\int_{-T_{ij}}^{0} K_{ij}(r)y_j(u+r)\,\mathrm{d}r+\sum_{j\in J_i} b_{ij}(u)y_i(u)\int_{-\tau_{ij}}^{0} f_{ij}(r)y_j(u+r)\,\mathrm{d}r\Bigg]\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{-T_{ii}}^{0}\int_{t+s}^{t}\int_{v}^{t}K_{ii}(s)a_{ii}(v-s)\left[x_i(u)\sum_{j=1}^{n}a_{ij}(u)\int_{-T_{ij}}^{0}K_{ij}(r)|x_j(u+r)-y_j(u+r)|\,\mathrm{d}r\right.\\
&+x_i(u)\sum_{j\in J_i}b_{ij}(u)x_i(u)\int_{-\tau_{ij}}^{0}f_{ij}(r)|x_j(u+r)-y_j(u+r)|\,\mathrm{d}r\\
&\left.+x_i(u)\sum_{j\in J_i}b_{ij}(u)\int_{-\tau_{ij}}^{0}f_{ij}(r)y_j(u+r)\,\mathrm{d}r|x_i(u)-y_i(u)|\right]\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s.
\end{aligned}
$$

Thus, for $t \geqslant T+\tau$, one has

$$
\begin{aligned}
&D^{+}V_{i1}(t)+\dot V_{i2}(t)\\
&\leqslant -a_{ii}(t)|x_i(t)-y_i(t)|+\sum_{j\in J_i}\int_{-T_{ij}}^{0}K_{ij}(s)a_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|\\
&\quad+\sum_{j\in J_i}(M_i+\varepsilon)\int_{-\tau_{ij}}^{0}f_{ij}(s)b_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|+\sum_{j\in J_i}b_{ij}(t)(M_j+\varepsilon)|x_i(t)-y_i(t)|\\
&\quad+\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s|x_i(t)-y_i(t)|\left[r_i(t)+\sum_{j=1}^{n}a_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(r)y_j(t+r)\,\mathrm{d}r\right.\\
&\quad\left.+\sum_{j\in J_i}b_{ij}(t)y_i(t)\int_{-\tau_{ij}}^{0}f_{ij}(r)y_j(t+r)\,\mathrm{d}r\right]+\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s\\
&\quad\times\left[x_i(t)\sum_{j=1}^{n}a_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(r)|x_j(t+r)-y_j(t+r)|\,\mathrm{d}r\right.\\
&\quad+x_i(t)\sum_{j\in J_i}b_{ij}(t)x_i(t)\int_{-\tau_{ij}}^{0}f_{ij}(r)|x_j(t+r)-y_j(t+r)|\,\mathrm{d}r\\
&\quad\left.+x_i(t)\sum_{j\in J_i}b_{ij}(t)\int_{-\tau_{ij}}^{0}f_{ij}(r)y_j(t+r)\,\mathrm{d}r|x_i(t)-y_i(t)|\right]\\
&\leqslant -a_{ii}(t)|x_i(t)-y_i(t)|+\sum_{j\in J_i}\int_{-T_{ij}}^{0}K_{ij}(s)a_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|\\
&\quad+\sum_{j\in J_i}(M_i+\varepsilon)\int_{-\tau_{ij}}^{0}f_{ij}(s)b_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|\\
&\quad+\sum_{j\in J_i}b_{ij}(t)(M_j+\varepsilon)|x_i(t)-y_i(t)|+\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s\\
&\quad\times\left[r_i(t)+\sum_{j=1}^{n}a_{ij}(t)(M_j+\varepsilon)+\sum_{j\in J_i}b_{ij}(t)(M_i+\varepsilon)(M_j+\varepsilon)\right]|x_i(t)-y_i(t)|
\end{aligned}
$$

$$+\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s\Bigg[(M_i+\varepsilon)\sum_{j=1}^{n}a_{ij}(t)\int_{-T_{ij}}^{0}K_{ij}(r)$$

$$\times|x_j(t+r)-y_j(t+r)|\,\mathrm{d}r+(M_i+\varepsilon)\sum_{j\in J_i}b_{ij}(t)(M_i+\varepsilon)\int_{-\tau_{ij}}^{0}f_{ij}(r)$$

$$\times|x_j(t+r)-y_j(t+r)|\,\mathrm{d}r+(M_i+\varepsilon)\sum_{j\in J_i}b_{ij}(t)(M_j+\varepsilon)|x_i(t)-y_i(t)|\Bigg].$$

We now define

$$V_i(t)=V_{i1}(t)+V_{i2}(t)+V_{i3}(t), \tag{3.7}$$

in which

$$V_{i3}(t)=(M_i+\varepsilon)\int_{-T_{ij}}^{0}\int_{t+r}^{t}\int_{-T_{ii}}^{0}\int_{l-r+s}^{l-r}K_{ii}(s)a_{ii}(v-s)\sum_{j=1}^{n}a_{ij}(l-r)K_{ij}(r)|x_j(l)-y_j(l)|\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}l\,\mathrm{d}r$$

$$+(M_i+\varepsilon)^2\int_{-\tau_{ij}}^{0}\int_{t+r}^{t}\int_{-T_{ii}}^{0}\int_{l-r+s}^{l-r}K_{ii}(s)a_{ii}(v-s)\sum_{j\in J_i}b_{ij}(l-r)f_{ij}(r)|x_j(l)-y_j(l)|\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}l\,\mathrm{d}r.$$

