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Algebraic integrability of foliations of the plane ☆

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Abstract

We give an algorithm to decide whether an algebraic plane foliation $\mathcal{F}$ has a rational first integral and to compute it in the affirmative case. The algorithm runs whenever we assume the polyhedrality of the cone of curves of the surface obtained after blowing-up the set $\mathcal{B}_{\mathcal{F}}$ of infinitely near points needed to get the dicritical exceptional divisors of a minimal resolution of the singularities of $\mathcal{F}$. This condition can be detected in several ways, one of them from the proximity relations in $\mathcal{B}_{\mathcal{F}}$ and, as a particular case, it holds when the cardinality of $\mathcal{B}_{\mathcal{F}}$ is less than 9.
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1. Introduction

The problem of deciding whether a complex polynomial differential equation on the plane is algebraically integrable goes back to the end of the nineteenth century when Darboux [12], Poincaré [37–39], Painlevé [35] and Autonne [1] studied it. In modern terminology and from a more algebraic point of view, it can be stated as deciding whether an algebraic foliation $\mathcal{F}$ with singularities on the projective plane over an algebraically closed field of characteristic zero (plane foliation or foliation on $\mathbb{P}^2$, in the sequel) admits a rational first integral and, if it is so, to compute it. In the main result of this paper, we shall give a satisfactory answer to that problem when the cone of curves of certain surface is polyhedral. This surface is obtained by blowing-up what we call dicritical points of a minimal resolution of the singularities of $\mathcal{F}$.

The existence of a rational first integral is equivalent to state that every invariant curve of $\mathcal{F}$ is algebraic. The fact that $\mathcal{F}$ admits algebraic invariant curves has interest for several reasons.

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For instance, it is connected with the center problem for quadratic vector fields [10,41], with problems related to solutions of Einstein's field equations in general relativity [22] or with the second part of the Hilbert's sixteenth problem, which looks for a bound of the number of limit cycles for a (real) polynomial vector field [29].

Coming back to the nineteenth century and in the analytic complex case, it was Poincaré [38] who observed that "to find out if a differential equation of the first order and of the first degree is algebraically integrable, it is enough to find an upper bound for the degree of the integral. Afterwards, we only need to perform purely algebraic computations". This observation gave rise to the so called Poincaré problem which, nowadays, is established as the one of bounding the degrees of the algebraic leaves of a foliation whether it is algebraically integrable or not. It was Poincaré himself who studied a particular case within the one where the singularities of the foliation are non-degenerated [38]. Carnicer [8] provided an answer for the nondicritical foliations. Bounds depending on the invariant curve have been obtained by Campillo and Carnicer [4] and, afterwards, improved by Esteves and Kleiman [15]. However, the classical Poincaré problem has a negative answer, that is the degree of a general irreducible invariant curve of a plane foliation $\mathcal{F}$ with a rational first integral cannot be bounded by a function on the degree of the foliation, and this happens even in the case of families of foliations where the analytic type of each singularity is constant [27]. An analogous answer is given in [27] for a close question posed by Painlevé in [35], which consists of recognizing the genus of the general solution of a foliation as above. Also Painlevé, proved in [34] that 12 is an upper bound of the degree of a general invariant curve for a plane foliation with a rational first integral which has, at most, 8 dicritical points that can be resolved by only one blow-up. Although with a different philosophy and since the cone of curves of the surface obtained after blowing-up those dicritical points is polyhedral, our main result can be regarded, in some sense, as a generalization of that result by Painlevé.

The so called d-extactic curves of a plane foliation $\mathcal{F}$, studied by Lagutinskii [13] and Pereira [36], provide a nice, but non-efficient, procedure to decide whether $\mathcal{F}$ has a rational first integral of some given degree. Moreover, two sufficient conditions so that $\mathcal{F}$ had a rational first integral are showed in [10].

The main result of this paper is to give an algorithm to decide whether plane foliations in a certain class have a rational first integral, which also allows to compute it. In fact, in the affirmative case, we obtain a primitive first integral, that allows to get any other rational first integral, and, trivially, a bound for the degree of the irreducible components of the invariant curves of the foliation. A well-known result is the so called resolution theorem [42] that asserts that after finitely many blow-ups at singular points (of the successively obtained foliations by those blow-ups), $\mathcal{F}$ is transformed in a foliation on another surface with finitely many singularities, all of them of a nonreducible by blowing-up type, called simple. The input of our algorithm will be the foliation and also a part of the mentioned configuration of infinitely near points that resolves its singularities. This part is what leads to obtain the so called dicritical exceptional divisors of the foliation. If we blow that configuration, we get a surface $Z_{\mathcal{F}}$, and the cone of curves $NE(Z_{\mathcal{F}})$ of that surface will be our main tool (such a cone is a basic object in the minimal model theory [26]). When $\mathcal{F}$ admits a rational first integral, it has associated an irreducible pencil of plane curves whose general fibers provide a divisor $D_{\mathcal{F}}$ on $Z_{\mathcal{F}}$. The class in the Picard group of $D_{\mathcal{F}}$ determines a face of $NE(Z_{\mathcal{F}})$ that contains essential information on the invariant by $\mathcal{F}$ curves (Theorem 1). This face has codimension 1 if, and only if, $\mathcal{F}$ has an independent system of algebraic solutions S (see Definition 3). If we get a system as S, then we can determine $D_{\mathcal{F}}$ (see Theorem 2 and Proposition 2). We devote Section 3 to explain and prove the above considerations and Section 2 to give the preliminaries and notations.

In Section 4, we show our main result, Theorem 3, by means of two algorithms, based on the above results, that one must jointly use: Algorithm 3 runs for foliations $\mathcal{F}$ whose cone of curves $NE(Z_{\mathcal{F}})$ is (finite) polyhedral and computes a system S as above (or discards the existence of a rational first integral), whereas Algorithm 2 uses S to compute a rational first integral (or newly discards its existence). Both algorithms can be implemented without difficulty from their inputs. Polyhedrality of the cone of curves happens in several cases. For instance, when the anti-canonical bundle on $Z_{\mathcal{F}}$ is ample by the Mori cone theorem [32]. Also whenever $Z_{\mathcal{F}}$ is obtained by blowing-up configurations of toric type [5,33], relative to pencils $H - \lambda Z^d$, where H is an homogeneous polynomial of degree d that defines a curve with a unique branch at infinity [7], or of less than nine points [6] (see also [31]) that are included within the much wider set of P-sufficient configurations (Definition 4), whose cone of curves is also polyhedral (see [17] for a proof). Notice that in this last case (P-sufficient configurations) and when Algorithm 3 returns an independent system of algebraic solutions, to compute a rational first integral of $\mathcal{F}$ is simpler than above and, furthermore, the Painlevé problem is solved (Propositions 2 and 5). This last section includes several illustrative examples of our results.

To conclude, recall that Prelle and Singer gave in [40] a procedure (implemented by Man [30]) to compute elementary first integrals of foliations $\mathcal{F}$ on the projective plane over the complex numbers. As a particular case, it uses results by Darboux and Jouanolou to deal with the computation of meromorphic first integrals of $\mathcal{F}$; however, the obstruction revealed by the Poincaré problem makes the above procedure be only a semi-decision one. In this paper, we also give an alternative algorithm (Algorithm 1) that decides, for each tentative degree d of a general irreducible invariant curve, whether $\mathcal{F}$ has a rational first integral of degree d and computes it in the affirmative case. Its input is $\mathcal{F}$ and the cited part of its resolution configuration, and it uniquely involves resolution of systems of linear equations.

2. Preliminaries and notations

Let k be an algebraically closed field of characteristic zero. An (algebraic singular) *foliation* $\mathcal{F}$ on a projective smooth surface (a surface in the sequel) X can be defined by the data $\{(U_i, \omega_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of X, ω_i is a nonzero regular differential 1-form on U_i with isolated zeros and, for each couple $(i, j) \in I \times I$,

$$\omega_i = g_{ij} w_j \quad \text{on } U_i \cap U_j,\ g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*. \tag{1}$$

Given $p \in X$, a *(formal) solution of $\mathcal{F}$ at p* will be an irreducible element $f \in \widehat{\mathcal{O}}_{X,p}$ (where $\widehat{\mathcal{O}}_{X,p}$ is the m_p-adic completion of the local ring $\mathcal{O}_{X,p}$ and m_p its maximal ideal) such that the local differential 2-form $\omega_p \wedge df$ is a multiple of f, w_p being a local equation of $\mathcal{F}$ at p. An element in $\widehat{\mathcal{O}}_{X,p}$ will be said to be *invariant by $\mathcal{F}$* if all its irreducible components are solutions of $\mathcal{F}$ at p. An *algebraic solution* of $\mathcal{F}$ will be an integral (i.e., reduced and irreducible) curve C on X such that its local equation at each point in its support is invariant by $\mathcal{F}$. Moreover, if every integral component of a curve D on X is an algebraic solution, we shall say that *D is invariant by $\mathcal{F}$*.

The transition functions g_{ij} of a foliation $\mathcal{F}$ define an invertible sheaf $\mathcal{N}$ on X, the *normal sheaf* of $\mathcal{F}$, and the relations (1) can be thought as defining relations of a global section of the sheaf $\mathcal{N} \otimes \Omega_X^1$, which has isolated zeros (because each ω_i has isolated zeros). This section is uniquely determined by the foliation $\mathcal{F}$, up to multiplication by a non zero element in k. Conversely, given an invertible sheaf $\mathcal{N}$ on X, any global section of $\mathcal{N} \otimes \Omega_X^1$ with isolated zeros defines a foliation $\mathcal{F}$ whose normal sheaf is $\mathcal{N}$. So, alternatively, we can define a foliation as a map of $\mathcal{O}_X$-modules $\mathcal{F}: \Omega^1_X \to \mathcal{N}$, $\mathcal{N}$ being some invertible sheaf as above. Set $Sing(\mathcal{F})$ the singular locus of $\mathcal{F}$, that is the subscheme of X, where $\mathcal{F}$ fails to be surjective.

Particularizing to the projective plane $\mathbb{P}^2_k = \mathbb{P}^2$, for a non-negative integer r, the Euler sequence, $0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \to 0$, allows to regard the foliation $\mathcal{F}: \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(r-1)$, in analytic terms, as induced by a homogeneous vector field

$$\mathbf{X} = U\partial/\partial X + V\partial/\partial Y + W\partial/\partial Z,$$

where U, V and W are homogeneous polynomials of degree r in homogeneous coordinates $(X:Y:Z)$ on $\mathbb{P}^2$. It is convenient to notice that two vector fields define the same foliation if, and only if, they differ by a multiple of the radial vector field. By convention, we shall say that $\mathcal{F}$ has degree r. We prefer to use forms to treat foliations and, so, equivalently, we shall give a foliation $\mathcal{F}$ on $\mathbb{P}^2$ of degree r, up to a scalar factor, by means of a projective 1-form

$$\Omega = A\,dX + B\,dY + C\,dZ,$$

where A, B and C are homogeneous polynomials of degree $r+1$ without common factors which satisfy the Euler's condition $XA + YB + ZC = 0$ (see [20]). Ω allows to handle easily the foliation in local terms and the singular points of $\mathcal{F}$ are the common zeros of the polynomials A, B and C. Moreover, a curve D on $\mathbb{P}^2$ is invariant by $\mathcal{F}$ if, and only if, G divides the projective 2-form $dG \wedge \Omega$, where $G(X:Y:Z) = 0$ is an homogeneous equation of D.

