Wavelet bases in the weighted Besov and Triebel–Lizorkin spaces with $A^\infty_p$-weights

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Abstract

We obtain wavelet characterizations of Besov spaces and the Triebel–Lizorkin spaces associated with $A^\infty_p$-weights. These characterizations are used to show that our wavelet bases are also greedy.

Keywords: Besov space; Triebel–Lizorkin space; Wavelet; Unconditional basis; Greedy basis

1. Introduction

Let $\{h_{v,m}\}_{v,m \in \mathbb{Z}, m \in \mathbb{Z}}$ be the Haar wavelet set and $1 < p < \infty$. Then, it is well known that, for $f \in L^p(\mathbb{R})$,

$$\left\| \sum_{v=-\infty}^{\infty} \sum_{m \in \mathbb{Z}} |(f, h_{v,m})_{L^2} h_{v,m}|^2 \right\|^{\frac{1}{2}}_{L^p(\mathbb{R})} \simeq \| f : L^p(\mathbb{R}) \|$$

and that $f$ can be expanded via $\{h_{v,m}\}_{v,m \in \mathbb{Z}}$ in the $L^p(\mathbb{R})$ norm

$$f = \sum_{v=-\infty}^{\infty} \sum_{m \in \mathbb{Z}} (f, h_{v,m})_{L^2} h_{v,m},$$

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where the convergence in (2) is in fact unconditional [22,23,27], see Definition 6 for unconditional basis and unconditional convergence.

In $\mathbb{R}^n$, this result, the characterization and the expansion of $L^p$ space in terms of wavelet basis, is extended for various wavelets [23,37] and for various function spaces (see [14] for Herz spaces and [35] for Besov spaces and Triebel–Lizorkin spaces). In particular, Lemarié-Rieusset [19] first constructed an unconditional basis for weighted $L^p$ spaces $L^p_w$ with $A_p$-local weights and proved the corresponding (1) and (2) for $f \in L^p_w$, where $w$ is called an $A_p$-local weight, if

$$
sup_{\ell(Q) \leq 1} \left( \frac{1}{|Q|} \int_Q w(y) dy \right) \left( \frac{1}{|Q|} \int_Q w(y)^{- \frac{1}{p-q}} dy \right)^{p-1} < \infty, \quad (3)$$

when $1 < p < \infty$ and, when $p = 1$, if

$$\text{ess sup}_{x \in \mathbb{R}^n} \left( w(x)^{-1} \sup_{\ell(Q) \leq 1} \frac{1}{|Q|} \int_Q w(y) dy \right) < \infty. \quad (4)$$

Here $Q$ runs over all cubes with edges parallel to the coordinate axes and $\ell(Q)$ is the side-length of $Q$. We denote the space of $A_p$-local weights by $A_p^{\text{loc}}$ and

$$A_\infty^{\text{loc}} = \bigcup_{1 \leq p < \infty} A_p^{\text{loc}}.$$  

On the other hand, Deng, Xu and Yan [4] have proved similar results for Triebel–Lizorkin space $F^{s,w}_{p,q}$ for $A_p$-weights (not for $A_p^{\text{loc}}$).

In this paper, we further extend these results for weighted Besov spaces $B^{s,w}_{p,q}$ and for weighted Triebel–Lizorkin spaces $F^{s,w}_{p,q}$ with $A_\infty^{\text{loc}}$-weights $w$ defined by Rychkov [26], see below for the precise definition. We construct unconditional basis for these spaces and characterize $f \in B^{s,w}_{p,q}$ or $f \in F^{s,w}_{p,q}$ in terms of the expansion coefficients with respect to the basis as in (1) and prove the unconditional expansion (2), see Definition 6 for the definition of the unconditional basis. We write $E = \{1, 2, \ldots, 2^n - 1\}$.

**Theorem 1.** Let $w$ be an $A_\infty^{\text{loc}}$-weight. Let $A^{s,w}_{p,q}$ be either $B^{s,w}_{p,q}$ or $F^{s,w}_{p,q}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Suppose the functions $\{\varphi, \psi^\varepsilon : \varepsilon \in E\}$ satisfy the following conditions for a sufficiently large $r \in \mathbb{N}$.

1. $\varphi$ and $\psi^\varepsilon, \varepsilon \in E$ are $C^r$-functions with compact supports.
2. $\psi^\varepsilon, \varepsilon \in E$ satisfy

$$\int_{\mathbb{R}^n} x^{\alpha} \psi^\varepsilon(x) dx = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^r. \quad (5)$$

3. The following system is an orthonormal basis of $L^2(\mathbb{R}^n)$:

$$\{\varphi_{0,k}, \psi^\varepsilon_{j,k} : \varepsilon \in E, j \in \mathbb{Z}^n, k \in \mathbb{Z}^n\}. \quad (6)$$

The following statements are satisfied:

1. If $0 < p < \infty, 0 < q \leq \infty$, then there exists a constant $c > 0$ such that

$$c^{-1} \| f \| : A^{s,w}_{p,q} \leq A^{s,w}_{p,q}(f) \leq c \| f \| : A^{s,w}_{p,q} \quad (7)$$

where $A^{s,w}_{p,q}(f)$ is defined in Definition 13 in terms of the expansion coefficients of $f$ via system (6).

2. If $0 < p, q < \infty$, then system (6) is an unconditional basis of $A^{s,w}_{p,q}$. 


We shall also prove in Theorem 16 that if we suitably normalize system (6):
\[
\left\{ \frac{\varphi_{0,k}}{\| \varphi_{0,k} : F_{p,q}^{s,w} \|}, \frac{\psi_{j,k}^\varepsilon}{\| \psi_{0,k} : F_{p,q}^{s,w} \|} : \varepsilon \in E, j \in \mathbb{Z}^n, k \in \mathbb{Z}^n \right\},
\]
then this becomes a greedy basis of \( F_{p,q}^{s,w} \) (see Definition 7 for the greedy basis).