Then for $t\geqslant T+\tau$, it follows that

$$D^{+}V_i(t)\leqslant -a_{ii}(t)|x_i(t)-y_i(t)|+\sum_{j\in J_i}\int_{-T_{ij}}^{0}K_{ij}(s)a_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|$$

$$+\sum_{j\in J_i}(M_i+\varepsilon)\int_{-\tau_{ij}}^{0}f_{ij}(s)b_{ij}(t-s)\,\mathrm{d}s|x_j(t)-y_j(t)|$$

$$+\sum_{j\in J_i}b_{ij}(t)(M_j+\varepsilon)|x_i(t)-y_i(t)|+\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s$$

$$\times\Bigg[r_i(t)+\sum_{j=1}^{n}a_{ij}(t)(M_j+\varepsilon)+\sum_{j\in J_i}b_{ij}(t)(M_i+\varepsilon)(M_j+\varepsilon)\Bigg]|x_i(t)-y_i(t)|$$

$$+\sum_{j=1}^{n}(M_i+\varepsilon)\int_{-T_{ij}}^{0}\int_{-T_{ii}}^{0}\int_{t-r+s}^{t-r}K_{ii}(s)a_{ii}(v-s)a_{ij}(t-r)K_{ij}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r$$

$$\times|x_j(t)-y_j(t)|+\sum_{j\in J_i}(M_i+\varepsilon)^2\int_{-\tau_{ij}}^{0}\int_{-T_{ii}}^{0}\int_{t-r+s}^{t-r}K_{ii}(s)a_{ii}(v-s)$$

$$\times b_{ij}(t-r)f_{ij}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r|x_j(t)-y_j(t)|+\sum_{j\in J_i}(M_i+\varepsilon)b_{ij}(t)(M_j+\varepsilon)$$

$$\times\int_{-T_{ii}}^{0}\int_{t+s}^{t}K_{ii}(s)a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s|x_i(t)-y_i(t)|.$$

Now we define a Lyapunov functional as follows:

$$V(t)=\sum_{i=1}^{n}\lambda_i V_i(t). \tag{3.8}$$

Then, for $t \geqslant T + \tau$, it follows that

$$D^+V(t) \leqslant -\sum_{i=1}^{n} A_i(t, \varepsilon)|x_i(t) - y_i(t)|, \tag{3.9}$$

where

$$\begin{aligned} A_i(t, \varepsilon) = {} & \lambda_i a_{ii}(t) - \sum_{i \in J_j} \lambda_j \int_{-T_{ji}}^{0} K_{ji}(s) a_{ji}(t-s)\,\mathrm{d}s - \sum_{i \in J_j} \lambda_j (M_j + \varepsilon) \\ & \times \int_{-\tau_{ji}}^{0} f_{ji}(s) b_{ji}(t-s)\,\mathrm{d}s - \lambda_i \sum_{j \in J_i} b_{ij}(t)(M_j + \varepsilon) - \lambda_i \int_{-T_{ii}}^{0} \int_{t+s}^{t} K_{ii}(s) \\ & \times a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s \left[r_i(t) + \sum_{j=1}^{n} a_{ij}(t)(M_j + \varepsilon) + \sum_{j \in J_i} b_{ij}(t)(M_i + \varepsilon)(M_j + \varepsilon) \right] \\ & - \sum_{j=1}^{n} \lambda_j (M_j + \varepsilon) \int_{-T_{ji}}^{0} \int_{-T_{jj}}^{0} \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) a_{ji}(t-r) K_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \\ & - \sum_{i \in J_j} \lambda_j (M_j + \varepsilon)^2 \int_{-\tau_{ji}}^{0} \int_{-T_{jj}}^{0} \int_{t-r+s}^{t-r} K_{jj}(s) a_{jj}(v-s) b_{ji}(t-r) f_{ji}(r)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}r \\ & - \lambda_i \sum_{j \in J_i} (M_i + \varepsilon) b_{ij}(t)(M_j + \varepsilon) \int_{-T_{ii}}^{0} \int_{t+s}^{t} K_{ii}(s) a_{ii}(v-s)\,\mathrm{d}v\,\mathrm{d}s. \end{aligned}$$

By the hypotheses in (H_5), we could choose $\varepsilon > 0$ small enough and a constant $T^* \geqslant T + \tau$ large enough such that

$$A_i(\varepsilon, t) \geqslant \varepsilon > 0 \quad \text{for } t \geqslant T^* \text{ and } i = 1, 2, \ldots, n. \tag{3.10}$$

Integrating both sides of (3.9) on interval $[T^*, t)$,

$$V(t) - V(T^*) \leqslant \int_{T^*}^{t} \sum_{i=1}^{n} A_i(\varepsilon, s)|x_i(s) - y_i(s)|\,\mathrm{d}s \quad \text{for } t \geqslant T^*. \tag{3.11}$$