To blow-up a surface at a closed point and the corresponding evolution of a foliation on it, will be an important tool in this paper. Thus, let us consider a sequence of morphisms

$$X_{n+1} \xrightarrow{\pi_n} X_n \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2, \tag{2}$$

where π_i is the blow-up of X_i at a closed point $p_i \in X_i$, $1 \leqslant i \leqslant n$. The associated set of closed points $\mathcal{K} = \{p_1, p_2, \ldots, p_n\}$ will be called a *configuration* over $\mathbb{P}^2$ and the variety X_{n+1} the *sky* of $\mathcal{K}$; we identify two configurations with $\mathbb{P}^2$-isomorphic skies. We shall denote by E_{p_i} (respectively $\widetilde{E}_{p_i}$ $(E^*_{p_i})$) the exceptional divisor appearing in the blow-up π_i (respectively its strict transform on X_{n+1} (its total transform on X_{n+1})). Also, given two points p_i, p_j in $\mathcal{K}$, we shall say that p_i is *infinitely near to* p_j (denoted $p_i \geqslant p_j$) if either $p_i = p_j$ or $i > j$ and $\pi_j \circ \pi_{j+1} \circ \cdots \circ \pi_{i-1}(p_i) = p_j$. The relation $\geqslant$ is a partial ordering among the points of the configuration $\mathcal{K}$. Furthermore, we say that a point p_i is *proximate* to other one p_j whenever p_i is in the strict transform of the exceptional divisor created after blowing up at p_j in the surface which contains p_i. As a visual display of a configuration $\mathcal{K}$, we shall use the so called *proximity graph* of $\mathcal{K}$, whose vertices represent those points in $\mathcal{K}$, and two vertices, $p, q \in \mathcal{K}$, are joined by an edge if p is proximate to q. This edge is dotted except when p is in the first infinitesimal neighborhood of q (here the edge is continuous). For simplicity sake, we delete those edges which can be deduced from others.

If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$, the sequence of morphisms (2) induces, for each $i = 2, 3, \ldots, n+1$, a foliation $\mathcal{F}_i$ on X_i, the strict transform of $\mathcal{F}$ on X_i (see [3], for instance). As we have said in the Introduction, Seidenberg's result of reduction of singularities [42] proves that there is a sequence of blow-ups as in (2) such that the strict transform $\mathcal{F}_{n+1}$ of $\mathcal{F}$ on the last obtained surface X_{n+1} has only certain type of singularities which cannot be removed by blowing-up, called simple singularities. Such a sequence of blow-ups is called a *resolution* of $\mathcal{F}$, and it will be *minimal* if it is so with respect to the number of involved blow-ups. Assuming that (2) is a minimal resolution of $\mathcal{F}$, we shall denote by $\mathcal{K}_{\mathcal{F}}$ the associated configuration $\{p_i\}_{i=1}^{n}$. Note that each point p_i is an ordinary (that is, not simple) singularity of the foliation $\mathcal{F}_i$.

Dicriticalness of divisors and points will be an essential concept to decide if certain plane foliations have a rational first integral, object of our study. Next we state the definitions.

Definition 1. An exceptional divisor E_{p_i} (respectively, a point $p_i \in \mathcal{K}_{\mathcal{F}}$) of a minimal resolution of a plane foliation $\mathcal{F}$ is called *nondicritical* if it is invariant by the foliation $\mathcal{F}_{i+1}$ (respectively, all the exceptional divisors E_{p_j}, with $p_j \geqslant p_i$, are nondicritical). Otherwise, E_{p_i} (respectively p_i) is said to be *dicritical*.

Along this paper, we shall denote by $\mathcal{B}_{\mathcal{F}}$ the configuration of dicritical points in $\mathcal{K}_{\mathcal{F}}$, and by $\mathcal{N}_{\mathcal{F}}$ the set of points $p_i \in \mathcal{B}_{\mathcal{F}}$ such that E_{p_i} is a nondicritical exceptional divisor.

Definition 2. We shall say that a plane foliation $\mathcal{F}$ has a *rational first integral* if there exists a rational function R of $\mathbb{P}^2$ such that $dR \wedge \Omega = 0$.

The existence of a rational first integral is equivalent to each one of the following three facts (see [23]): $\mathcal{F}$ has infinitely many algebraic solutions, all the solutions of $\mathcal{F}$ are restrictions of algebraic solutions and there exists a unique irreducible pencil of plane curves $\mathcal{P}_{\mathcal{F}} := \langle F, G\rangle \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$, for some $d \geqslant 1$, such that the algebraic solutions of $\mathcal{F}$ are exactly the integral components of the curves of the pencil. *Irreducible pencil* means that its general elements are integral curves.

Two generators F and G of $\mathcal{P}_{\mathcal{F}}$ give rise to a rational first integral $R = \frac{F}{G}$ of $\mathcal{F}$ and, if T is whichever rational function of $\mathbb{P}^1$, then $T(R)$ is also a rational first integral of $\mathcal{F}$ and any rational first integral is obtained in this way. We shall consider *primitive* rational first integrals, that is, those arising from the unique pencil $\mathcal{P}_{\mathcal{F}}$.

Until the end of this section, we shall assume that $\mathcal{F}$ has a rational first integral. $\mathcal{P}_{\mathcal{F}}$ has finitely many base points, since F and G have no common factor. Set $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^2}$ the ideal sheaf supported at the base points of $\mathcal{P}_{\mathcal{F}}$ and such that $\mathcal{I}_p = (F_p, G_p)$ for each such a point p, where F_p and G_p are the natural images of F and G in $\mathcal{O}_{\mathbb{P}^2,p}$. There exists a sequence of blow-ups centered at closed points

$$X_{m+1} \xrightarrow{\pi_m} X_m \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2 \tag{3}$$

such that, if $\pi_{\mathcal{F}}$ denotes the composition morphism $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_m$, the sheaf $\mathcal{I}\mathcal{O}_{X_{m+1}}$ becomes an invertible sheaf of X_{m+1} [5]. We denote by $\mathcal{C}_{\mathcal{F}}$ the set of centers of the blow-ups that appear in a minimal sequence with this property and by $Z_{\mathcal{F}}$ the sky of $\mathcal{C}_{\mathcal{F}}$. This sequence can also be seen as a minimal sequence of blow-ups that eliminates the indeterminacies of the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ induced by the pencil $\mathcal{P}_{\mathcal{F}}$. Hence, there exists a morphism $h_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^1$ factorizing through $\pi_{\mathcal{F}}$. Notice that this morphism is essentially unique, up to composition with an automorphism of $\mathbb{P}^1$.

Let F and G be two general elements of the pencil $\mathcal{P}_{\mathcal{F}}$ and, for each $p_i \in \mathcal{K}_{\mathcal{F}}$, assume that f (respectively g) gives a local equation at p_i of the strict transform of the curve on $\mathbb{P}^2$ defined by F (respectively G). Then, the local solutions of $\mathcal{F}_i$ at p_i are exactly the irreducible components of the elements of the (local) pencil in $\widehat{\mathcal{O}}_{X_i,p_i}$ generated by f and g [19]. As a consequence, the following result is clear.

Proposition 1. *If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ with a rational first integral, then the configurations $\mathcal{C}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{F}}$ coincide.*

3. Foliations with a rational first integral

Along this paper, we shall consider a foliation $\mathcal{F}$ on $\mathbb{P}^2$ such that its associated configuration $\mathcal{B}_{\mathcal{F}}$ has cardinality larger than 1 and we keep the notations as in the previous section. Notice that in case that $\mathcal{B}_{\mathcal{F}}$ be empty, it is obvious that the foliation $\mathcal{F}$ has no rational first integral and, when $\mathcal{B}_{\mathcal{F}}$ consists of a point, only quotients of linear homogeneous polynomials defining transversal lines which pass through that point can be primitive rational first integrals.

From now on, $Z_{\mathcal{F}}$ will denote the sky of $\mathcal{B}_{\mathcal{F}}$ (it will be the one of $\mathcal{C}_{\mathcal{F}}$ whenever $\mathcal{F}$ has a rational first integral). Denote by $A(Z_{\mathcal{F}})$ the real vector space (endowed with the usual real topology) $\mathrm{Pic}(Z_{\mathcal{F}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{m+1}$, where $\mathrm{Pic}(Z_{\mathcal{F}})$ stands for the Picard group of the surface $Z_{\mathcal{F}}$. The *cone of curves* (respectively *nef cone*) of $Z_{\mathcal{K}}$, which we shall denote by $NE(Z_{\mathcal{F}})$ (respectively $P(Z_{\mathcal{F}})$), is defined to be the convex cone of $A(Z_{\mathcal{F}})$ generated by the images of the effective (respectively nef) classes in $\mathrm{Pic}(Z_{\mathcal{F}})$. The $\mathbb{Z}$-bilinear form $\mathrm{Pic}(Z_{\mathcal{F}}) \times \mathrm{Pic}(Z_{\mathcal{F}}) \to \mathbb{Z}$ given by intersection theory induces a nondegenerate $\mathbb{R}$-bilinear pairing

$$A(Z_{\mathcal{F}}) \times A(Z_{\mathcal{F}}) \to \mathbb{R}. \tag{4}$$

For each pair $(x, y) \in A(Z_{\mathcal{F}}) \times A(Z_{\mathcal{F}})$, $x \cdot y$ will denote its image by the above bilinear form.

On the other hand, given a convex cone C of $A(Z_{\mathcal{F}})$, its *dual cone* is defined to be $C^{\vee} := \{x \in A(Z_{\mathcal{F}}) \mid x \cdot y \geqslant 0 \text{ for all } y \in C\}$, and a *face* of C is a subcone $D \subseteq C$ such that $a + b \in D$ implies that $a, b \in D$, for all pair of elements $a, b \in C$. The 1-dimensional faces of C are also called *extremal rays* of C. Note that $P(Z_{\mathcal{F}})$ is the dual cone of $NE(Z_{\mathcal{F}})$, and that it is also the dual cone of $\overline{NE}(Z_{\mathcal{F}})$, the closure of $NE(Z_{\mathcal{F}})$ in $A(Z_{\mathcal{F}})$.

Given a divisor D on $Z_{\mathcal{F}}$, we shall denote by $[D]$ its class in the Picard group of $Z_{\mathcal{F}}$ and also its image into $A(Z_{\mathcal{F}})$. For a curve C on $\mathbb{P}^2$, $\widetilde{C}$ (respectively C^*) will denote its strict (respectively total) transform on the surface $Z_{\mathcal{F}}$ via the sequence of blow-ups given by $\mathcal{B}_{\mathcal{F}}$. It is well known that the set $\{[L^*]\} \cup \{[E_q^*]\}_{q \in \mathcal{B}_{\mathcal{F}}}$ is a $\mathbb{Z}$-basis (respectively $\mathbb{R}$-basis) of $\mathrm{Pic}(Z_{\mathcal{F}})$ (respectively $A(Z_{\mathcal{F}})$), where L denotes a general line of $\mathbb{P}^2$.