Finally we describe the organization of the present paper. First, we always place ourselves in the setting of function spaces coming with an \( A^\text{loc}_\infty \)-weight \( w \). In Section 2 we recall the definition of the function spaces \( B_{p,q}^{s,w} \) and \( F_{p,q}^{s,w} \) with \( w \in A^\text{loc}_\infty \). As well as defining the function spaces and formulating the atomic decomposition, we make a brief review of the maximal operator and the notion of bases in quasi-Banach spaces. Section 3 is devoted to the wavelet characterizations. Finally in Section 4 we obtain unconditional bases in weighted Besov and Triebel–Lizorkin spaces. Furthermore we also construct a greedy basis in weighted Triebel–Lizorkin spaces.

2. Preliminaries

2.1. Weight class

Let \( 1 < p < \infty \). Then we have a series of characterization of \( L_p \) spaces by way of various wavelets. Starting from the pioneering work by Lemarié-Rieusset [19], several people including the first author of the present paper became aware of the fact that the results are extendible to the weighted cases. For example, Deng, Xu and Yan characterized weighted Triebel–Lizorkin spaces \( F_{p,q}^{s,w} \) for \( w \in A_p \) (see [4]). However, returning to the inhomogeneous characterization (see p16, Definition 13) by the Haar wavelets, we notice that the condition need not be global: It suffices to assume that the weight is locally dyadic \( A_p \) on any interval with the same constant. Thus, we have only to assume that
\[
\sup_{\nu \geq 0, m \in \mathbb{Z}} \left( \frac{1}{|Q_{vm}|} \int_{Q_{vm}} w(x)dx \right) \cdot \left( \frac{1}{|Q_{vm}|} \int_{Q_{vm}} w(x)^{-\frac{1}{p-1}}dx \right)^{\frac{1}{p-1}} < \infty,
\]
if \( 1 < p < \infty \). Hence, it is sufficient for weights to be locally dyadic-regular in the above sense.

In the present paper we are concerned with the function spaces reflecting smoothness as well as integrability. Therefore, the assumption that the weight belongs to the class of local dyadic \( A_p \) is too weak. Therefore, it seems appropriate that the weight is locally regular, which is defined more precisely below. We shall find some bases on \( A^{s,w}_{p,q} \), the weighted Besov space and the weighted Triebel–Lizorkin space, where \( w \) is an \( A_p \)-local weight. This class first appeared in Lemarié-Rieusset’s work [19], where Lemarié-Rieusset considered a characterization and a construction of unconditional basis of weighted \( L_p \) spaces in terms of scaling functions and wavelets. After that, Rychkov [26] developed the theory of \( A_p \)-local weights through the study of weighted Besov and Triebel–Lizorkin spaces. As is well known, in the celebrated paper [24] the class \( A_p \) was defined in connection with the Hardy–Littlewood maximal operator given by
\[
Mf(x) := \sup_{x \in Q} m_Q(|f|) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy, \quad f \in L^\text{loc}_1
\]
where \( Q \) runs over all cubes whose edges are parallel to the coordinate axis and \( m_Q(g) \) denotes the average of \( g \) over a cube \( Q \). Below, simply by a “cube” we mean a compact cube whose edges are parallel to the coordinate axis. A “weight” is a non-negative measurable function locally integrable and non-vanishing dx-almost everywhere on \( \mathbb{R}^n \). A weight \( w \) is said to be an \( A_1 \)-weight, if there exists a constant \( c > 0 \) such that \( Mw(x) \leq c \, w(x) \) for dx-almost every
Let $x \in \mathbb{R}^n$. Let $p > 1$. A weight $w$ is said to be an $A_p$-weight if

$$\sup_Q m_Q(w) \cdot m_Q \left( w^{\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where $Q$ runs over all cubes. We refer, for example, to [7] for more information of this class of weights. The present paper deals with its local version. To formulate the local counterpart of $A_p$-class, first we introduce the following maximal operator due to Rychkov [26]. Let us define

$$M_{\text{loc}} f(x) := \sup_{x \in Q} m_Q(|f|),$$

Here $Q$ runs over all cubes whose edges are parallel to the coordinate axis and which have side-length less than 1. The class of $A_p$-local weights is defined as follows:

$$A^1_{\text{loc}} := \{ w : w \text{ is a weight with } A^1_{\text{loc}}(w) < \infty \}, \quad 1 \leq p < \infty,$$

where

$$A^1_{\text{loc}}(w) := \text{esssup}_{x \in \mathbb{R}^n} \frac{M_{\text{loc}} w(x)}{w(x)}.$$  

$$A^p_{\text{loc}}(w) := \sup_{\ell(Q) \leq 1} m_Q(w) \cdot m_Q \left( w^{\frac{1}{p-1}} \right)^{p-1}, \quad p > 1.$$

In analogy with the classical case, Rychkov also took up $A_{\infty} := \bigcup_{1 \leq p < \infty} A^p_{\text{loc}}$ in his paper [26]. This class of weights enjoys properties analogous to $A_p$ such as the openness property and the (local) reverse Hölder inequality.  

In [26] Rychkov defined weighted Besov spaces $B^s_{p,q}$ and weighted Triebel–Lizorkin spaces $F^s_{p,q}$ for $w \in A^\infty_{\text{loc}}$. We remark that these weighted function spaces are investigated for some other classes of weights. We list [28,29], the papers due to Schott as the first crucial papers dealing with the exponential weights. Schott considered the class of weights $w$ satisfying

$$0 < w(x) \leq c \exp(c |x - y|)w(y), \quad x, y \in \mathbb{R}^n$$

for some $c > 0$. In his subsequent paper [30], Schott investigated the boundedness of the pseudodifferential operators as well. Returning to the function spaces coming with $A^\infty_{\text{loc}}$-weights, the first and second authors investigated the atomic decomposition of this space [13]. For the unweighted case we refer to the pioneering works [5,35].

2.2. Function spaces $A^{s,w}_{p,q}$ with $w \in A^\infty_{\text{loc}}$

Based mainly on [26], let us state the results needed in the present paper.

Let us describe precisely function spaces $A^{s,w}_{p,q}$. Throughout the present paper, unless additionally stated, we assume that $w$ is an $A^\infty_{\text{loc}}$-weight, or more precisely we assume that $w \in A^u_{\text{loc}}$ for some $1 \leq u < \infty$. We maintain the following notations.