It follows from (3.10) that

$$V(t) + \varepsilon \int_{T^*}^{t} \sum_{i=1}^{n} |x_i(s) - y_i(s)|\,\mathrm{d}s \leqslant V(T^*) \quad \text{for } t \geqslant T^*. \tag{3.12}$$

Therefore, $V(t)$ is bounded on $[T^*, +\infty)$ and satisfies

$$\int_{T^*}^{t} \sum_{i=1}^{n} |x_i(s) - y_i(s)|\,\mathrm{d}s < +\infty. \tag{3.13}$$

By Theorem 2.1, $|x_i(t) - y_i(t)|$ $(i = 1, 2, \ldots, n)$ are bounded on $[T^*, +\infty)$. On the other hand, it is easy to see that $\dot{x}_i(t)$ and $\dot{y}_i(t)$ are bounded for $t \geqslant T^*$. Therefore, $|x_i(t) - y_i(t)|$ $(i = 1, 2, \ldots, n)$ are uniformly continuous on $[T^*, +\infty)$. By Barbalat's Lemma [3], one can conclude that

$$\lim_{t \to +\infty} \sum_{i=1}^{n} |x_i(t) - y_i(t)| = 0.$$

And so,

$$\lim_{t\to+\infty} |x_i(t) - y_i(t)| = 0, \quad i = 1, 2, \ldots, n.$$

This ends the proof of Theorem 3.1. □

4. Extinction

In this section, we will consider the following two-dimensional system:

$$\dot{x}_1(t) = x_1(t)\left[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)\int_{-T_{12}}^{0} K_{12}(s)x_2(t+s)\,ds - b_{12}(t)x_1(t)\int_{-\tau_{12}}^{0} f_{12}(s)x_2(t+s)\,ds\right],$$

$$\dot{x}_2(t) = x_2(t)\left[r_2(t) - a_{21}(t)\int_{-T_{21}}^{0} K_{21}(s)x_1(t+s)\,ds - a_{22}(t)x_2(t) - b_{21}(t)x_2(t)\int_{-\tau_{21}}^{0} f_{21}(s)x_1(t+s)\,ds\right] \tag{4.1}$$

together with the initial conditions

$$x_i(\theta) = \phi_i(\theta) \geqslant 0, \quad \theta \in (-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, \tag{4.2}$$

where $\tau = \max_{i,j}\{T_{ij}, \tau_{ij}, i, j = 1, 2\}$, ϕ_i are continuous on $[-\tau, 0]$.

Before stating the main results of this section, we introduce a set of conditions:

(H_7)

$$r_1^l a_{21}^l > a_{11}^u r_2^u, \quad r_1^l a_{22}^l \geqslant r_2^u a_{12}^u \quad \text{and} \quad r_1^l b_{21}^l \geqslant r_2^u b_{12}^u.$$

Theorem 4.1. *Assume that* (H_1)–(H_2), (H_7) *hold, and let* col$(x_1(t), x_2(t))$ *be any solution of system* (4.1) *with initial condition* (4.2). *Then* $\lim_{t\to\infty} x_2(t) = 0$ *and* $\lim_{t\to\infty}[x_1(t) - x_1^*(t)] = 0$, *where* $x_1^*(t)$ *is the unique solution of the logistic equation*

$$\dot{x}(t) = x(t)[r_1(t) - a_{11}(t)x(t)], \tag{4.3}$$

such that $0 < \delta \leqslant x_1^*(t) \leqslant \Delta < \infty$ *for certain number* δ *and* Δ.

We will prove Theorem 4.1 by adapting the idea of Montes de Oca and Vivas [31] and Li and Chen [27]. We first state and prove some lemmas which will be useful in the proof of Theorem 4.1.

Lemma 4.1. *Let* col$(x_1(t), x_2(t))$ *be any solution of system* (4.1) *with initial conditions* (4.2). *Then* $x_i(t) > 0$, $i = 1, 2$ *and*

$$x_i(t) \leqslant \max\{r_i^u/a_{ii}^l, \phi_i(0)\}, \quad i = 1, 2.$$

Lemma 4.2. *Let* col$(x_1(t), x_2(t))$ *be any solution of system* (4.1) *with initial conditions* (4.2), *then* $\limsup_{t\to\infty} x_i(t) \leqslant r_i^u/a_{ii}^l$, $i = 1, 2$.

The proofs of Lemmas 4.1 and 4.2 are similar to that of the proofs of Lemmas 1 and 2 of [31], respectively, and we omit the detail here.

Lemma 4.3 (*Fluctuation lemma [31, Lemma 4]*). *Let $x(t)$ be a bounded differentiable function on (α, ∞). Then there exist sequences $\tau_n \to \infty$, $\sigma_n \to \infty$ such that*:

(a) $\dot{x}(\tau_n) \to 0$ *and* $x(\tau_n) \to \limsup_{t\to\infty} x(t) = \overline{x}$ *as* $n \to \infty$,
(b) $\dot{x}(\sigma_n) \to 0$ *and* $x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x}$ *as* $n \to \infty$.