Now assume that the foliation $\mathcal{F}$ has a rational first integral. Then, we define the following divisor on the surface $Z_{\mathcal{F}}$:

$$D_{\mathcal{F}} := dL^* - \sum_{q \in \mathcal{B}_{\mathcal{F}}} r_q E_q^*,$$

where d is the degree of the curves in $\mathcal{P}_{\mathcal{F}}$ and r_q the multiplicity at q of the strict transform of a general curve of $\mathcal{P}_{\mathcal{F}}$ on the surface that contains q. Notice that the image on $A(Z_{\mathcal{F}})$ of the strict transform of a general curve of $\mathcal{P}_{\mathcal{F}}$ coincides with $[D_{\mathcal{F}}]$ and, moreover, $D_{\mathcal{F}}^2 = 0$ by Bézout theorem.

Since $|D_{\mathcal{F}}|$ is a base-point-free complete linear system, $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ is the face of the cone $NE(Z_{\mathcal{F}})$ spanned by the images, in $A(Z_{\mathcal{F}})$, of those curves on $Z_{\mathcal{F}}$ contracted by the morphism $Z_{\mathcal{F}} \to \mathbb{P}|D_{\mathcal{F}}|$ determined by a basis of $|D_{\mathcal{F}}|$. This will be a useful fact in this paper. In order to study that face, we shall apply Cayley–Bacharach theorem [14, CB7], which deals with residual schemes with respect to complete intersections.

Lemma 1. *If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ with a rational first integral, then the following equality, involving the projective space of one dimensional quotients of global sections of a sheaf of $\mathcal{O}_{\mathbb{P}^2}$-modules, holds*:

$$\mathbb{P}H^0\big(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})\big) = \mathcal{P}_{\mathcal{F}}.$$

Proof. Consider the ideal sheaf $\mathcal{J}$ defined locally by the equations of the curves corresponding to two general elements of the pencil $\mathcal{P}_{\mathcal{F}}$ and its associated zero dimensional scheme Γ. $\pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(-\sum_{q\in\mathcal{B}_{\mathcal{F}}} r_q E_q^*)$ coincides with the integral closure of $\mathcal{J}$, $\overline{\mathcal{J}}$, that is, the ideal sheaf such that in any point $p \in \mathbb{P}^2$, the stalk $\overline{\mathcal{J}}_p$ is the integral closure of the ideal $\mathcal{J}_p$ of $\mathcal{O}_{\mathbb{P}^2,p}$. Applying Cayley–Bacharach theorem to Γ and the subschemes defined by $\overline{\mathcal{J}}$ and $\mathcal{J}' = \mathrm{Ann}(\overline{\mathcal{J}}/\mathcal{J})$, one gets:

$$h^1\big(\mathbb{P}^2, \overline{\mathcal{J}}(d)\big) = h^0\big(\mathbb{P}^2, \mathcal{J}'(d-3)\big) - h^0\big(\mathbb{P}^2, \mathcal{J}(d-3)\big),$$

where h^i means $\dim H^i$ $(0 \leqslant i \leqslant 1)$. The last term of the above equality vanishes by Bézout theorem and $\overline{\mathcal{J}}(d)$ coincides with the sheaf $\pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})$. Moreover, $\mathcal{J}'$ is nothing but the conductor sheaf of $\mathcal{J}$ (i.e., the sheaf that satisfies that for all $p \in \mathbb{P}^2$, the stalk of $\mathcal{J}'$ at p is the common conductor ideal of the generic elements of $\mathcal{J}_p$). Hence, $h^0(\mathbb{P}^2, \mathcal{J}'(d-3))$ coincides with p_g, the geometric genus of a general curve of $\mathcal{P}_{\mathcal{F}}$. So, we have the following equality:

$$h^1\big(Z_{\mathcal{F}}, \mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})\big) = h^1\big(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})\big) = p_g.$$

By Bertini's theorem, the strict transform on $Z_{\mathcal{F}}$ of any general curve of $\mathcal{P}_{\mathcal{F}}$ is smooth and so, its geometric and arithmetic genus coincide. Therefore, using the adjunction formula, we obtain:

$$h^1\big(Z_{\mathcal{F}}, \mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})\big) = 1 + (K_{Z_{\mathcal{F}}} \cdot D_{\mathcal{F}})/2,$$

where $K_{Z_{\mathcal{F}}}$ denotes a canonical divisor of $Z_{\mathcal{F}}$. Finally, the result follows by applying Riemann–Roch theorem to the divisor $D_{\mathcal{F}}$. $\square$

Theorem 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ with a rational first integral. Then,*

(a) *The image in $A(Z_{\mathcal{F}})$ of a curve $C \hookrightarrow Z_{\mathcal{F}}$ belongs to $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ if, and only if, $C = D + E$ where E is a sum* (*may be empty*) *of strict transforms of non-dicritical exceptional divisors and, either $D = 0$, or D is the strict transform on $Z_{\mathcal{F}}$ of an invariant by $\mathcal{F}$ curve.*
(b) *If C is a curve on $Z_{\mathcal{F}}$ that belongs to $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$, then $C^2 \leqslant 0$. Moreover, $C^2 = 0$ if, and only if, C is linearly equivalent to $r D_{\mathcal{F}}$ for some positive rational number r.*

Proof. The morphism $h_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^1$ defined by the sequence (3) is induced by a linear system $V \subseteq |D_{\mathcal{F}}|$ such that the direct image by $\pi_{\mathcal{F}}$ of rational functions induces a one-to-one correspondence between V and $\mathcal{P}_{\mathcal{F}}$ [2, Theorem II.7]. Hence, by Lemma 1, $V = |D_{\mathcal{F}}|$. Now, clause (a) follows from the fact that the integral curves contracted by $h_{\mathcal{F}}$ are the integral components of the strict transforms of the curves belonging to the pencil $\mathcal{P}_{\mathcal{F}}$ and the strict transforms of the nondicritical exceptional divisors (see [19, Proposition 2.5.2.1] and [9, Exercise 7.2]).

To show clause (b), consider an ample divisor H on $Z_{\mathcal{F}}$ and the set $\Theta := \{z \in A(Z_{\mathcal{F}}) \mid z^2 > 0$ and $[H] \cdot z > 0\}$. Θ is contained in $NE(Z_{\mathcal{F}})$ by [26, Corollary 1.21] and, by clause (a), $[C]$ belongs to the boundary of $NE(Z_{\mathcal{F}})$. Hence, the first statement of clause (b) holds.

Finally, in order to prove the second statement of (b), assume that $C^2 = 0$. By [21, Remark V.1.9.1], there exists a basis of $A(Z_{\mathcal{F}})$ in terms of which the topological closure of Θ, $\overline{\Theta}$, gets the shape of a half-cone over an Euclidean ball of dimension the cardinality of $\mathcal{B}_{\mathcal{F}}$. The strict convexity of this Euclidean ball implies that the intersection of the hyperplane $[D_{\mathcal{F}}]^{\perp}$ with $\overline{\Theta}$ is just the ray $\mathbb{R}_{\geqslant 0}[D_{\mathcal{F}}]$, where $\mathbb{R}_{\geqslant 0}$ denotes the set of non-negative real numbers. Therefore, $[C] \in \mathbb{R}_{\geqslant 0}[D_{\mathcal{F}}]$ by (a) and so, $[C] = r[D_{\mathcal{F}}]$ for some positive rational number r. $\square$

Remark 1. As a consequence of Theorem 1, next we give two conditions which, in case that one of them be satisfied, allow to discard the existence of a rational first integral for a foliation on $\mathbb{P}^2$:

1. There exists an invariant curve, C, such that $\widetilde{C}^2 > 0$.
2. There exist two invariant curves, C_1 and C_2, such that $\widetilde{C}_1^2 = 0$ and $\widetilde{C}_1 \cdot \widetilde{C}_2 \neq 0$.

An essential concept for this paper is introduced in the following definition.

Definition 3. An *independent system of algebraic solutions of a foliation $\mathcal{F}$ on $\mathbb{P}^2$*, which needs not to have a rational first integral, is a set of algebraic solutions of $\mathcal{F}$, $S = \{C_1, C_2, \ldots, C_s\}$, where s is the number of dicritical exceptional divisors appearing in the minimal resolution of $\mathcal{F}$, such that $\widetilde{C}_i^2 \leqslant 0$ $(1 \leqslant i \leqslant s)$ and the set of classes $\mathcal{A}_S := \{[\widetilde{C}_1], [\widetilde{C}_2], \ldots, [\widetilde{C}_s]\} \cup \{[\widetilde{E}_q]\}_{q \in \mathcal{N}_{\mathcal{F}}} \subseteq A(Z_{\mathcal{F}})$ is $\mathbb{R}$-linearly independent.

Remark 2. Note that, when $\mathcal{F}$ has a rational first integral, the existence of an independent system of algebraic solutions is an equivalent fact to say that the face of the cone of curves of $Z_{\mathcal{F}}$ given by $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ has codimension 1.

Assume now that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ admits an independent system of algebraic solutions $S = \{C_1, C_2, \ldots, C_s\}$. Set $\mathcal{B}_{\mathcal{F}} = \{q_1, q_2, \ldots, q_m\}$, $\mathcal{N}_{\mathcal{F}} = \{q_{i_1}, q_{i_2}, \ldots, q_{i_l}\}$ and stand $c_i := (d_i, -a_{i1}, \ldots, -a_{im})$ (respectively, $e_{q_{i_k}} := (0, b_{k1}, \ldots, b_{km})$) for the coordinates of the classes of the strict transforms on $Z_{\mathcal{F}}$, $[\widetilde{C}_i]$ (respectively $[\widetilde{E}_{q_{i_k}}]$) of the curves C_i, $1 \leqslant i \leqslant s$ (respectively nondicritical exceptional divisors $E_{q_{i_k}}$, $1 \leqslant k \leqslant l$), in the basis of $A(Z_{\mathcal{F}})$ given by $\{[L^*], [E^*_{q_1}], [E^*_{q_2}], \ldots, [E^*_{q_m}]\}$.

Notice that d_i is the degree of C_i, a_{ij} the multiplicity of the strict transform of C_i at q_j, and b_{kj} equals 1 if $j = i_k$, -1 if q_j is proximate to q_{i_k} and 0, otherwise.

Consider the divisor on $Z_{\mathcal{F}}$:

$$T_{\mathcal{F},S} := \delta_0 L^* - \sum_{j=1}^{m} \delta_j E^*_{q_j}, \tag{5}$$

where $\delta_j := \delta'_j / \gcd(\delta'_0, \delta'_1, \ldots, \delta'_m)$, δ'_j being the absolute value of the determinant of the matrix obtained by removing the $(j+1)$th column of the $m \times (m+1)$-matrix defined by the rows $c_1, \ldots, c_s, e_{q_{i_1}}, \ldots, e_{q_{i_l}}$. Also, the set

$$\Sigma(\mathcal{F}, S) := \{\lambda \in \mathbb{Z}_+ \mid h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\lambda T_{\mathcal{F},S})) \geqslant 2\},$$

where h^0 means $\dim H^0$ and $\mathbb{Z}_+$, is the set of positive integers.