1. We set $\mathbb{N}_0 := \{0, 1, \ldots \}$ and $\mathbb{N} := \{1, 2, \ldots \}$.
2. $S_\varepsilon$ denotes the set of all smooth functions $\varphi$ satisfying

$$q_N(\varphi) := \sup_{x \in \mathbb{R}^n} \left( \sum_{|\alpha| \leq N} \varepsilon^{N|x|} |\partial^\alpha \varphi(x)| \right) < \infty$$

for all $N \in \mathbb{N}_0$. Topologize $S_\varepsilon$ with $\{q_N\}_{N \in \mathbb{N}_0}$. $S_\varepsilon'$ denotes the topological dual of $S_\varepsilon$. 
(3) Let \( B(R) := \{ x \in \mathbb{R}^n : |x| < R \} \).

(4) Define
\[
\| f : L_p^w \| := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty
\]
for a measurable function \( f \).

(5) Let \( 0 < p < \infty \) and \( 0 < q \leq \infty \). Given a sequence of measurable functions \( \{ f_j \}_{j \in \mathbb{N}_0} \), we define
\[
\| \{ f_j \}_{j \in \mathbb{N}_0} : L_p^w \| := \left( \sum_{j \in \mathbb{N}_0} \| f_j : L_p^w \|^q \right)^{\frac{1}{q}},
\]
\[
\| \{ f_j \}_{j \in \mathbb{N}_0} : L_p^w (l_q) \| := \left\| \left( \sum_{j \in \mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} : L_p^w \right\|.
\]

Based on this notation, we define the weighted function spaces \( A_{p,q}^{s,w} \).

**Definition 2.** Let \( L \in \mathbb{Z} \) and \( s \in \mathbb{R} \). Take a sequence
\[
\{ \varphi_j \}_{j \in \mathbb{N}_0} \subset C_c^{\infty}
\]
satisfying the following conditions.

1. \( L \geq \max(-1, [s]) \).
2. \( \text{supp}(\varphi_0) \subset B(1) \).
3. \( \varphi_1(x) = 2^n \varphi_0(2x) - \varphi_0(x) \).
4. \( \varphi_j(x) = 2^{(j-1)n} \varphi_1(2^{j-1}x) \) for \( j \in \mathbb{N} \).
5. \( \int_{\mathbb{R}^n} x^\beta \varphi_1(x) dx = 0 \) for all \( \beta \in \mathbb{N}_0^n \) with \( |\beta| \leq L \). If \( L = -1 \), then this means no condition.
6. \( \int_{\mathbb{R}^n} \varphi_0(x) dx \neq 0 \).

Rychkov named this condition the \( M_s \) condition in his paper [26]. However, for the sake of brevity we do not use the term \( M_s \) condition and instead we say that \( \varphi_0 \) satisfies the moment condition of order \( \max(-1, [s]) \).

Using the sequence \( \{ \varphi_j \}_{j \in \mathbb{N}_0} \) above, Rychkov defined the function space \( A_{p,q}^{s,w} \) as follows:

**Definition 3.** Suppose that the parameters \( p, q, s \) satisfy
\[
0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}
\]
and that \( w \in A_{\text{loc}}^{\infty} \).

Then define
\[
\| f : B_{p,q}^{s,w} \| := \| \{ 2^j s \varphi_j * f \}_{j \in \mathbb{N}_0} : l_q (L_p^w) \|
\]
\[
\| f : F_{p,q}^{s,w} \| := \| \{ 2^j s \varphi_j * f \}_{j \in \mathbb{N}_0} : L_p^w (l_q) \|
\]
for \( f \in \mathcal{S}_c' \). To unify the notation, here and below \( A_{p,q}^{s,w} \) is used to denote either \( B_{p,q}^{s,w} \) or \( F_{p,q}^{s,w} \).
Rychkov proved the following theorem for \( w \in A^{\text{loc}}_w \) in his key paper [26].

**Theorem 4 ([26])**. Suppose that the parameters \( p, q, s \) satisfy
\[
0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}
\]
and \( w \in A^{\text{loc}}_w \). Then a different admissible choice of \( \{\varphi_j\}_{j \in \mathbb{N}_0} \) in Definition 2 will yield equivalent norms in Definition 3, provided \( \varphi_0 \) satisfies the moment condition of order \( \max(-1, [s]) \).

As for weighted function spaces, we remark that \( h_p^w \) was investigated by Bui when \( w \in A_p \) which was equivalent to \( F_{p,2}^0 \) (see [1]). In [26] the result was extended to the case when \( w \in A^{\text{loc}}_p \).

We shall utilize the following atomic decomposition for these function spaces, which is a key tool in the present paper. To formulate the result, we need some more notations. Let
\[
Q_{v,m} := \prod_{j=1}^n \left[ \frac{m_j}{2^v} - \frac{m_j + 1}{2^v} \right], \quad v \in \mathbb{N}_0, \quad m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n.
\]

Denote by \( \chi_{(p)}^{v,m} \) the \( p \)-normalized indicator of this cube:
\[
\chi_{v,m}^{(p)}(x) := 2^{rac{nv}{p}} \chi_{Q_{v,m}}(x).
\]

Given a doubly indexed sequence \( \lambda = \{\lambda_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \), define
\[
\|\lambda : b_{p,q}^w \| := \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \chi_{v,m}^{(p)} : l_q(L_p^w) \right\},
\]
\[
\|\lambda : f_{p,q}^w \| := \left\{ \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \chi_{v,m}^{(p)} : L_w^w(l_q) \right\}.
\]

Denoting by 1 the constant weight, let us define
\[
\sigma_p(w) := \inf \left\{ n \left( \frac{u}{\min(p, u)} - 1 \right) + (u - 1)n : u \in [1, \infty), \ w \in A^{\text{loc}}_w \right\}
\]
\[
\sigma_q := \sigma_q(1)
\]
\[
\sigma_{p,q}(w) := \max(\sigma_p(w), \sigma_q).
\]

Let us formulate the atomic decomposition.