Lemma 4.4. *Let* $\mathrm{col}(x_1(t), x_2(t))$ *be any solution of system* (4.1) *with initial conditions* (4.2). *Under the assumption of Theorem* 4.1, *there exists* $\alpha > 0$ *such that* $x_1(t) \geqslant \alpha$ *for all* $t \geqslant 0$.

Proof. The proof of Lemma 4.4 is similarly to the proof of Lemma 3.5 of Li and Chen [27] and we omit the detail here. □

Lemma 4.5. *Let* $\mathrm{col}(x_1(t), x_2(t))$ *be any solution of system* (4.1) *with initial conditions* (4.2). *Under the assumption of Theorem* 4.1, $\lim_{t\to\infty} x_2(t) = 0$.

Proof. It follows from Lemma 4.1 that $x_i(t),\ i = 1, 2$ are bounded and positive for all $t \geqslant 0$. Let $\underline{x}_1 = \liminf_{t\to\infty} x_1(t)$ and $\overline{x}_2 = \limsup_{t\to\infty} x_2(t)$. From Lemma 4.4 we know that $\underline{x}_1 \geqslant \alpha > 0$, obviously $\overline{x}_2 \geqslant 0$. To end the proof of Lemma 4.5, it suffices to show that $\overline{x}_2 = 0$. In order to get a contradiction, suppose that $\overline{x}_2 > 0$. According to Fluctuation lemma, there exists sequences $\tau_n \to \infty$, $\sigma_n \to \infty$ such that $\dot{x}_1(\tau_n) \to 0$, $\dot{x}_2(\sigma_n) \to 0$, $x_1(\tau_n) \to \underline{x}_1$ and $x_2(\sigma_n) \to \overline{x}_2$. Since the functions $\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)\,\mathrm{d}s$, $\int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)\,\mathrm{d}s$, $\int_{-T_{21}}^0 K_{21}(s)x_1(t+s)\,\mathrm{d}s$ and $\int_{-\tau_{21}}^0 f_{21}(s)x_1(t+s)\,\mathrm{d}s$ are bounded, we can assume that

$$\int_{-T_{12}}^0 K_{12}(s)x_2(\tau_n+s)\,\mathrm{d}s \to \alpha_1,$$
$$\int_{-\tau_{12}}^0 f_{12}(s)x_2(\tau_n+s)\,\mathrm{d}s \to \alpha_2,$$
$$\int_{-T_{21}}^0 K_{21}(s)x_1(\sigma_n+s)\,\mathrm{d}s \to \beta_1,$$
$$\int_{-\tau_{21}}^0 f_{21}(s)x_1(\sigma_n+s)\,\mathrm{d}s \to \beta_2.$$

It is clear that $\alpha_i \leqslant \overline{x}_2$, $\beta_i \geqslant \underline{x}_1$, $i = 1, 2$. Therefore, it follows from (4.1) that

$$0 \geqslant \underline{x}_1[r_1^l - a_{11}^u\underline{x}_1 - a_{12}^u\overline{x}_2 - b_{12}^u\underline{x}_1\overline{x}_2],$$
$$0 \leqslant \overline{x}_2[r_2^u - a_{21}^l\underline{x}_1 - a_{22}^l\overline{x}_2 - b_{21}^l\underline{x}_1\overline{x}_2].$$

Since $\underline{x}_1 \geqslant \alpha > 0$ and $\overline{x}_2 > 0$, it follows that

$$r_1^l \leqslant a_{11}^u\underline{x}_1 + a_{12}^u\overline{x}_2 + b_{12}^u\underline{x}_1\overline{x}_2, \tag{4.4}$$

$$r_2^u \geqslant a_{21}^l\underline{x}_1 + a_{22}^l\overline{x}_2 + b_{21}^l\underline{x}_1\overline{x}_2. \tag{4.5}$$

Now, by applying third inequality in (H_7) to (4.4), we get

$$r_1^l \leqslant a_{11}^u\underline{x}_1 + a_{12}^u\overline{x}_2 + \frac{b_{21}^l}{r_2^u} r_1^l\underline{x}_1\overline{x}_2. \tag{4.6}$$

Multiplying (4.5) by $-r_1^l/r_2^u$ and adding the corresponding inequality to (4.6), we obtain

$$0 \leqslant \left(a_{11}^u - \frac{r_1^l}{r_2^u}a_{21}^l\right)\underline{x}_1 + \left(a_{12}^u - \frac{r_1^l}{r_2^u}a_{22}^l\right)\overline{x}_2,$$

that is

$$\left(a_{12}^u-\frac{r_1^l}{r_2^u}a_{22}^l\right)\overline{x}_2\geqslant\left(\frac{r_1^l}{r_2^u}a_{21}^l-a_{11}^u\right)\underline{x}_1. \tag{4.7}$$