When the foliation $\mathcal{F}$ has a rational first integral, the set $\mathcal{A}_S$ spans the hyperplane $[D_{\mathcal{F}}]^{\perp}$ and, therefore, $\sum_{i=0}^{m} \delta_i x_i = 0$ is an equation for it, whenever $(x_0, x_1, \ldots, x_m)$ are coordinates in $A(Z_{\mathcal{F}})$ with respect to the basis $\{[L^*]\} \cup \{[E_q^*]\}_{q\in\mathcal{B}_{\mathcal{F}}}$. Hence, in this case, the divisor $D_{\mathcal{F}}$ is a positive multiple of $T_{\mathcal{F},S}$. In fact, $[T_{\mathcal{F},S}]$ is the primitive element of the ray in $A(Z_{\mathcal{F}})$ spanned by $[D_{\mathcal{F}}]$ in the sense that every divisor class belonging to this ray is the product of $[T_{\mathcal{F},S}]$ by a positive integer. Therefore the divisor $T_{\mathcal{F},S}$ does not depend on the choice of the independent system of algebraic solutions S.

Lemma 2. *Consider a foliation $\mathcal{F}$ on $\mathbb{P}^2$, with a rational first integral, such that it admits an independent system of algebraic solutions $S = \{C_j\}_{j=1}^{s}$. Then, $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ is a simplicial cone if the decomposition of the class $[T_{\mathcal{F},S}]$ as a linear combination of the elements in the set $\mathcal{A}_S$ contains every class in $\mathcal{A}_S$ and all its coefficients are strictly positive.*

Proof. It follows from the fact that the convex cone $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ is spanned by the classes in $\mathcal{A}_S$. Indeed, by Theorem 1, it is enough to prove that, for each algebraic solution $D \hookrightarrow \mathbb{P}^2$ which does not belong to S, the class $[\widetilde{D}]$ can be written as a positive linear combination of the above mentioned classes. But, since $D_{\mathcal{F}} \cdot \widetilde{D} = 0$ and $D_{\mathcal{F}}$ is nef and a positive multiple of $T_{\mathcal{F},S}$, one has that $\widetilde{C}_i \cdot \widetilde{D} = 0$ for all $i = 1, 2, \ldots, s$ and $\widetilde{E}_q \cdot \widetilde{D} = 0$ for each $q \in \mathcal{N}_{\mathcal{F}}$. Then, $[\widetilde{D}]$ belongs to the subspace of $A(Z_{\mathcal{F}})$ orthogonal to $\mathcal{A}_S$ and, therefore, it must be a positive multiple of $[T_{S,\mathcal{F}}]$, fact that proves the statement. □

The following result shows how to obtain the divisor $D_{\mathcal{F}}$ from an independent system of algebraic solutions.

Theorem 2. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ with a rational first integral. Assume that $\mathcal{F}$ admits an independent system of algebraic solutions $S = \{C_i\}_{i=1}^{s}$ and set*

$$[T_{\mathcal{F},S}] = \sum_{i=1}^{s} \alpha_i \big[\widetilde{C}_i\big] + \sum_{q\in\mathcal{N}_{\mathcal{F}}} \beta_q \big[\widetilde{E}_q\big] \tag{6}$$

the decomposition of $[T_{\mathcal{F},S}]$ as a linear combination of the classes in $\mathcal{A}_S$. Then, the following properties hold:

(a) *$D_{\mathcal{F}} = \alpha_{\mathcal{F}} T_{\mathcal{F},S}$, where $\alpha_{\mathcal{F}}$ is the minimum of $\Sigma(\mathcal{F}, S)$.*
(b) *Assume that the coefficients α_i $(1 \leqslant i \leqslant s)$ and β_q $(q \in \mathcal{N}_{\mathcal{F}})$ of the decomposition* (6) *are positive. Let r be the minimum positive integer such that $r\alpha_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, s$. Then,*

$$\alpha_{\mathcal{F}} = \delta_{\mathcal{F}} := \frac{r(\deg(\mathcal{F}) + 2 - \sum_{i=1}^{s} \deg(C_i))}{\gcd\big(\sum_{i=1}^{s} r\alpha_i \deg(C_i), \deg(\mathcal{F}) + 2 - \sum_{i=1}^{s} \deg(C_i)\big)},$$

where $\deg(\mathcal{F})$ *denotes de degree of the foliation $\mathcal{F}$ and* $\deg(C_i)$ *the one of the curve C_i $(1 \leqslant i \leqslant s)$.*

Proof. Let μ be the positive integer such that $D_{\mathcal{F}} = \mu T_{\mathcal{F},S}$.

In order to prove (a) we shall reason by contradiction assuming that $\alpha_{\mathcal{F}} < \mu$. Taking into account that $D_{\mathcal{F}} \cdot T_{\mathcal{F},S} = 0$ and $D_{\mathcal{F}}$ is nef, and applying Theorem 1, it is deduced that all the elements of the linear system $\mathbb{P}H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}}T_{\mathcal{F},S}))$ are invariant curves. So, its integral components must be also integral components of the fibers of the pencil $\mathcal{P}_{\mathcal{F}}$. It is clear that there exist infinitely many integral components of elements in $\mathbb{P}H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}}T_{\mathcal{F},S}))$ whose strict transforms have the same class in the Picard group of $Z_{\mathcal{F}}$. Finally, since $\mathcal{P}_{\mathcal{F}}$ is an irreducible pencil, the unique class in $\mathrm{Pic}(Z_{\mathcal{K}})$ corresponding to an infinite set of integral curves is that of the general fibers of $\mathcal{P}_{\mathcal{F}}$, which is a contradiction, because the degree of these general fibers is larger than the degree of the curves in $\mathbb{P}H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}}T_{\mathcal{F},S}))$.

Next, we shall prove (b). By Lemma 2, $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ is the simplicial convex cone spanned by the classes in $\{[\widetilde{C}_i]\}_{1\leqslant i\leqslant s} \cup \{[\widetilde{E}_q]\}_{q\in\mathcal{N}_{\mathcal{F}}}$. Firstly, we shall show that $\sum_{i=1}^{s} \mu\alpha_i C_i$ is the unique curve of $\mathcal{P}_{\mathcal{F}}$ containing some curve of S as a component.

To do it, assume that Q is a fiber of $\mathcal{P}_{\mathcal{F}}$ satisfying the mentioned condition. Then, by clause (a) of Theorem 1, there exists a sum of strict transforms of nondicritical exceptional divisors, E, such that $\widetilde{Q} + E$ is a divisor linearly equivalent to $D_{\mathcal{F}}$. Taking the decomposition of Q as a sum of integral components, one gets

$$\widetilde{Q} + E = \sum_{i=1}^{s} a_i \widetilde{C}_i + \sum_{j=1}^{t} b_j \widetilde{D}_j + \sum_{q\in\mathcal{N}_{\mathcal{F}}} c_q \widetilde{E}_q, \tag{7}$$

where all coefficients are non-negative integers, some a_i is positive and the elements in $\{D_j\}_{1\leqslant j\leqslant t}$ are integral curves on $\mathbb{P}^2$ which are not in S. If some $\widetilde{D}_j$ had negative self-intersection, its image in $A(Z_{\mathcal{F}})$ would span an extremal ray of the cone $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$, contradicting the fact that this cone is simplicial. Hence, by Theorem 1, $\widetilde{D}_j^2 = 0$ for all $j = 1, 2, \ldots, t$ and all the classes $[\widetilde{D}_j]$ belong to the ray spanned by $[D_{\mathcal{F}}]$. As a consequence, one has the following decomposition:

$$\begin{aligned}
[\widetilde{Q}] + [E] &= \sum_{i=1}^{s} a_i [\widetilde{C}_i] + \sum_{j=1}^{t} b'_j \left(\sum_{i=1}^{s} \alpha_i [\widetilde{C}_i] + \sum_{q\in\mathcal{N}_{\mathcal{F}}} \beta_q [\widetilde{E}_q] \right) + \sum_{q\in\mathcal{N}_{\mathcal{F}}} c_q [\widetilde{E}_q] \\
&= \sum_{i=1}^{s} \left(a_i + \alpha_i \sum_{j=1}^{t} b'_j \right) [\widetilde{C}_i] + \sum_{q\in\mathcal{N}_{\mathcal{F}}} \left(c_q + \beta_q \sum_{j=1}^{t} b'_j \right) [\widetilde{E}_q],
\end{aligned}$$

where, for each $j = 1, 2, \ldots, t$, b'_j is b_j times a positive integer. As $[\widetilde{Q}] + [E] = [D_{\mathcal{F}}] = \mu(\sum_{i=1}^{s} \alpha_i [\widetilde{C}_i] + \sum_{q\in\mathcal{N}_{\mathcal{F}}} \beta_q [\widetilde{E}_q])$, $a_i = (\mu - \sum_{j=1}^{t} b'_j)\alpha_i$ for all $i = 1, 2, \ldots, s$ and $c_q = (\mu - \sum_{j=1}^{t} b'_j)\beta_q$ for all $q \in \mathcal{N}_{\mathcal{F}}$. Then, since some a_i does not vanish and all the rational numbers α_i are different from zero, it follows that $a_i \neq 0$ for all $i = 1, 2, \ldots, s$. Now, from the inequality

$$s - 1 \geqslant \sum_{R} (n_R - 1), \tag{8}$$

where the sum is taken over the set of fibers R of $\mathcal{P}_{\mathcal{F}}$ and n_R denotes the number of distinct integral components of R [24], we deduce that Q cannot have integral components different from those in S and, therefore, we must take, in the equality (7), $b_j = 0$ for each $j = 1, 2, \ldots, t$, $a_i = \mu\alpha_i$ $(1 \leqslant i \leqslant s)$ and $c_q = \mu\beta_q$ for all $q \in \mathcal{N}_{\mathcal{F}}$. As a consequence, we have proved the mentioned property of the curve $\sum_{i=1}^{s} \mu\alpha_i C_i$ and, moreover, that the integer r defined in the statement divides μ.