**Proposition 5 ([13])**. Let
\[
0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}, \quad 1 \leq u < \infty,
\]
Take an integer \( L \) with
\[
L \geq \max(-1, [\sigma_p(w) - s])
\]
when \( A \) denotes \( B \) and
\[
L \geq \max(-1, [\sigma_{p,q}(w) + n(u - 1)])
\]
when \( A \) denotes \( F \). Assume in addition \( w \in A^{\text{loc}}_w \). Then there exist constants \( c_0, c_1, c_2 > 0 \) and \( c_\alpha > 0, \ \alpha \in \mathbb{N}_0^\nu \) with the following property.
(1) Let $f \in A_{p,q}^{s,w}$. Then $f$ admits the following decomposition.

$$ f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} a_{vm}, $$

where the coefficient $\lambda = \{\lambda_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ and $\{a_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ satisfy conditions (8)–(11) given below.

$$ \|\lambda : a_{p,q}^w\| \leq c_0 \|f : A_{p,q}^{s,w}\| \quad \text{(8)} $$

$$ \text{supp}(a_{vm}) \subset c_1 Q_{v,m} \quad \text{(9)} $$

$$ a_{vm} \in C^\infty, \|\partial^\alpha a_{vm} : L_\infty\| \leq c_\alpha 2^{-v\left(s - \frac{n}{p}\right) + v|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}_0^n \quad \text{(10)} $$

$$ \int_{\mathbb{R}^n} x^\beta a_{vm}(x) dx = 0 \quad \text{for all } v \in \mathbb{N}, m \in \mathbb{Z}^n, |\beta| \leq L. \quad \text{(11)} $$

If $L = -1$, then (11) means no condition.

(2) Assume that the coefficient $\lambda = \{\lambda_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ and $\{a_{vm}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ satisfy conditions (9), (11), (12) and (13). Here conditions (12) and (13) are given below.

$$ a_{vm} \in C^{(1+[s])_+}, \|\partial^\alpha a_{vm} : L_\infty\| \leq c_\alpha 2^{-v\left(s - \frac{n}{p}\right) + v|\alpha|} \quad \text{(12)} $$

$$ \|\lambda : a_{p,q}^w\| < \infty. \quad \text{(13)} $$

for all $\alpha \in \mathbb{N}_0$ with $|\alpha| \leq (1 + [s])_+$. Then

$$ f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} a_{vm} $$

converges in $S'_c$ and satisfies the norm estimate

$$ \|f : A_{p,q}^{s,w}\| \leq c_2 \|\lambda : a_{p,q}^w\|. $$

The aim of the present paper is to characterize these weighted function spaces in terms of wavelets. What is new about the present paper can be summarized as follows: We characterize the function spaces $A_{p,q}^{s,w}$ for all admissible $p, q, s$ and locally regular weights $w \in A_{loc}^{1,w}$ by means of the atomic decompositions obtained in [13]. Our method is applicable to the dotted function spaces $\hat{A}_{p,q}^{s,w}(\mathbb{R}^n, w)$ with $w \in A_p$ for all admissible parameters $p, q, s$ as well. We integrate what has been known about the inhomogeneous wavelet characterizations of the weighted function spaces.

### 2.3. Bases

Finally to formulate our results, let us recall the definition of unconditional bases. Let $(X, \| \cdot \|_X)$ be a quasi-Banach space and $A$ a countable index set until the end of this section.

Greedy bases, which was introduced by Konyagin and Temlyakov in [18], give a nonlinear approximation of a function in terms of a finite sum of the expansion and their study has begun recently. The precise definition will be described in Section 2.3 later. Applying the wavelet characterizations, we also attempt to obtain unconditional basis or greedy basis. The attempts have been done in [8–12,34,38]. It is known that wavelets construct a greedy basis in some function spaces. In fact, Temlyakov [34] showed that suitable wavelets give greedy bases in $L_p$ spaces for $1 < p < \infty$. Additionally if wavelets have sufficient smoothness and decay, then they also give greedy bases in Sobolev and Triebel–Lizorkin spaces ([8,10]). Furthermore the first author
has constructed them in weighted $L_p$ and Sobolev spaces ([11,12]). But we cannot always obtain greedy bases. Soardi [32] has established wavelet characterization of Orlicz spaces $L^\Phi(\mathbb{R}^n)$ and proved that a wavelet basis forms an unconditional basis on $L^\Phi(\mathbb{R}^n)$ using wavelets with proper decay and smoothness. After that, recently Wojtaszczyk [38] and Garrigós, Hernández and Martell [9] have independently obtained the following rigidity property of greediness: The normalized wavelet basis is greedy if and only if $L^\Phi(\mathbb{R}^n)$ coincides with $L_p(\mathbb{R}^n)$ for some constant $1 < p < \infty$. In the present paper we show that the construction of greedy basis is applicable to weighted Triebel–Lizorkin spaces, however it does not generally hold for the Besov spaces.

Let us provide a brief view of the history of the wavelet characterization of Besov spaces and Triebel–Lizorkin spaces. We remark that [5,6,25,31,35] are pioneering works of this attempt with $w \equiv 1$. For more details about wavelet characterization on various function spaces we refer to [23,36,37] when $w \equiv 1$. For the globally regular weight, that is, $w \in A_p$, Deng, Xu and Yan obtained some characterization of one-dimensional dotted function spaces $\tilde{F}^{s}_{\rho,q}(\mathbb{R}, w)$ with $|s|$ restricted to very small values [4]. As for the periodic case Kazaryan and Lizorkin obtained unconditional bases for weighted function spaces [16]. Their method is based on the boundedness of the singular integral operators. The boundedness of singular integral operators on weighted periodic function spaces dates back to Kokilashvili [17].

We recall the definitions of unconditional, Schauder, greedy and democratic bases. The first one is unconditional basis. It is known that there are several equivalent definitions of unconditional basis in Banach spaces [15,21,37]. Returning to the quasi-Banach space $X$, let $\{x_m\}_{m \in A}$ be a sequence in $X$. We recall that the series $\sum_{m \in A} x_m$ is said to converge unconditionally in $X$, if $\sum_{i=1}^{\infty} x_{\sigma(i)}$ converges for every bijection $\sigma$ from $\mathbb{N}$ to $A$. Some of them are defined by Konyagin and Temlyakov [18], which we now follow closely.

**Definition 6.** $\{x_k\}_{k=1}^{\infty} \subset X$ is said to be a Schauder basis if there exists a unique sequence $\{c_k(x)\}_{k=1}^{\infty} \subset \mathbb{C}$ such that $x = \sum_{k=1}^{\infty} c_k(x)x_k$ in $X$ for all $x \in X$. Furthermore, if the convergence above is always unconditional, then the basis is said to be unconditional.