From the first inequality in condition (H_7) and $\underline{x}_1\geqslant\alpha>0$, we get $((r_1^l/r_2^u)a_{21}^l-a_{11}^u)\underline{x}_1>0$. Therefore, (4.7) leads to

$$\left(a_{12}^u-\frac{r_1^l}{r_2^u}a_{22}^l\right)\overline{x}_2>0. \tag{4.8}$$

From the second inequality in condition (H_7) we have

$$a_{12}^u-\frac{r_1^l}{r_2^u}\,a_{22}^l\leqslant 0, \tag{4.9}$$

and (4.8) together with (4.9) leads to $\overline{x}_2\leqslant 0$, which is a contradiction. This completes the proof of Lemma 4.5. □

Lemma 4.6 (*Montes De Oca and Vivas [31, Lemma 7]*). *There exists a unique solution $x_1^*(t)$ of the logistic equation* (4.3) *such that $\delta\leqslant x_1^*(t)\leqslant\Delta$ on $(-\infty,\infty)$, where Δ and δ are any numbers satisfying the inequalities $0<\delta<r_1^l/a_{11}^u$ and $r_1^u/a_{11}^l<\Delta$.*

Proof of Theorem 4.1. Lemma 4.5 shows that under the assumption of Theorem 4.1, $\lim_{t\to\infty}x_2(t)=0$. We define $w(t)=1/x_1(t),\ w^*(t)=1/x_1^*(t)$ and $z(t)=w(t)-w^*(t)$. Then $0<w(t)\leqslant 1/\alpha$ and $1/\Delta\leqslant w^*(t)\leqslant 1/\delta$, so it follows that

$$w'(t)=\,-r_1(t)w(t)+a_{11}(t)+a_{12}(t)w(t)\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)\,\mathrm{d}s+b_{12}(t)\int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)\,\mathrm{d}s,$$

$$w^{*\prime}(t)=-r_1(t)w^*(t)+a_{11}(t),$$

and for all $t\geqslant 0$

$$z'(t)=-r_1(t)z(t)+a_{12}(t)w(t)\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)\,\mathrm{d}s+b_{12}(t)\int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)\,\mathrm{d}s.$$

Because $z(t)$ is a bounded differentiable function, by the Fluctuation lemma there exists sequences $\tau_n\to\infty,\sigma_n\to\infty$ such that $z(\tau_n)\to\overline{z};\ z'(\tau_n)\to 0,\ z(\sigma_n)\to\underline{z};\ z'(\sigma_n)\to 0$. Now we shall show that $\underline{z}=\overline{z}=0$. From the above inequalities we obtain

$$z(t)=\frac{a_{12}(t)w(t)\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)\,\mathrm{d}s}{r_1(t)}+\frac{b_{12}(t)\int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)\,\mathrm{d}s}{r_1(t)}-\frac{z'(t)}{r_1(t)}. \tag{4.10}$$

Since

$$0<\frac{a_{12}(t)w(t)}{r_1(t)}\leqslant\frac{a_{12}^u/\alpha}{r_1^l},\quad 0<\frac{b_{12}(t)}{r_1(t)}\leqslant\frac{b_{12}^u}{r_1^l},\quad 0<\frac{1}{r_1(t)}\leqslant\frac{1}{r_1^l},$$

$$\lim_{t\to\infty}\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)\,\mathrm{d}s=0,\quad \lim_{t\to\infty}\int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)\,\mathrm{d}s=0,$$

it follows that

$$
\begin{aligned}
\lim_{n\to\infty}\frac{a_{12}(\tau_n)w(\tau_n)\int_{-T_{12}}^{0}K_{12}(s)x_2(\tau_n+s)\,\mathrm{d}s}{r_1(\tau_n)} &= \lim_{n\to\infty}\frac{b_{12}(\tau_n)\int_{-\tau_{12}}^{0}f_{12}(s)x_2(\tau_n+s)\,\mathrm{d}s}{r_1(\tau_n)}\\
&= \lim_{n\to\infty}\frac{z'(\tau_n)}{r_1(\tau_n)}\\
&= \lim_{n\to\infty}\frac{a_{12}(\sigma_n)w(\sigma_n)\int_{-T_{12}}^{0}K_{12}(s)x_2(\sigma_n+s)\,\mathrm{d}s}{r_1(\sigma_n)}\\
&= \lim_{n\to\infty}\frac{b_{12}(\sigma_n)\int_{-\tau_{12}}^{0}f_{12}(s)x_2(\sigma_n+s)\,\mathrm{d}s}{r_1(\sigma_n)}\\
&= \lim_{n\to\infty}\frac{z'(\sigma_n)}{r_1(\sigma_n)}=0.
\end{aligned}
$$

Therefore, it follows from (4.10) that $\underline{z}=\overline{z}=0$, that is $\lim_{t\to\infty}z(t)=0$. Noticing that

$$|x_1(t)-x_1^*(t)|=|w^*(t)-w(t)|x_1(t)x_1^*(t)$$

and $x_1(t)$ and $x_1^*(t)$ are bounded functions, we obtain the desired result, that is, $\lim_{t\to\infty}(x_1(t)-x_1^*(t))=0$. This completes the proof of Theorem 4.1. □