Secondly, from the above paragraph it can be deduced that the non-integral fibers of $\mathcal{P}_{\mathcal{F}}$, except $\sum_{i=1}^{s} \mu\alpha_i C_i$, have the form nD, where $n > 1$ is a positive integer, D is an integral curve and $\widetilde{D}^2 = 0$. Set $m := \mu/r$. Recall that Poincaré proved in [38] that if a pencil of plane curves $\langle F, G\rangle$ is irreducible, then $\varrho F + \varsigma G$ is a nontrivial power for, at most, two values $(\varrho : \varsigma) \in \mathbb{P}^1$. Thus, if $m \neq 1$, we can pick as generators of the pencil $\mathcal{P}_{\mathcal{F}}$, $\sum_{i=1}^{s} \mu\alpha_i C_i$ and nD, n being a positive integer and D some integral curve, and in addition, the inequality (8) shows that the remaining curves in $\mathcal{P}_{\mathcal{F}}$ are reduced. Notice that $\gcd(m, n) = 1$, because the pencil $\mathcal{P}_{\mathcal{F}}$ is irreducible. Now, if we apply the formula that asserts that, if a plane foliation $\mathcal{F}$ has a rational first integral with general algebraic solution of degree d, then $2d - \deg(\mathcal{F}) - 2 = \sum (e_R - 1)\deg(R)$, where the sum is taken over the set of integral components R of the curves in $\mathcal{P}_{\mathcal{F}}$ and e_R denotes the multiplicity of R as a component of the correspondent fiber, one gets:

$$\begin{aligned}\deg(\mathcal{F}) + 2 &= n\deg(D) + \sum_{i=1}^{s} \mu\alpha_i \deg(C_i) - (n-1)\deg(D) - \sum_{i=1}^{s} (\mu\alpha_i - 1)\deg(C_i)\\ &= \deg(D) + \sum_{i=1}^{s} \deg(C_i).\end{aligned}$$

Thus,

$$m = \frac{\deg(D)}{(1/n)\sum_{i=1}^{s} r\alpha_i \deg(C_i)} = \frac{\deg(\mathcal{F}) + 2 - \sum_{i=1}^{s}\deg(C_i)}{\gcd\bigl(\sum_{i=1}^{s} r\alpha_i \deg(C_i), \deg(\mathcal{F}) + 2 - \sum_{i=1}^{s}\deg(C_i)\bigr)}$$

and therefore,

$$\mu = \alpha_{\mathcal{F}} = \delta_{\mathcal{F}}.$$

Finally, we notice that this equality also holds when $m = 1$. □

Stands $K_{Z_{\mathcal{F}}}$ for a canonical divisor on $Z_{\mathcal{F}}$. The next result follows from Bertini's theorem and the adjunction formula, and it shows that the condition $K_{Z_{\mathcal{F}}} \cdot T_{\mathcal{F},S} < 0$ makes easy to check whether $\mathcal{F}$ has or not a rational first integral, and to compute it (using Lemma 1).

Proposition 2. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ admitting an independent system of algebraic solutions S. Assume that $K_{Z_{\mathcal{F}}} \cdot T_{\mathcal{F},S} < 0$ and $\mathcal{F}$ has a rational first integral. Then, the general elements of the pencil $\mathcal{P}_{\mathcal{F}}$ are rational curves and $D_{\mathcal{F}} = T_{\mathcal{F},S}$.*

4. Computation of rational first integrals. Algorithms and examples

With the above notations, recall that by Theorem 1, for each foliation $\mathcal{F}$ on $\mathbb{P}^2$ with a rational first integral, the divisor $D_{\mathcal{F}}$ satisfies $D_{\mathcal{F}}^2 = 0$ and $D_{\mathcal{F}} \cdot \widetilde{E}_q = 0$ (respectively $D_{\mathcal{F}} \cdot \widetilde{E}_q > 0$) for all $q \in \mathcal{N}_{\mathcal{F}}$ (respectively $q \in \mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$). These facts and Lemma 1 support the following decision algorithm for the problem of deciding whether an arbitrary foliation $\mathcal{F}$ has a rational first integral of a fixed degree d. In fact, it allows to compute it in the affirmative case.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ and d a positive integer.

Algorithm 1.

Input: d, a projective 1-form Ω defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$.
Output: Either a rational first integral of degree d, or "0" if there is no such first integral.

1. Consider the finite set Γ of divisors $D = dL^* - \sum_{q\in\mathcal{B}_{\mathcal{F}}} e_q E_q^*$ such that
   (a) $D^2 = 0$,
   (b) $D \cdot \widetilde{E}_q = 0$ for all $q \in \mathcal{N}_{\mathcal{F}}$, and
   (c) $D \cdot \widetilde{E}_q > 0$ for all $q \in \mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$.
2. Pick $D \in \Gamma$.
3. If the dimension of the k-vector space $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D))$ is 2, then take a basis $\{F, G\}$ and check the condition $d(F/G) \wedge \Omega = 0$. If it is satisfied, then return F/G.
4. Set $\Gamma := \Gamma \setminus \{D\}$.
5. If the set Γ is not empty, go to step 2. Else return "0."

This is, in some sense, an alternative algorithm to the particular case of that given by Prelle and Singer [40] (and implemented by Man [30]) to compute first integrals of foliations. Man uses packages involving Groebner basis to detect inconsistency as well as to solve consistent systems of equations [30, 3.3]. Note that our algorithm needs, as an input, the sets $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$, that can be computed resolving $\mathcal{F}$, what may not be easy, but after it only involves integer arithmetic and resolution of systems of linear equations (step 3) and it does not need to use Groebner bases.

Now, and again supported in results of the previous section, we give an algorithm to decide, under certain conditions, whether an *arbitrary foliation* $\mathcal{F}$ on $\mathbb{P}^2$ has a rational first integral (of arbitrary degree). As above, the algorithm computes it in the affirmative case.

First, we state the needs in order that the algorithm works. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ that admits an independent system of algebraic solutions S which satisfies *at least one* of the following conditions:

(1) $T^2_{\mathcal{F},S} \neq 0$.
(2) The decomposition of the class $[T_{\mathcal{F},S}]$ as a linear combination of those in the set $\mathcal{A}_S$, given in Definition 3, contains all the classes in $\mathcal{A}_S$ with positive coefficients. In this case, it will be useful Theorem 2 and the number $\delta_{\mathcal{F}}$ defined there.
(3) The set $\Sigma(\mathcal{F}, S)$ is not empty.

Algorithm 2.

Input: A projective 1-form Ω defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$, $\mathcal{N}_{\mathcal{F}}$ and an independent system of algebraic solutions S satisfying at least one of the above conditions (1), (2) and (3).
Output: Either a rational first integral for $\mathcal{F}$, or "0" if there is no such first integral.

1. If (1) holds, then return "0."
2. If condition (2) is satisfied and $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\delta_{\mathcal{F}} T_{\mathcal{F},S})) \leqslant 1$, then return "0."
3. Consider $\alpha_{\mathcal{F}}$, that is the minimum of the set $\Sigma(\mathcal{F}, S)$.
4. If $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}} T_{\mathcal{F},S})) > 2$, then return "0."
5. Take a basis $\{F, G\}$ of $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}} T_{\mathcal{F},S}))$ and check the equality $d(F/G) \wedge \Omega = 0$. If it is satisfied, then return F/G. Else, return "0."

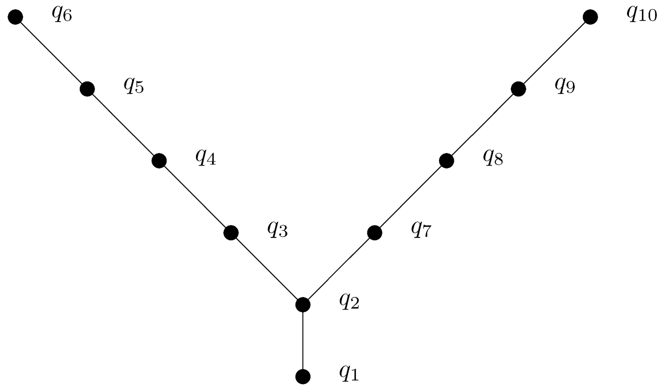

Fig. 1. The proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 1.

Example 1. Let $\mathcal{F}$ be the foliation on the projective plane over the complex numbers defined by the projective 1-form $\Omega = A\,dX + B\,dY + C\,dZ$, where

$$\begin{aligned}
A &= X^3Y + 4Y^4 + 2X^3Z - X^2YZ - 4X^2Z^2 - XYZ^2 + 2XZ^3 + YZ^3,\\
B &= -X^4 - 4XY^3 + 3X^3Z + 4Y^3Z - 3X^2Z^2 + XZ^3,\\
C &= -2X^4 - 2X^3Y - 4Y^4 + 4X^3Z + 4X^2YZ - 2X^2Z^2 - 2XYZ^2.
\end{aligned}$$

Resolving $\mathcal{F}$, we have computed the configuration $\mathcal{K}_{\mathcal{F}}$, which coincides with the configuration $\mathcal{B}_{\mathcal{F}} = \{q_j\}_{j=1}^{10}$ of dicritical points. The proximity graph is given in Fig. 1. $\mathcal{N}_{\mathcal{F}} = \{q_1, q_2, q_3, q_4, q_5, q_7, q_8, q_9\}$ and $S = \{C_1, C_2\}$ is an independent system of algebraic solutions, where C_1 (respectively C_2) is the line (respectively conic) given by the equation $X - Z = 0$ (respectively $(8i-1)X^2 + 4iXY + 8Y^2 + (2-8i)XZ - 4iYZ - Z^2 = 0$). The divisor $T_{\mathcal{F},S}$ is $4L^* - 2E_1^* - 2E_2^* - \sum_{j=3}^{10} E_j^*$. Clearly, condition (1) above is not satisfied. (2) does not hold either, because

$$[T_{\mathcal{F},S}] = 4[\widetilde{C}_1] + 2[\widetilde{E}_1] + 4[\widetilde{E}_2] + 3[\widetilde{E}_3] + 2[\widetilde{E}_4] + [\widetilde{E}_5] + 3[\widetilde{E}_7] + 2[\widetilde{E}_8] + [\widetilde{E}_9].$$

The space of global sections $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},S}))$ has dimension 2 and is spanned by $F = X^2Z^2 - 2X^3Z + X^4 + XYZ^2 - 2X^2YZ + X^3Y + Y^4$ and $G = (X-Z)^4$. Therefore, condition (3) happens and, if $\mathcal{F}$ admits rational first integrals, one of them must be $R := F/G$. The equality $\Omega \wedge dR = 0$ shows that R is, in fact, a rational first integral of $\mathcal{F}$.

Remark 3. Assume that $\mathcal{F}$ has a rational first integral and admits an independent system of algebraic solutions. Set $k(T_{\mathcal{F},S}) = \operatorname{tr}\deg(\bigoplus_{n\geqslant 0} H^0(\mathcal{O}_{Z_{\mathcal{F}}}(nT_{\mathcal{F},S}))) - 1$ the $T_{\mathcal{F},S}$-dimension of $Z_{\mathcal{F}}$. $k(T_{\mathcal{F},S}) \leqslant 2$ and, by using results in [11,43], it can be proved that condition (3) given before Algorithm 2 only happens when $k(T_{\mathcal{F},S}) = 1$.

To decide whether an independent system of algebraic solutions satisfies one of the above mentioned conditions (1) or (2) is very simple, but to check condition (3) may be more difficult. However, when $K_{Z_{\mathcal{F}}} \cdot T_{\mathcal{F},S} < 0$, we should not be concerned about these conditions since, by

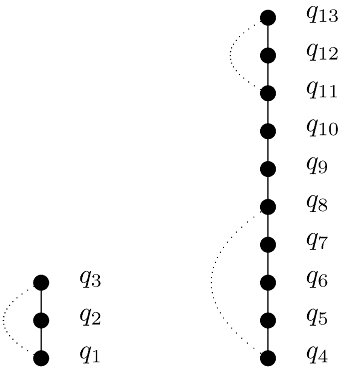

Fig. 2. The proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 2.