**Definition 7.** Let $\{x_k\}_{k=1}^{\infty}$ be a Schauder basis in $X$ normalized as $\|x_k\|_X = 1$ for all $k \in \mathbb{N}$.

1. $\{x_k\}_{k=1}^{\infty}$ is said to be greedy for $X$ if there exists a constant $c > 0$ with the following property: For every $x \in X$ there exists a permutation $\rho$ of $\mathbb{N}$ which satisfies

   $$|c_{\rho(1)}(x)| \geq |c_{\rho(2)}(x)| \geq \cdots \geq |c_{\rho(N)}(x)| \geq \cdots$$

   and

   $$\|x - \sum_{k=1}^{N} c_{\rho(k)}(x)x_{\rho(k)}\|_X \leq c \inf_{y \in \Sigma_N} \|x - y\|_X,$$

   for every $N \in \mathbb{N}$, where $\Sigma_N := \{\sum_{\alpha \in A} \alpha \omega_x : \alpha \omega_x \in \mathbb{C}, \|\alpha\|_A \leq N, A \subset \mathbb{N}\}$.

2. $\{x_k\}_{k=1}^{\infty}$ is said to be democratic for $X$ if there exists a constant $c > 0$ and a function $\mu : \mathbb{N} \to \mathbb{R}$ such that

   $$\mu(\sharp P) \leq \left\| \sum_{k \in P} x_k \right\|_X \leq c \mu(\sharp P),$$

   for all $P \subset \mathbb{N}$.

We remark that “unconditional” and “democratic” are independent notions. Indeed, in [18, Section 3] Konyagin and Temlyakov gave some examples of bases, which are not democratic but
unconditional, or which are not unconditional but democratic. However, if we assume that the basis is unconditional, then the above two notions agree. That is, we have the following.

**Proposition 8 ([18, Section 1]).** Let \( \{x_k\}_{k=1}^{\infty} \) be a Schauder basis in \( X \) such that \( \|x_k\|_X = 1 \) for all \( k \in \mathbb{N} \). Then \( \{x_k\}_{k=1}^{\infty} \) is greedy if and only if it is unconditional and democratic.

### 3. Wavelet characterizations of \( A_{p,q}^{s,w} \)

In this section we characterize certain function spaces in terms of wavelets. Now let us recall some terminologies. Let \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \).

1. Given a function \( \psi : \mathbb{R}^n \to \mathbb{C} \), we define \( \psi_{j,k}(x) := 2^{jn/2} \psi(2^j x - k) \).
2. We define an index set \( E \) by \( E := \{1, 2, \ldots, 2^n - 1\} \).

According to wavelet theory, there exists a collection of compactly supported \( C^r \)-functions \( \{\phi, \psi^\varepsilon : \varepsilon \in E\} \) such that

\[
\int_{\mathbb{R}^n} x^\alpha \psi^\varepsilon(x) dx = 0 \quad \text{for all } \varepsilon \in E \quad \text{and all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq r,
\]

and that the sequence

\[
\{\phi_{0,k}, \psi_{j,k}^\varepsilon : \varepsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \}
\]

forms an orthonormal basis in \( L^2 \) (see [23,37]). We often say that the function \( \phi \) is a scaling function and that each \( \psi^\varepsilon \) is a wavelet. In particular, we can even arrange that \( \phi \) and each \( \psi^\varepsilon \) be real-valued and that supp \( \phi = \text{supp } \psi^\varepsilon = [0, 2N - 1]^n \) for any given positive integer \( N \geq 2 \). In our actual construction, the constant \( r \) can be chosen as an increasing function of \( N \) [3,20,23].

To characterize function spaces, the scaling functions are indispensable as well as the wavelets. We refer to [19,23] for more information. Throughout the present paper, we consider a set of functions \( \{\phi, \psi^\varepsilon : \varepsilon \in E\} \) satisfying the following three conditions:

(A) \( \phi \) and each \( \psi^\varepsilon \) are compactly supported \( C^r \)-functions for some \( r \in \mathbb{N}_0 \).
(B) Each \( \psi^\varepsilon \) satisfies condition (14).
(C) System (15) forms an orthonormal basis in \( L^2 \).

Let \( w \in A_{loc}^\infty \), more specifically, \( w \in A_{u}^\text{loc} \).

**Lemma 9 ([26, Lemma 2.15 (2.31)]).** Suppose that the parameters \( p, q, s \) satisfy

\[
0 < p < \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}.
\]

Assume in addition that \( w \in A_{u}^\text{loc} \). Let us set

\[
L_0 = \left[\frac{n u}{p} - s\right],
\]

where \([\cdot]\) denotes the Gauss sign. Then there exists a constant \( B \gg 0 \) such that

\[
|\langle f, \gamma \rangle| \leq c \|f : A_{p,q}^{s,w}\| \cdot \sup_{x \in \mathbb{R}^n} \left( \sum_{|\alpha| \leq L_0} e^{B|x|} |\partial^\alpha \gamma(x)| \right)
\]

for all \( f \in A_{p,q}^{s,w} \) and \( \gamma \in C_\varepsilon^\infty \) supported in a cube of side-length 1.

**Proof.** This is just a re-examination of the proof with \( L \) in [26, (2.32)] specified.

For the sake of convenience of the readers we outline the proof.
Let us pick a sequence of functions \( \{ \psi_j \}_{j=0}^{\infty} \subset C_c^\infty \) so that
\[
\psi_j(x) = \psi_1(2^{j-1}x), \quad j \in \mathbb{N}, \quad \sum_{j=0}^{\infty} \psi_j \ast \varphi_j = \delta \quad \text{in } S'_e, \quad \int x^\beta \psi_1(x) dx = 0
\]
for all \( \beta \) with \( |\beta| \leq L_0 - 1 \). Let us assume that \( \gamma \) is supported on a cube centered at \( x_0 \) of side-length 1. Let us set \( \delta(x) = \gamma(x_0 + x) \). In [26] Rychkov established
\[
\sup_{y \in \mathbb{R}^n} \frac{|\delta \ast f(x - y)|}{e^B |y|} \leq c \sup_{|\alpha| \leq L_0} |\partial^\alpha \delta| \sum_{k=0}^{\infty} 2^{-(L_0 - A + s)k} 2^{ks} \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k \ast f(x - y)|}{(1 + 2^j |y|)^A e^B |y|}.
\]
In the aforementioned paper it is shown that
\[
\left\| \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k \ast f(x - y)|}{(1 + 2^j |y|)^A e^B |y|} : L^u_p \right\| \leq c \left\| \varphi_k \ast f : L^u_p \right\|,
\]
if \( B \gg 0 \) and \( A > \frac{nu}{p} \). Assuming that \( L_0 > \frac{nu}{p} - s \), we obtain (17).