Now let us consider the following system:

$$
\begin{aligned}
\dot{x}_1(t)&=x_1(t)\left[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)\int_{-\infty}^{0}K_{12}(s)x_2(t+s)\,\mathrm{d}s-b_{12}(t)x_1(t)\int_{-\infty}^{0}f_{12}(s)x_2(t+s)\,\mathrm{d}s\right],\\
\dot{x}_2(t)&=x_2(t)\left[r_2(t)-a_{21}(t)\int_{-\infty}^{0}K_{21}(s)x_1(t+s)\,\mathrm{d}s-a_{22}(t)x_2(t)\right.\\
&\qquad\left.-b_{21}(t)x_2(t)\int_{-\infty}^{0}f_{21}(s)x_1(t+s)\,\mathrm{d}s\right],
\end{aligned}
\tag{4.11}
$$

together with the initial conditions

$$x_i(\theta)=\phi_i(\theta)\geqslant 0,\quad \theta\in(-\infty,0],\qquad \phi_i(0)>0,\quad i=1,2, \tag{4.12}$$

where ϕ_i are continuous on $(-\infty,0]$. We introduce a condition

(H_2') $K_{ij}\in C((-\infty,0],(0,+\infty))$ and $\int_{-\infty}^{0}K_{ij}(s)\,\mathrm{d}s=1$, $f_{ij}\in C((-\infty,0],(0,+\infty))$ $(i\neq j)$ and $\int_{-\infty}^{0}f_{ij}(s)\,\mathrm{d}s=1$ $(i,j=1,2)$.

Theorem 4.2. *Assume that* (H_1), (H_2') *and* (H_7) *hold, let* $\mathrm{col}(x_1(t),x_2(t))$ *be any solution of system* (4.11) *with initial conditions* (4.12). *Then* $\lim_{t\to\infty}x_2(t)=0$ *and* $\lim_{t\to\infty}[x_1(t)-x_1^*(t)]=0$, *where* $x_1^*(t)$ *is the unique solution of* (4.3).

Remark 4.1. Under the assumption $b_{21}(t)=b_{12}(t)=0$, Theorem 4.2 implies the main theorem of [31], thus we generalize the main results of [31].

Now let us consider the following system:

$$
\begin{aligned}
\dot{x}_1(t)&=x_1(t)[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)x_2(t-\tau_{12}(t))-b_{12}(t)x_1(t)x_2(t-\eta_{12}(t))],\\
\dot{x}_2(t)&=x_2(t)[r_2(t)-a_{21}(t)x_1(t-\tau_{21}(t))-a_{22}(t)x_2(t)-b_{21}(t)x_2(t)x_1(t-\eta_{21}(t))],
\end{aligned}
\tag{4.13}
$$

together with the initial conditions

$$x_i(\theta) = \phi_i(\theta) \geqslant 0, \quad \theta \in [-\tau, 0], \quad \phi_i(0) > 0, \quad i = 1, 2, \tag{4.14}$$

where $\tau_{ij}(t), \eta_{ij}(t), i, j = 1, 2$ are nonnegative continuous bounded functions, $\tau = \max_t \{\tau_{ij}(t), \eta_{ij}(t), \ i, j = 1, 2\}$, ϕ_i are continuous on $[-\tau, 0]$.

Theorem 4.3. *Assume that* (H_1) *and* (H_7) *hold, and let* $\mathrm{col}(x_1(t), x_2(t))$ *be any solution of system* (4.13) *with initial condition* (4.14). *Then* $\lim_{t\to\infty} x_2(t) = 0$ *and* $\lim_{t\to\infty} [x_1(t) - x_1^*(t)] = 0$, *where* $x_1^*(t)$ *is the unique solution of* (4.3).

The proofs of Theorems 4.2 and 4.3 are similarly to that of the proof of Theorem 4.1, we omit the detail here.

Now let us consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - b_{12}(t)x_1(t)x_2(t)], \\ \dot{x}_2(t) &= x_2(t)[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - b_{21}(t)x_2(t)x_1(t)]. \end{aligned} \tag{4.15}$$

As a direct corollary of Theorem 4.3, we have:

Corollary 4.1. *Assume that* (H_1) *and* (H_7) *hold, and let* $\mathrm{col}(x_1(t), x_2(t))$ *be any solution of system* (4.15) *with initial values* $x_i(t_0) > 0$, $i = 1, 2$. *Then* $\lim_{t\to\infty} x_2(t) = 0$ *and* $\lim_{t\to\infty} [x_1(t) - x_1^*(t)] = 0$, *where* $x_1^*(t)$ *is the unique solution of* (4.3).

Remark 4.2. Corollary 4.1 is the main theorem of [27], thus Theorems 4.1–4.3 generalize the main result of [27] to the continuous delay (infinite delay or finite delay) and discrete delay cases.

Remark 4.3. Unlike Sections 2 and 3, where we show that delays play important roles on the permanence and the stability of system (4.1), Theorems 4.1–4.3 and Corollary 4.1 show that only if the coefficients of the system satisfy certain inequalities, delays have no influence on determining the extinction behavior of some components of the system. Also, condition (H_7) shows that toxic substance plays important role on the extinction of the species.