Proposition 2, there is no need to take all the steps in Algorithm 2. Indeed, it suffices to check whether $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},S})) = 2$; in the affirmative case, we shall go to step 5 and, otherwise, $\mathcal{F}$ has no rational first integral. Next, we shall show an enlightening example.

Example 2. Set $\mathcal{F}$ the foliation on the complex projective plane defined by the 1-form

$$\Omega = 2YZ^5dX + \left(-7Y^5Z - 3XZ^5 + YZ^5\right)dY + \left(7Y^6 + XYZ^4 - Y^2Z^4\right)dZ.$$

The configuration $\mathcal{K}_{\mathcal{F}}$ coincides with the one of dicritical points $\mathcal{B}_{\mathcal{F}}$. It has 13 points and its proximity graph is given in Fig. 2. The dicritical exceptional divisors are E_{q_3} and $E_{q_{13}}$. The set S given by the lines with equations $Y = 0$ and $Z = 0$ is an independent system of algebraic solutions. Its associated divisor $T_{\mathcal{F},S}$ is

$$10L^* - 2E^*_{q_1} - E^*_{q_2} - E^*_{q_3} - 8E^*_{q_4} - 2\sum_{i=5}^{11} E^*_{q_i} - E^*_{q_{12}} - E^*_{q_{13}}.$$

Now, by Proposition 2, if $\mathcal{F}$ has a rational first integral, then the pencil $\mathcal{P}_{\mathcal{F}}$ is

$$\mathbb{P}H^0\left(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},S})\right).$$

A basis of this projective space is given by $F_1 = Y^{10} - 2XY^5Z^4 + 2Y^6Z^4 + X^2Z^8 - 2XYZ^8 + Y^2Z^8$ and $F_2 = Y^3Z^7$. Finally, the equality $d(F_1/F_2) \wedge \Omega = 0$ shows that F_1/F_2 is a rational first integral of $\mathcal{F}$.

Not all foliations on $\mathbb{P}^2$ with a rational first integral admit an independent system of algebraic solutions, for instance, those associated with a Lefschetz pencil of degree larger than 1. Next, we show another example.

Example 3. Consider the foliation $\mathcal{F}$ on the projective plane over the complex numbers given by the projective 1-form that defines the derivation on the following rational function:

$$\frac{XZ^2 + 3YZ^2 - Y^3}{YZ^2 + 3XZ^2 - X^3}.$$

The configuration of dicritical points $\mathcal{B}_\mathcal{F}$ has 9 points, all in $\mathbb{P}^2$, and therefore, $\mathcal{N}_\mathcal{F} = \emptyset$. Moreover, the divisor $D_\mathcal{F}$ is given by $3L^* - \sum_{q\in\mathcal{B}_\mathcal{F}} E_q^*$. By Theorem 1, the intersection product of the strict transform on $Z_\mathcal{F}$ of whichever invariant curve of $\mathcal{F}$ times $D_\mathcal{F}$ vanishes, and so the nongeneral algebraic solutions are among the lines passing through three points in $\mathcal{B}_\mathcal{F}$ and the irreducible conics passing through six points in $\mathcal{B}_\mathcal{F}$. Simple computations show that these are, exactly, the 8 curves given by the equations:

$$X - Y = 0; \qquad X + Y = 0; \qquad 2X + \big(\sqrt{5}+3\big)Y = 0; \qquad -2X + (\sqrt{5}-3)Y = 0;$$
$$X^2 - XY + Y^2 - 4Z^2 = 0; \qquad X^2 + XY + Y^2 - 2Z^2 = 0;$$
$$2X^2 + \big(\sqrt{5}-3\big)XY - \big(3\sqrt{5}-7\big)Y^2 + \big(8\sqrt{5}-24\big)Z^2 = 0;$$
$$-2X^2 + \big(\sqrt{5}+3\big)XY - \big(3\sqrt{5}+7\big)Y^2 + \big(8\sqrt{5}+24\big)Z^2 = 0.$$

Hence, $\mathcal{F}$ does not admit an independent system of algebraic solutions.

The following result will help to state an algorithm, for foliations $\mathcal{F}$ whose cone $NE(Z_\mathcal{F})$ is polyhedral, that either computes an independent system of algebraic solutions or discards that $\mathcal{F}$ has a rational first integral. In the sequel, for each subset W of $A(Z_\mathcal{F})$, $\mathrm{con}(W)$ will denote the convex cone of $A(Z_\mathcal{F})$ spanned by W.

Proposition 3. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ having a rational first integral and such that $NE(Z_\mathcal{F})$ is polyhedral. Let G be a non-empty finite set of integral curves on $\mathbb{P}^2$ such that $x^2 \geqslant 0$ for each element x in the dual cone $\mathrm{con}(W)^\vee$, W being the following subset of $A(Z_\mathcal{F})$: $W = \{[\widetilde{Q}] \mid Q \in G\} \cup \{[\widetilde{E}_q]\}_{q\in\mathcal{B}_\mathcal{F}}$. Then, $\mathcal{F}$ admits an independent system of algebraic solutions S such that $S \subseteq G$.*

Proof. The conditions of the statement imply $\mathrm{con}(W)^\vee \subseteq \overline{\Theta}$, where $\overline{\Theta}$ is the topological closure of the set $\Theta = \{x \in A(Z_\mathcal{F}) \mid x^2 > 0 \text{ and } [H]\cdot x > 0\}$ given in the proof of Theorem 1. Thus, $\overline{\Theta}^\vee \subseteq (\mathrm{con}(W)^\vee)^\vee = \mathrm{con}(W)$, where the equality is due to the fact that $\mathrm{con}(W)$ is closed. Now, $[D_\mathcal{F}] \in P(Z_\mathcal{F}) \subseteq \overline{\Theta}^\vee$ by [26, Corollary 1.21], and so $[D_\mathcal{F}] \in \mathrm{con}(W)$. Note that the cones $\mathrm{con}(W)$ and $NE(Z_\mathcal{F})$ have the same dimension and, as $[D_\mathcal{F}]$ is in the boundary of $NE(Z_\mathcal{F})$, it also belongs to the boundary of $\mathrm{con}(W)$. Moreover $[D_\mathcal{F}] \in \mathrm{con}(W)^\vee$. As a consequence, $\mathrm{con}(W) \cap [D_\mathcal{F}]^\perp$ is a face of $\mathrm{con}(W)$ which, in addition, contains the class $[D_\mathcal{F}]$.

Let R be the maximal proper face of $\mathrm{con}(W)$ containing $\mathrm{con}(W) \cap [D_\mathcal{F}]^\perp$. Since $\mathrm{con}(W)$ is polyhedral, there exists $y \in \mathrm{con}(W)^\vee \setminus \{0\}$ such that $R = \mathrm{con}(W) \cap y^\perp$. Note that $y^2 \geqslant 0$ and $y \in [D_\mathcal{F}]^\perp$.

Recalling the shape of $\overline{\Theta}$ (see the proof of Theorem 1), it is clear that the hyperplane $[D_\mathcal{F}]^\perp$ is tangent to the boundary of the half-cone $\overline{\Theta}$. Thus, for each $x \in [D_\mathcal{F}]^\perp \setminus \{0\}$, $x^2 \leqslant 0$ and the following equivalence holds:

$$x^2 < 0 \quad \text{if and only if,} \quad x \text{ does not belong to the line } \mathbb{R}[D_\mathcal{F}]. \tag{9}$$

Therefore y belongs to the ray $\mathbb{R}_{\geqslant 0}[D_\mathcal{F}]$ and so the equality $R = \mathrm{con}(W) \cap [D_\mathcal{F}]^\perp$ is satisfied, which concludes the proof by taking into account clause (a) of Theorem 1 and that R is a face of codimension one. $\square$

Corollary 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ having a rational first integral and such that $NE(Z_{\mathcal{F}})$ is polyhedral. Then, $\mathcal{F}$ admits an independent system of algebraic solutions S. Moreover, S can be taken such that $\widetilde{C}^2 < 0$ for all $C \in S$.*

Proof. The cone $NE(Z_{\mathcal{F}})$ is strongly convex (by Kleiman's ampleness criterion [25]) and closed, so it is spanned by its extremal rays. Thus, setting G the set of integral curves on $\mathbb{P}^2$ whose strict transforms on $Z_{\mathcal{F}}$ give rise to generators of extremal rays of $NE(Z_{\mathcal{F}})$ and applying Proposition 3, we prove our first statement. The second one follows simply by taking into account that, since the cardinality of $\mathcal{B}_{\mathcal{F}}$ is assumed to be larger than 1, the extremal rays of $NE(Z_{\mathcal{F}})$ are just those spanned by the classes of the integral curves on $Z_{\mathcal{F}}$ with negative self-intersection (due to the polyhedrality of $NE(Z_{\mathcal{F}})$). $\square$

Now, we state the announced algorithm, where $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ such that the cone $NE(Z_{\mathcal{F}})$ is polyhedral.

Algorithm 3.

Input: A projective 1-form Ω defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$.
Output: Either "0" (which implies that $\mathcal{F}$ has no rational first integral) or an independent system of algebraic solutions.

1. Define $V := \mathrm{con}(\{[\widetilde{E}_q]\}_{q\in\mathcal{B}_{\mathcal{F}}})$, $G := \emptyset$ and let Γ be the set of divisors $C = dL^* - \sum_{q\in\mathcal{B}_{\mathcal{F}}} e_q E_q^*$ satisfying the following conditions:
   (a) $d > 0$ and $0 \leqslant e_q \leqslant d$ for all $q \in \mathcal{B}_{\mathcal{F}}$.
   (b) $C \cdot \widetilde{E}_q \geqslant 0$ for all $q \in \mathcal{B}_{\mathcal{F}}$.
   (c) Either $C^2 = K_{Z_{\mathcal{F}}} \cdot C = -1$, or $C^2 < 0$, $K_{Z_{\mathcal{F}}} \cdot C \geqslant 0$ and $C^2 + K_{Z_{\mathcal{F}}} \cdot C \geqslant -2$.
2. Pick $D \in \Gamma$ such that $D \cdot L^*$ is minimal.
3. If D satisfies the conditions
   (a) $[D] \notin V$,
   (b) $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D)) = 1$,
   (c) $[D] = [\widetilde{Q}]$, where Q is the divisor of zeros of a global section of $\pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D)$,
   then set $V := \mathrm{con}(V \cup \{[D]\})$. If, in addition, Q is an invariant curve of $\mathcal{F}$, no curve in G is a component of Q and $\{[\widetilde{R}] \mid R \in G\} \cup \{[D]\} \cup \{[\widetilde{E}_q]\}_{q\in\mathcal{N}_{\mathcal{F}}}$ is a $\mathbb{R}$-linearly independent system of $A(Z_{\mathcal{F}})$, then set $G := G \cup \{Q\}$.
4. Let $\Gamma := \Gamma \setminus \{D\}$.
5. Repeat the steps 2, 3 and 4 while the following two conditions are satisfied:
   (a) $\mathrm{card}(G) < \mathrm{card}(\mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}})$, where card stands for cardinality.
   (b) There exists $x \in V^\vee$ such that $x^2 < 0$.
6. If $\mathrm{card}(G) < \mathrm{card}(\mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}})$, then return "0". Else, return G.