Note that
\[
|\gamma \ast f(0)| = |\delta \ast f(-x_0)| \leq c e^{B|x_0|} \inf_{|x| \leq 1} \left( \sup_{y \in \mathbb{R}^n} e^{-B|y|} |\delta \ast f(x - y)| \right).
\]
In view of (16) and (17) we obtain
\[
|\gamma \ast f(0)|^p \leq \int_{|x| \leq 1} w(x)^p dx \left( \sup_{y \in \mathbb{R}^n} \frac{|\delta \ast f(x - y)|}{e^B |y|} \right)^p dx \leq \int_{|x| \leq 1} w(x)^p dx \sup_{|\alpha| \leq L_0} |\partial^\alpha \delta(x)| \sum_{k=0}^{\infty} 2^{k(s - e)} \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k \ast f(x - y)|}{(1 + 2^j |y|)^A e^B |y|} : L^u_p \right\| \leq c \sup_{|\alpha| \leq L_0} e^{B_p|x|} |\partial^\alpha \delta(x)|.
\]
This is the desired result. \( \square \)

**Lemma 10.** Maintain the same assumption as Lemma 9. Given \( \Psi \in S_e \), we set
\[
\psi^{(M)} := \sum_{|k| \leq M} \langle \psi, \varphi_{0,k} \rangle \varphi_{0,k} + \sum_{e \in E} \sum_{j=0}^{M} \sum_{|k| \leq M} \langle \psi, \varphi_{e,j,k} \rangle \varphi_{e,j,k}.
\]
Let \( B \) be a constant from Lemma 9. Then we have
\[
\lim_{M \to \infty} \sup_{|\alpha| \leq \left\lceil \frac{nu}{p} - s \right\rceil} \sup_{x \in \mathbb{R}^n} e^{B|x|} |\partial^\alpha [\psi - \psi^{(M)}]|(x) = 0.
\]

**Proof.** Using the moment condition of \( \psi^e \), we can prove the assertion easily. \( \square \)

With the help of the two lemmas above, we can prove the assertion. This corollary is auxiliary and crucial for the whole paper.
Corollary 11. Maintain the same assumption as Lemmas 9 and 10. Then one has

$$f = \sum_{k \in \mathbb{Z}^n} (f, \varphi_{0,k}) \varphi_{0,k} + \sum_{s \in E} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} (f, \psi_{j,k}^e) \psi_{j,k}^e$$  \hspace{1cm} (18)

in the topology of $S'$. Furthermore, for any $\eta \in C^r$ with compact support, the functional

$$f \in S_c \mapsto \int_{\mathbb{R}^n} f(x)\overline{\eta(x)}dx$$

can be extended to a bounded linear functional on $A^{s,w}_{p,q}$.

**Proof.** The second statement is clear because of Lemma 9. As for the first statement we pick $\varphi \in S_c$. Then we have

$$\lim_{M \to \infty} \sum_{|k| \leq M} (f, \varphi_{0,k}) \cdot (\varphi_{0,k}, \eta) + \sum_{s \in E} \sum_{j = 0}^M \sum_{|k| \leq M} (f, \psi_{j,k}^e) \cdot (\psi_{j,k}^e, \eta)$$

$$= \lim_{M \to \infty} \left( f, \sum_{|k| \leq M} (\varphi_{0,k}, \eta) \cdot \varphi_{0,k} + \sum_{s \in E} \sum_{j = 0}^M \sum_{|k| \leq M} (\psi_{j,k}^e, \eta) \cdot \psi_{j,k}^e \right)$$

$$= (f, \eta),$$

where we have used Lemmas 9 and 10 for the second equality. Therefore, the proof is complete. \hfill $\square$

Lemma 12. We maintain the same assumption on the parameters $p$, $q$, $s$, $w$ as Lemma 9. Assume the functions $\varphi, \psi^e \in C^{(1+|s|)}$ are compactly supported functions such that

$$\int_{\mathbb{R}^n} x^\beta \psi^e(x)dx = 0$$

for all $\varepsilon \in E$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq [s]$. Define $m_{j, A, B}(y) := (1 + 2^j |y|)^A 2^B |y|$, where $A > \frac{n + s}{p}$ and $B$ are sufficiently large. Then we have

$$\left\| \sup_{y \in \mathbb{R}^n} \frac{|(f, \varphi(-x + y))|}{m_{0, A, B}(y)} : L^w_p \right\| \leq c \| f : A^{s,w}_{p,q} \|$$

and

$$\left\| \sup_{s \in E} \left\{ \left( \frac{2^{(s+\frac{q}{2})} \sup_{y \in \mathbb{R}^n} \frac{|(f, \psi_{j,0}(-x + y))|}{m_{j, A, B}(y)} \right) \right\} : l^w_q(L^w_p) \right\| \leq c \| f : B^{s,w}_{p,q} \|$$

$$\left\| \sup_{s \in E} \left\{ \left( \frac{2^{(s+\frac{q}{2})} \sup_{y \in \mathbb{R}^n} \frac{|(f, \psi_{j,0}(-x + y))|}{m_{j, A, B}(y)} \right) \right\} : L^w_p(l^q) \right\| \leq c \| f : F^{s,w}_{p,q} \|$$

for all $f \in S_c$. Here the $L^w_p$ norms are taken with respect to $x$.