Corresponding to condition (H_7), we now further propose following two set of assumptions:

(H_8)

$$r_1^l a_{21}^l > a_{11}^u r_2^u, \quad r_1^l a_{22}^l \geqslant r_2^u a_{12}^u \quad \text{and} \quad b_{12}^u a_{22}^l \leqslant a_{12}^u b_{21}^l.$$

(H_9)

$$r_1^l a_{21}^l > a_{11}^u r_2^u, \quad r_1^l a_{22}^l \geqslant r_2^u a_{12}^u, \quad b_{12}^u a_{22}^l \leqslant a_{12}^u b_{21}^l \quad \text{and} \quad a_{12}^u a_{21}^l \leqslant a_{11}^u a_{22}^l.$$

Mahhuba [30] showed that under the assumptions (H_1) and (H_9), the conclusion of Corollary 3.1 holds. Recently, we [27] gave an example which satisfies condition (H_7) but not satisfies condition (H_9) (indeed, the last inequality in (H_9) is not satisfied). At first sight, condition (H_7) and (H_9) seem independent of each other. Are they really? Indeed, we have the following interesting results:

Theorem 4.4. *Under the assumption* (H_1), *as far as system* (4.15) *is concerned*, (H_9) $\Leftrightarrow$ (H_8), (H_8) $\Rightarrow$ (H_7).

Proof. Obviously, (H_9) $\Rightarrow$ (H_8), since (H_8) is the special case of (H_9). To show that (H_8) $\Rightarrow$ (H_9), it is enough to show that if (H_8) holds, then

$$a_{12}^u a_{21}^l \leqslant a_{11}^u a_{22}^l \tag{4.16}$$

holds.

We mention here that since (H_8) holds, the conclusion of Lemma 4.4 and (4.4), (4.5) hold. Now, multiplying (4.5) by $-a_{12}^u/a_{22}^l$ and adding the result to (4.4), we obtain

$$r_1^l - \frac{a_{12}^u}{a_{22}^l} r_2^u \leqslant \left(a_{11}^u - \frac{a_{12}^u}{a_{22}^l} a_{21}^l\right) \underline{x}_1 + \left(b_{12}^u - \frac{a_{12}^u}{a_{22}^l} b_{21}^l\right) \underline{x}_1 \overline{x}_2. \tag{4.17}$$

By (H_8), we have

$$r_1^l - \frac{a_{12}^u}{a_{22}^l} r_2^u > 0, \quad b_{12}^u - \frac{a_{12}^u}{a_{22}^l} b_{21}^l \leqslant 0.$$

Thus, from $\underline{x}_1 \geqslant \alpha$, $\overline{x}_2 \geqslant 0$ we have

$$\left(a_{11}^u - \frac{a_{12}^u}{a_{22}^l} a_{21}^l\right) \underline{x}_1 > 0.$$

That is,

$$a_{11}^u - \frac{a_{12}^u}{a_{22}^l} a_{21}^l > 0,$$

and so, (4.16) holds.

To end the proof of Theorem 4.4, it is enough to show that (H_8) $\Rightarrow$ (H_7). Indeed, from the second inequality of (H_8) it follows that

$$a_{22}^l \geqslant \frac{r_2^u a_{12}^u}{r_1^l}. \tag{4.18}$$

Therefore, from the third inequality of (H_8), we have

$$a_{12}^u b_{21}^l \geqslant b_{12}^u a_{22}^l \geqslant b_{12}^u \frac{r_2^u a_{12}^u}{r_1^l}.$$

That is,

$$r_1^l b_{21}^l \geqslant r_2^u b_{12}^u,$$

which shows that the third inequality in (H_7) holds. This ends the proof of Theorem 4.4. □

5. Examples

Example 5.1. Consider the following system:

$$\begin{aligned}
\dot{x}_1(t) = x_1(t) \Bigg[& 80 - 100 \int_{-T_{11}}^{0} K_{11}(s) x_1(t+s)\,\mathrm{d}s - \frac{1}{10} \int_{-T_{12}}^{0} K_{12}(s) x_2(t+s)\,\mathrm{d}s \\
& - \left(\frac{1}{20} + \frac{\cos(t)}{20}\right) x_1(t) \int_{-\tau_{12}}^{0} f_{12}(s) x_2(t+s)\,\mathrm{d}s \Bigg], \\
\dot{x}_2(t) = x_2(t) \Bigg[& 80 - \frac{1}{10} \int_{-T_{21}}^{0} K_{21}(s) x_1(t+s)\,\mathrm{d}s - 100 \int_{-T_{22}}^{0} K_{22}(s) x_2(t+s)\,\mathrm{d}s \\
& - \left(\frac{1}{20} + \frac{\cos(t)}{20}\right) x_2(t) \int_{-\tau_{21}}^{0} f_{21}(s) x_1(t+s)\,\mathrm{d}s \Bigg],
\end{aligned} \tag{5.1}$$

where $T_{ij}=\tau_{ij}=\frac{1}{160\,000}$. In this case, corresponding to system (1.5), we have $r_i(t)=80$, $a_{11}(t)=a_{22}(t)=100$, $a_{12}(t)=a_{21}(t)=\frac{1}{10}$, $b_{12}(t)=b_{21}(t)=\frac{1}{20}+\cos(t)/20$. Also, we assume that

$$\int_{-T_{ij}}^{0} K_{ij}(s)\,\mathrm{d}s=1,\quad \int_{-\tau_{ij}}^{0} f_{ij}(s)\,\mathrm{d}s=1,\quad i,j=1,2.$$