Explanation. This algorithm computes elements of a strictly increasing sequence of convex cones $V_0 \subset V_1 \subset \cdots$ such that $V_0 = \mathrm{con}(\{[\widetilde{E}_q]\}_{q\in\mathcal{B}_{\mathcal{F}}})$ and $V_i = \mathrm{con}(\{V_{i-1} \cup [\widetilde{Q}_i]\})$ for $i \geqslant 1$, where $Q_1, Q_2, \ldots$ is the sequence of curves on $\mathbb{P}^2$ (ordered in such a way that $i < j$ implies $\deg(Q_i) \leqslant \deg(Q_j)$) satisfying the following conditions:

(1) Either $\widetilde{Q}_i^2 = K_{Z_{\mathcal{F}}} \cdot \widetilde{Q}_i = -1$, or $\widetilde{Q}_i^2 < 0$, $K_{Z_{\mathcal{F}}} \cdot \widetilde{Q}_i \geqslant 0$ and $\widetilde{Q}_i^2 + K_{Z_{\mathcal{F}}} \cdot \widetilde{Q}_i \geqslant -2$.
(2) $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\widetilde{Q}_i)) = 1$.

Notice that, since the cone $NE(Z_{\mathcal{F}})$ is polyhedral, it has a finite number of extremal rays and, moreover, they are generated by the classes of the integral curves of $Z_{\mathcal{F}}$ having negative self-intersection, which are either strict transforms of exceptional divisors, or strict transforms of curves of $\mathbb{P}^2$ satisfying the above conditions (1) and (2) (the second condition is obvious and the first one is a consequence of the adjunction formula). Then, it is clear that for k large enough, $NE(Z_{\mathcal{F}}) \subseteq V_k$ and, so $V_k^{\vee} \subseteq P(Z_{\mathcal{F}})$. Therefore, after repeating the steps 2–4 finitely many times, condition (b) of step 5 will not be satisfied and, hence, the process described in the algorithm stops.

Finally, our algorithm is justified by bearing in mind Proposition 3 and the fact that the final set G is a maximal subset of $\{Q_1, Q_2, \ldots\}$ among those whose elements are invariant curves and $\{[\widetilde{R}] \mid R \in G\} \cup \{[\widetilde{E}_q]\}_{q \in \mathcal{N}_{\mathcal{F}}}$ is a linearly independent subset of $A(Z_{\mathcal{F}})$. Observe that all the curves in G are integral by the condition given in step 3 and the fact that they have been computed with degrees in increasing order.

Remark 4. Polyhedrality of the cone $NE(Z_{\mathcal{F}})$ ensures that Algorithm 3 ends. Notwithstanding, one can apply it anyway and, if it stops after a finite number of steps, one gets an output as it is stated in the algorithm. When an independent system of algebraic solutions S is obtained, one might decide whether, or not, $\mathcal{F}$ admits a rational first integral either by using Algorithm 2 (when some of the conditions (1), (2) and (3) stated before it held) or by applying results from Section 3, as Proposition 2 or Remark 1. Let us see an example.

Example 4. Let $\mathcal{F}$ be the foliation on $\mathbb{P}^2_{\mathbb{C}}$ defined by the projective 1-form $\Omega = A\,dX + B\,dY + C\,dZ$, where:

$$A = -3X^2Y^3 + 9X^2Y^2Z - 9X^2YZ^2 + 3X^2Z^3, \quad B = 3X^3Y^2 - 6X^3YZ - 5Y^4Z + 3X^3Z^2$$
$$\text{and} \quad C = -3X^3Y^2 + 5Y^5 + 6X^3YZ - 3X^3Z^2.$$

The configuration $\mathcal{K}_{\mathcal{F}}$ consists of the union of two chains $\{q_i\}_{i=1}^{19} \cup \{q_i\}_{i=20}^{23}$ with the following additional proximity relations: q_4 is proximate to q_2, q_{22} to q_{20} and q_{23} to q_{21}. Since the unique dicritical exceptional divisor is $E_{q_{19}}$, we have that $\mathcal{B}_{\mathcal{F}} = \{q_i\}_{i=1}^{19}$ and $\mathcal{N}_{\mathcal{F}} = \{q_i\}_{i=1}^{18}$.

A priori, we do no know whether $NE(Z_{\mathcal{F}})$ is, or not, polyhedral. However, if we run Algorithm 3, it ends; providing the independent system of algebraic solutions $S = \{C\}$, where C is the line defined by the equation $Y - Z = 0$. Therefore, $T_{\mathcal{F},S} = 5L^* - 2E_{q_1}^* - 2E_{q_2}^* - \sum_{i=3}^{19} E_{q_i}^*$ and, since $[T_{\mathcal{F},S}] = 5[\widetilde{C}] + 3[\widetilde{E}_{q_1}] + 6[\widetilde{E}_{q_2}] + 10[\widetilde{E}_{q_3}] + \sum_{i=4}^{18}(19 - i)[\widetilde{E}_{q_i}]$, condition (2) before Algorithm 2 is satisfied and, hence, we can apply this algorithm. The value $\delta_{\mathcal{F}}$ of Theorem 2 equals 1 and the algorithm returns the rational first integral F_1/F_2, given by $F_1 = Y^5 - X^3Y^2 + 2X^3YZ - X^3Z^2$ and $F_2 = (Y - Z)^5$.

The next proposition and the fact that Algorithm 3 stops when the cone $NE(Z_{\mathcal{F}})$ is polyhedral allow either discard the existence of a rational first integral for $\mathcal{F}$ or to compute an input to run Algorithm 2 and, so, either to get a rational first integral for $\mathcal{F}$ or, newly, to discard its existence.

Proposition 4. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ such that $NE(Z_{\mathcal{F}})$ is a polyhedral cone. Let S be an independent system of algebraic solutions obtained by calling Algorithm* 3. *Then, S satisfies one of the conditions* (1), (2) *or* (3) *described before Algorithm* 2.

Proof. Assume that condition (1) does not hold. Let V be the convex cone obtained by calling Algorithm 3. Since $y^2 \geqslant 0$ for all $y \in V^\vee$, a similar reasoning to that given in the proof of Proposition 3 (taking $T_{\mathcal{F},S}$ instead of $D_{\mathcal{F}}$ and V in place of $\mathrm{con}(W)$) shows that $V \cap [T_{\mathcal{F},S}]^\perp$ is a face of V which contains $[T_{\mathcal{F},S}]$ and that, for all $x \in [T_{\mathcal{F},S}]^\perp \setminus \{0\}$, $x^2 \leqslant 0$ and so $x^2 < 0$ if, and only if, x is not a (real) multiple of $[T_{\mathcal{F},S}]$.

From the above facts, it is straightforward to deduce that the class $[T_{\mathcal{F},S}]$ does not belong to any proper face of $V \cap [T_{\mathcal{F},S}]^\perp$. Thus $[T_{\mathcal{F},S}]$ is in the relative interior of $V \cap [T_{\mathcal{F},S}]^\perp$, since each nonzero element of a polyhedral convex cone belongs to the relative interior of a unique face. Then, if $V \cap [T_{\mathcal{F},S}]^\perp$ is a simplicial cone, it is clear that condition (2) given before Algorithm 2 is satisfied. Otherwise, $[T_{\mathcal{F},S}]$ admits, at least, two different decompositions as linear combination (with rational positive coefficients) of classes of irreducible curves on $Z_{\mathcal{F}}$ belonging to $V \cap [T_{\mathcal{F},S}]^\perp$. Therefore, $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\lambda T_{\mathcal{F},S})) \geqslant 2$ for some positive integer λ and condition (3) holds. □

Now, we are ready to state our main result which we have proved already.

Theorem 3. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ such that $NE(Z_{\mathcal{F}})$ is a polyhedral cone. Then, calling Algorithms* 3 *and* 2, *one can decide whether $\mathcal{F}$ has a rational first integral and, in the affirmative case, to compute it. The unique data we need in that procedure are the following*: *a projective* 1*-form Ω defining $\mathcal{F}$, the configuration of dicritical points $\mathcal{B}_{\mathcal{F}}$ and the subset $\mathcal{N}_{\mathcal{F}}$ of $\mathcal{B}_{\mathcal{F}}$ defined in Definition* 1.

In [17], we gave conditions which imply the polyhedrality of the cone of curves of the smooth projective rational surface obtained by blowing-up at the infinitely near points of a configuration $\mathcal{C}$ over $\mathbb{P}^2$. One of them only depends on the proximity relations among the points of $\mathcal{C}$ and it holds for a wide range of surfaces whose anticanonical bundle is not ample. Now, we shall recall this condition, but first we introduce some necessary notations.

Consider the morphism $f: Z \to \mathbb{P}^2$ given by the composition of the blow-ups centered at the points of $\mathcal{C}$ (in a suitable order). For every $p \in \mathcal{C}$, the exceptional divisor E_p obtained in the blow-up centered at p defines a valuation of the fraction field of $\mathcal{O}_{\mathbb{P}^2, p_0}$, where p_0 is the image of p on $\mathbb{P}^2$ by the composition of blowing-ups which allow to obtain p. So, p is associated with a simple complete ideal I_p of the ring $\mathcal{O}_{\mathbb{P}^2, p_0}$ [28]. Denote by $D(p)$ the exceptionally supported divisor of Z such that $I_p\mathcal{O}_Z = \mathcal{O}_Z(-D(p))$ and by K_Z a canonical divisor on Z. Let $G_{\mathcal{C}} = (g_{p,q})_{p,q\in\mathcal{C}}$ be the square symmetric matrix defined as follows: $g_{p,q} = -9D(p)\cdot D(q) - (K_Z \cdot D(p))(K_Z \cdot D(q))$ (see [17], for an explicit description of $G_{\mathcal{C}}$ in terms of the proximity graph of $\mathcal{C}$).

Definition 4. A configuration $\mathcal{C}$ as above is called to be *P-sufficient* if $\mathbf{x}G_{\mathcal{C}}\mathbf{x}^t > 0$ for all nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$ with non-negative coordinates (where r denotes the cardinality of $\mathcal{C}$).

This definition gives the cited condition, thus *if $\mathcal{C}$ is a P-sufficient configuration, then the cone of curves of the surface Z obtained by blowing-up their points is polyhedral* [17, Theorem 2]. Notice that using the criterion given in [16], it is possible to decide whether a configuration is P-sufficient. Moreover, when the configuration $\mathcal{C}$ is a *chain* (that is, the partial ordering $\geqslant$ defined in Section 2 is a total ordering), a very simple to verify criterion can be given: *$\mathcal{C}$ is P-sufficient if* *the last entry of the matrix $G_{\mathcal{C}}$ is positive* [17, Proposition 6]. Also, it is worthwhile to add that whichever configuration of cardinality less than 9 is P-sufficient.