**Proof.** The proof is obtained by using the fact that $\psi_{j,k}^e$ has a vanishing moment of order up to $L \geq [s])$. The proof is similar to the one in [26, Theorem 2.5] and we omit it. \hfill $\square$

**Definition 13.** Suppose that the parameters $p$, $q$, $s$ and the weight $w$ satisfy

$$0 < p < \infty, 0 < q \leq \infty, s \in \mathbb{R}, \ w \in A^{\text{loc}}_p.$$
Assume in addition that \( \{ \varphi, \psi^\varepsilon : \varepsilon \in E \} \) satisfy conditions (A), (B) and (C) with \( r \) sufficiently large. Then define

\[
B_{p,q}^{s,w}(f) := \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| + \sum_{s \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\varepsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0} \right\|_{L_p^w} \]

\[
F_{p,q}^{s,w}(f) := \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| + \sum_{s \in E} \left\| \left\{ 2^{j(s+\frac{n}{2})} \langle f, \psi_{j,k}^\varepsilon \rangle \chi_{Q_{j,k}} \right\}_{j \in \mathbb{N}_0} \right\|_{L_p^w} \]

for \( f \in B_{p,q}^{s,w} \) and \( f \in F_{p,q}^{s,w} \) respectively. To simplify our formulation, denote by \( A_{p,q}^{s,w}(f) \) either \( B_{p,q}^{s,w}(f) \) or \( F_{p,q}^{s,w}(f) \).

**Theorem 14.** Suppose that the parameters \( p, q, s \) and the weight \( w \) satisfy

\[
0 < p < \infty, 0 < q \leq \infty, s \in \mathbb{R}, w \in A_{\infty}^{\text{loc}}.
\]

Assume in addition that the functions \( \{ \varphi, \psi^\varepsilon : \varepsilon \in E \} \) satisfy conditions (A), (B) and (C) with \( r \) sufficiently large. Then there exists a constant \( c > 0 \) such that

\[
c^{-1} \left\| f : A_{p,q}^{s,w} \right\| \leq A_{p,q}^{s,w}(f) \leq c \left\| f : A_{p,q}^{s,w} \right\|
\]

for all \( f \in A_{p,q}^{s,w} \).

**Proof.** Since we have shown

\[
f = \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \varphi_{0,k} + \sum_{s \in E} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{j,k}^\varepsilon \rangle \psi_{j,k}^\varepsilon
\]

in the topology of \( S' \), it is easy to see that

\[
\left\| f : A_{p,q}^{s,w} \right\| \leq c A_{p,q}^{s,w}(f)
\]

by Proposition 5. Conversely, for the estimation of the coefficients we make use of Lemma 12. Using Lemma 12, we can easily obtain

\[
A_{p,q}^{s,w}(f) \leq c \left\| f : A_{p,q}^{s,w} \right\|,
\]

which proves the theorem. \( \square \)

**4. Wavelet bases in** \( A_{p,q}^{s,w} \)

Having obtained wavelet characterization of the function spaces, now let us show that the basis is unconditional.

**4.1. Unconditional bases in** \( A_{p,q}^{s,w} \)

**Theorem 15.** Let \( 0 < p, q < \infty, s \in \mathbb{R}, w \in A_{\infty}^{\text{loc}}. \) Assume the functions \( \{ \varphi, \psi^\varepsilon : \varepsilon \in E \} \) satisfy conditions (A), (B) and (C) with \( r \) sufficiently large. Then the sequence

\[
\{ \varphi_{0,k}, \psi_{j,k}^\varepsilon : \varepsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \}
\]

forms an unconditional basis in \( A_{p,q}^{s,w} \).
Theorem 14

However, once we reduce the matter to

Let us establish the first assertion. The definition of \( A_{p,q}^{s,w} \) gives

\[
A_{p,q}^{s,w}(T_{A,B} f) \leq A_{p,q}^{s,w}(f).
\]

Combining this with Theorem 14, we obtain

\[
c^{-1} \left\| T_{A,B} f : A_{p,q}^{s,w} \right\| \leq A_{p,q}^{s,w}(T_{A,B} f) \leq A_{p,q}^{s,w}(f) \leq c \left\| f : A_{p,q}^{s,w} \right\|,
\]

where \( c \geq 1 \) is the constant appearing in Theorem 14.

Next we turn to the second condition. It suffices to prove that

\[
\lim_{B \to E \times N_0 \times Z^n} A_{p,q}^{s,w}(f - T_{A,B} f) = 0
\]

for all \( f \in A_{p,q}^{s,w} \), because we have established that

\[
\left\| f - T_{A,B} f : A_{p,q}^{s,w} \right\| \leq c A_{p,q}^{s,w}(f - T_{A,B} f)
\]

in Theorem 14. However, once we reduce the matter to \( A_{p,q}^{s,w}(f - T_{A,B} f) \), it is easy by the Lebesgue convergence theorem to see that \( A_{p,q}^{s,w}(f - T_{A,B} f) \) tends to 0 as \( A \to Z^n \) and \( B \to E \times N_0 \times Z^n \). Therefore, the second assertion is immediate. \( \square \)

4.2. Greedy bases in \( F_{p,q}^{s,w} \)

If \( A = F \), then we can even say that the basis obtained above is greedy. Precisely speaking, let us prove the following.

Theorem 16. Consider the normalization of \( \varphi_{0,k} \) and \( \psi_{j,k}^{e} \) in \( F_{p,q}^{s,w} \). Define

\[
\widetilde{\varphi}_{0,k} := \frac{\varphi_{0,k}}{\| \varphi_{0,k} : F_{p,q}^{s,w} \|}, \quad \widetilde{\psi}_{j,k}^{e} := \frac{\psi_{j,k}^{e}}{\| \psi_{j,k}^{e} : F_{p,q}^{s,w} \|}.
\]

Then under the same condition as Theorem 15 the sequence

\[
\{ \widetilde{\varphi}_{0,k}, \widetilde{\psi}_{j,k}^{e} : e \in E, j \in N_0, k \in Z^n \}
\]

forms a greedy basis in \( F_{p,q}^{s,w} \).

To prove Theorem 16, we need some lemmas.