By computation, we have

$$M_i=\frac{r_i^u}{a_{ii}^l}\exp\{r_i^u T_{ii}\}=\frac{80}{100}\exp\left\{80\cdot\frac{1}{160\,000}\right\}$$
$$\leqslant\frac{80}{100}3^{1/2000}\approx\frac{80}{100}\times 1.000549457\leqslant 1,\quad i=1,2.$$

And so,

$$r_i^l-\sum_{j\in J_i}a_{ij}^u M_j-\sum_{j\in J_i}b_{ij}^u M_i M_j>80-\frac{1}{5}-\frac{1}{10}>0,\quad i=1,2.$$

Above inequality shows that condition (H_3) holds, thus, by Theorems 2.2 and 2.3 we know that system (5.1) is permanent and admits at least one positive 2π-periodic solution.

Now, we take $\lambda_i=1>0$, $i=1,2$. By simple computation, corresponding to Corollary 3.1, we have

$$A_i^*\geqslant 20>0,\quad i=1,2.$$

Therefore, from Corollary 3.1 we know that any positive solution of system (5.1) is globally attractive.

Example 5.2. Consider the following system:

$$\begin{aligned}\dot{x}_1(t)&=x_1(t)[4-(1.5+\sin(10t))x_1(t)-(1+0.5\sin(10t))x_2(t-\tfrac{1}{10})-(\sqrt{2}+\cos(10t))x_1(t)x_2(t-\tfrac{1}{10})],\\ \dot{x}_2(t)&=x_2(t)[2-(3+0.5\cos(10t))x_1(t-\tfrac{1}{20})-(3.5+0.5\sin(10t))x_2(t)-3x_1(t-\tfrac{1}{20})x_2(t)].\end{aligned}\tag{5.2}$$

In this case, corresponding to system (4.13), we have

$$r_1(t)=4,\quad a_{11}(t)=1.5+\sin(10t),\quad a_{12}(t)=1+0.5\sin(10t),$$
$$r_2(t)=2,\quad a_{21}(t)=3+0.5\cos(10t),\quad a_{22}(t)=3.5+0.5\sin(10t),$$
$$b_{12}(t)=\sqrt{2}+\cos(10t),\quad b_{21}(t)=3,$$
$$\tau_{12}(t)=\eta_{12}(t)=\tfrac{1}{10},\quad \tau_{21}(t)=\eta_{21}(t)=\tfrac{1}{20}.$$

By simple computation, we have

$$r_1^l a_{21}^l=4\times 2.5=10>a_{11}^u r_2^u=2\times 2=4,$$
$$r_1^l a_{22}^l=4\times 3=12\geqslant r_2^u a_{12}^u=2\times 1.5=3,$$
$$r_1^l b_{21}^l=4\times 3=12>6=2\times 3\geqslant r_2^u b_{12}^u=2\times(\sqrt{2}+1).$$

Thus, condition (H_7) is satisfied. From Theorem 4.3 it follows that $\lim_{t\to\infty}x_2(t)=0$ and $\lim_{t\to\infty}[x_1(t)-x_1^*(t)]=0$, where $x_1^*(t)$ is the unique positive periodic solution of $\dot{x}_1(t)=x_1(t)[4-(1.5+\sin(10t))x_1(t)]$.

However, in this case, we have

$$b_{12}^u a_{22}^l=(\sqrt{2}+1)\times 3\geqslant a_{12}^u b_{21}^l=1.5\times 3,$$

and therefore, the third inequality in (H_8) could not hold. Thus, our results improve the main results of Mahhuba [30]. Fig. 1 is the numeric simulation of the solution of system (5.2) with the initial condition $(x_1(\theta),x_2(\theta))=(3,4)$, $\theta\in[-\frac{1}{10},0]$.

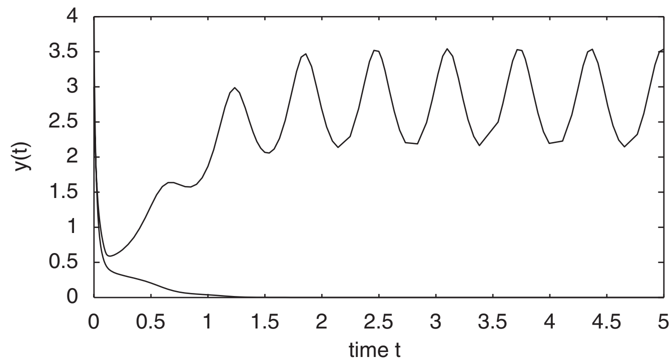

Fig. 1. Numeric simulation of the solution of system (5.2) with initial condition $(x_1(\theta), x_2(\theta)) = (3, 4)$, $\theta \in [-\frac{1}{10}, 0]$ and $t \in [0, 5]$.

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