Taking into account the above facts and applying [18, Lemma 3], it can be proved the following result which shows that, if the configuration of dicritical points of an arbitrary foliation $\mathcal{F}$ on $\mathbb{P}^2$ is P-sufficient, then Algorithm 3 and Proposition 2 can be applied to decide whether $\mathcal{F}$ has a rational first integral (and to compute it).

Proposition 5. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ such that the configuration $\mathcal{B}_{\mathcal{F}}$ is P-sufficient. Then*:

(a) *The cone $NE(Z_{\mathcal{F}})$ is polyhedral.*
(b) *If S is an independent system of algebraic solutions obtained by calling Algorithm* 3*, then either $T^2_{\mathcal{F},S} \neq 0$, $T_{\mathcal{F},S} \cdot \widetilde{E}_q < 0$ for some $q \in \mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$ or $K_{Z_{\mathcal{F}}} \cdot T_{\mathcal{F},S} < 0$.*

To end this paper, we give two illustrative examples where we have applied the above results and algorithms.

Example 5. Let $\{\mathcal{F}_a\}_{a\in\mathbb{C}}$ be the one-parameter family of foliations on the projective plane over the complex numbers $\mathbb{C}$ defined by the projective 1-form $\Omega = A\,dX + B\,dY + C\,dZ$, where:

$$A = Z\big(aXZ - Y^2 + Z^2\big), \qquad B = Z\big(X^2 - Z^2\big) \quad \text{and}$$
$$C = XY^2 - aX^2Z - XZ^2 - X^2Y + YZ^2.$$

Set $Q := \{\frac{q^2-p^2}{q^2} \mid p, q \in \mathbb{Z},\ p \neq 0 \neq q\}$ and consider the following points on $\mathbb{P}^2_{\mathbb{C}}$: $U = (1:0:0)$, $W = (0:1:0)$, $R_a = (1 : \sqrt{1+a} : 1)$ when $-a \in Q$ and $S_a = (1 : \sqrt{1-a} : -1)$ whenever $a \in Q$. In order to determine the foliations of the family $\{\mathcal{F}_a\}_{a\in\mathbb{C}}$ with a (or without any) rational first integral, we shall distinguish three cases:

Case 1: a and $-a$ are not in Q. In this case, $\mathcal{K}_{\mathcal{F}_a} = \{U, W\}$, $\mathcal{B}_{\mathcal{F}_a} = \{W\}$ and $\mathcal{N}_{\mathcal{F}_a} = \emptyset$. Since the cardinality of $\mathcal{B}_{\mathcal{F}_a}$ is 1, the inequality $d(X/Z) \wedge \Omega \neq 0$ shows that $\mathcal{F}_a$ has no rational first integral.

Case 2: $a \neq 0$ and either a or $-a$ belong to Q. Now $\mathcal{K}_{\mathcal{F}_a} = \{U, W\} \cup \mathfrak{D}_a$ and $\mathcal{B}_{\mathcal{F}_a} = \{W\} \cup \mathfrak{D}'_a$, where $\mathfrak{D}_a$ and $\mathfrak{D}'_a$ stand for configurations (may be empty) whose unique (possible) point on $\mathbb{P}^2$ is S_a (respectively, R_a) whenever $a \in Q$ (respectively, $-a \in Q$). $\mathfrak{D}_a$ and $\mathfrak{D}'_a$ strongly depend on the value a; we shall illustrate it taking three specific values for a and running our algorithms in order to decide about the existence of a rational first integral in each case.

If $a = 5/9$, $\mathfrak{D}_a = \mathfrak{D}'_a$ consists of a chain of 3 points $\{q_1, q_2, q_3\}$, where q_3 is proximate to q_1 and provides the unique dicritical exceptional divisor associated with points in $\mathfrak{D}_a$. Since the configuration $\mathcal{B}_{\mathcal{F}_a}$ is P-sufficient, the cone $NE(Z_{\mathcal{F}})$ is polyhedral and if we run Algorithm 3 for the foliation $\mathcal{F}_a$, it will end. The sequence of convex cones $V_0 \subset V_1$ computed by the algorithm (see the explanation in page 626) is the following: $V_0 = \mathrm{con}(\{[\widetilde{E}_W], [\widetilde{E}_{q_1}], [\widetilde{E}_{q_2}], [\widetilde{E}_{q_3}]\})$ and $V_1 = \mathrm{con}(V_0 \cup \{[L^* - E^*_W - E^*_{q_1} - E^*_{q_2}]\})$. The dual cone of V_1 is spanned by the classes $[L^*]$, $[L^* - E^*_W]$, $[L^* - E^*_{q_1}]$, $[2L^* - E^*_{q_1} - E^*_{q_2}]$ and $[3L^* - 2E^*_{q_1} - E^*_{q_2} - E^*_{q_3}]$. Since all these generators have non-negative self-intersection, (b) of step 5 of the above mentioned algorithm is not satisfied for V_1 and, then, the algorithm ends. The unique element of the obtained set G is the line with equation $X + Z = 0$, so step 6 returns "0" and, therefore, $\mathcal{F}_a$ has no rational first integral.

If $a = -\frac{861}{100}$, $\mathfrak{D}_a = \mathfrak{D}'_a$ is a chain of 13 points $\{q_i\}_{i=1}^{13}$, characterized by the fact that q_{13} is proximate to q_3, and $\mathcal{N}_{\mathcal{F}} = \{q_i\}_{i=1}^{12}$. Since the configuration $\mathcal{B}_{\mathcal{F}_a}$ is also P-sufficient, Algorithm 3 will end. The sequence of obtained convex cones is $V_0 \subset V_1 \subset V_2$, where $V_0 = \mathrm{con}(\{[\widetilde{E}_q]\}_{q\in\mathcal{B}_{\mathcal{F}_a}})$, $V_1 = \mathrm{con}(V_0 \cup \{[L^* - E^*_W - E^*_{q_1}]\})$ and $V_2 = \mathrm{con}(V_1 \cup \{[L^* - E^*_{q_1} - E^*_{q_2}]\})$. The dual cone of V_2 has 27 extremal rays which are spanned by classes with non-negative self-intersection. Moreover, the set G is the same as above. So, Algorithm 3 returns "0" and, thus, $\mathcal{F}_a$ has no rational first integral.

If $a = -3$, $\mathfrak{D}'_a = \emptyset$, and, as in case 1, $\mathcal{F}_a$ has no rational first integral.

It is possible to discard the existence of a rational first integral for any foliation $\mathcal{F}_a$ corresponding to the current case. Indeed, reasoning by contradiction, assume that $\mathcal{F}_a$ has a rational first integral and consider the algebraic solution C of $\mathcal{F}_a$ defined by the line with equation $Z = 0$. Since the self-intersection of the class of its strict transform on $Z_{\mathcal{F}_a}$ vanishes, by Theorem 1 it is easy to deduce that $D_{\mathcal{F}_a} = L^* - E^*_W$. The set $\{X, Z\}$ is a basis of $H^0(\mathbb{P}^2, \pi_{\mathcal{F}_a *}\mathcal{O}_{Z_{\mathcal{F}_a}}(D_{\mathcal{F}_a}))$ and, since $d(X/Z) \wedge \Omega \neq 0$, we get a contradiction.

Case 3: $a = 0$. Here, $\mathcal{K}_{\mathcal{F}_0} = \mathcal{B}_{\mathcal{F}_0} = \{R = R_0, S = S_0, U, W\}$, $\mathcal{N}_{\mathcal{F}_0} = \emptyset$. Applying Algorithm 3 we get the following independent system of algebraic solutions for $\mathcal{F}_0$:

$$G = \{l_{R,S}, l_{S,U}, l_{U,W}, l_{S,W}\},$$

where $l_{p,q}$ denotes the line joining p and q ($p, q \in \mathcal{B}_{\mathcal{F}_0}$). Its associated divisor $T_{\mathcal{F}_0,G}$ is $2L^* - E^*_R - E^*_S - E^*_U - E^*_W$. By Proposition 2, we get that if $\mathcal{F}_0$ admits rational first integral, then $D_{\mathcal{F}_0}$ should coincide with $T_{\mathcal{F}_0,G}$. The space of global sections $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F}_0,G}))$ has dimension 2 and it is generated by $F_1 = (X + Z)(Z - Y)$ and $F_2 = Z(Y - X)$. So we conclude, after checking it, that the rational function F_1/F_2 is a rational first integral of $\mathcal{F}_0$.

Example 6. Now, let $\mathcal{F}$ be a foliation as above defined by the 1-form $\Omega = A\,dX + B\,dY + C\,dZ$, where

$$\begin{aligned}
A &= X^4Y^3Z + 5X^3Y^4Z + 9X^2Y^5Z + 7XY^6Z + 2Y^7Z + X^4Z^4 - X^3YZ^4,\\
B &= -3X^5Y^2Z - 13X^4Y^3Z - 21X^3Y^4Z - 15X^2Y^5Z - 4XY^6Z + 2X^4Z^4 \quad \text{and}\\
C &= 2X^5Y^3 + 8X^4Y^4 + 12X^3Y^5 + 8X^2Y^6 + 2XY^7 - X^5Z^3 - X^4YZ^3.
\end{aligned}$$

Resolving $\mathcal{F}$, we get $\mathcal{K}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}} = \{q_i\}_{i=1}^{9}$. Its proximity graph is given in Fig. 3. $\mathcal{B}_{\mathcal{F}}$ is P-sufficient (use the above mentioned result given in [16]) and $\mathcal{N}_{\mathcal{F}} = \{q_1, q_3, q_7, q_8\}$. The out-

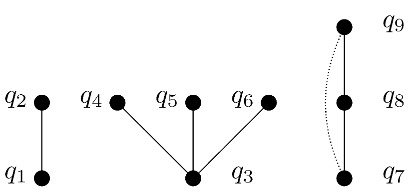

Fig. 3. The proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 6.

put G of Algorithm 3 is given by the curves with equations $X=0$, $X+Y=0$, $Z=0$, $XY+Y^2+XZ=0$ and $jXY+jY^2+XZ$, where j is a primitive cubic root of unity. So, G is an independent system of algebraic solutions. Its associated divisor $T_{\mathcal{F},G}$ is

$$6L^* - 3\sum_{i=1}^{3} E_{q_i}^* - \sum_{i=4}^{6} E_{q_i}^* - 2E_{q_7}^* - \sum_{i=8}^{9} E_{q_i}^*.$$

By Proposition 2, if $\mathcal{F}$ has a rational first integral, then the pencil $\mathcal{P}_{\mathcal{F}}$ is

$$\mathbb{P}H^0\big(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},G})\big).$$

A basis of this projective space is given by $F_1=(X+Y)^2X^2Z^2$ and $F_2=(X+Y)^3Y^3+X^3Z^3$. Finally, the equality $d(F_1/F_2)\wedge\Omega=0$ shows that F_1/F_2 is a rational first integral of $\mathcal{F}$.

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