Lemma 17. Let \( w \in A_{\infty}^{pec} \). Then there exists a constant \( 1 < d < \infty \) such that for all dyadic cubes \( Q, Q' \) satisfying \( Q' \subseteq Q \) and \( |Q'|, |Q| \leq 1, \)

\[
d w(Q') \leq w(Q).
\]
Proof. The assertion is a local counterpart which can be found in [7, p.141] and [33, Proof of Corollary 1.1]. Going through an argument similar to the ones in these literatures, we can prove Lemma 17. □

We need another characterization of $F_{p,q}^{s,w}$ in order to obtain greedy bases in terms of wavelets. Let us write

$$
\tilde{F}_1(f) := \left\{ \langle f, \varphi_{0,k} \rangle \left\| \varphi_{0,k} : F_{p,q}^{s,w} \right\|_{k \in \mathbb{Z}^n} : l_p(\mathbb{Z}^n) \right\}
$$

$$
\tilde{F}_2(f) := \sum_{s \in E} \left\{ w(Q_{j,k})^{-\frac{1}{p}} \left\| \psi_{j,k}^e : F_{p,q}^{s,w} \right\| \left\langle f, \psi_{j,k}^e \right\rangle \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : f_{p,q}^w \right\}.
$$

Lemma 18. Under the same condition as Theorem 15 there exists a constant $c > 0$ such that

$$
c^{-1} \tilde{F}_1(f) \leq \left\| \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{0,k} \rangle \chi_{Q_{0,k}} : L_p^w \right\| \leq c \tilde{F}_1(f) \tag{19}
$$

and that

$$
c^{-1} \tilde{F}_2(f) \leq \sum_{s \in E} \left\{ 2^{j(s+n/2)} \left\langle f, \psi_{j,k}^e \right\rangle \right\}_{j \in \mathbb{N}_0, k \in \mathbb{Z}^n} : f_{p,q}^w \right\} \leq c \tilde{F}_2(f) \tag{20}
$$

for all $f \in F_{p,q}^{s,w}$.

Proof (Proof of Lemma 18). We prove (20), the proof of (19) being simpler. By Theorem 14, there exists a constant $c \geq 1$ such that for each $s \in E$, $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^n$,

$$
c^{-1} \left\| \psi_{j,k}^e : F_{p,q}^{s,w} \right\| \leq F_{p,q}^{s,w}(\psi_{j,k}^e) \leq c \left\| \psi_{j,k}^e : F_{p,q}^{s,w} \right\|.
$$

Meanwhile, we have $F_{p,q}^{s,w}(\psi_{j,k}^e) = 2^{j(s+n/2)} w(Q_{j,k})^{\frac{1}{p}}$, yielding (20). □

Having set down some auxiliary estimates, let us refer back to the proof of Theorem 16.

Proof (Proof of Theorem 16). Let $f \in F_{p,q}^{s,w}$. The proof is based on that of [2, Lemma 4.1]. Proposition 8 and Theorem 15 reduce the matter to proving that the sequence

$$
\left\{ \varphi_{0,k}, \psi_{j,k}^e : e \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \right\}
$$

is democratic. We see that $\{\varphi_{0,k}, \psi_{j,k}^e : e \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}$ forms an unconditional basis for $F_{p,q}^{s,w}$ by Theorem 15. Note that (18) can be rephrased as

$$
f = \sum_{k \in \mathbb{Z}^n} a_k(f) \varphi_{0,k} + \sum_{e \in E} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} b_{j,k}^e(f) \psi_{j,k}^e,
$$

where

$$
a_k(f) := \langle f, \varphi_{0,k} \rangle \left\| \varphi_{0,k} : F_{p,q}^{s,w} \right\|
$$

$$
b_{j,k}^e(f) := \langle f, \psi_{j,k}^e \rangle \left\| \psi_{j,k}^e : F_{p,q}^{s,w} \right\|.
$$
Combining Theorem 14 and Lemma 18, we see that the norm
\[
\left( \sum_{k \in \mathbb{Z}^n} |a_k(f)|^p \right)^{1/p} + \sum_{e \in E} \left\| \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| w(Q_{j,k}) \frac{1}{r} b_{j,k}^e (f) \chi_{Q_{j,k}} \right|^q \right)^{1/q} : L_w^{q,r} \right\|
\tag{21}
\]
is equivalent to $\|f : F_{p,q}^{s,w}\|$. Let us denote $\tilde{\varphi}_Q := \varphi_{j,k}$ and $\tilde{\psi}_Q := \psi_{j,k}$ for a dyadic cube $Q = Q_{j,k}$. We now take a pair of finite subsets
\[
A \subset \{(0,k) : k \in \mathbb{Z}^n\}, \quad A \subset E \times \{(j,k) : j \in \mathbb{N}_0, k \in \mathbb{Z}^n\}
\]
and set $g := \sum_{J \in A} \tilde{\varphi}_J + \sum_{(e,J) \in A} \tilde{\psi}_J$. Below we use $\alpha \simeq \beta$ to express the fact that there exists a constant $c > 0$ independent of $A$, $\Lambda$ such that $c^{-1} \alpha \leq \beta \leq c \alpha$. Our claim is that
\[
(\sharp A + \sharp \Lambda)^{1/p} \simeq \|g : F_{p,q}^{s,w}\|. \tag{22}
\]
Let $B := \{ Q_{j,k} : (\epsilon, Q_{j,k}) \in A \text{ for some } \epsilon \in E \}$. In view of the trivial inequality $\sharp B \leq \sharp A \leq (2^n - 1) \sharp B$ and Theorem 14, for the purpose of proving (22), we can replace $\Lambda$ with $B$.

Using (21), we obtain
\[
\|g : F_{p,q}^{s,w}\| \simeq (\sharp A)^{1/p} + \left\| \left( \sum_{(e,J) \in B} \left| w(J) \frac{1}{r} \chi_J \right|^q \right)^{1/q} : L_w^{q,r} \right\|
\]
\[
= (\sharp A)^{1/p} + \left\{ \int \bigcup_{J^\prime \in B} J^\prime \left( \sum_{J \in B} \left| w(J) \right|^q \chi_J(x) \right)^{\frac{p}{q}} w(x)dx \right\}^{1/p}.
\]

Fix $x \in \bigcup_{J \in B} J$ for the time being. Let $J(x)$ be the minimal dyadic cube in $B$ containing $x$. We claim that
\[
w(J(x))^{-\frac{q}{p}} \leq \sum_{J \in B} w(J)^{-\frac{q}{p}} \chi_J(x) \leq c w(J(x))^{-\frac{q}{p}}. \tag{23}
\]
The left inequality being trivial, let us prove the right inequality.

Let $J_0 := J(x)$. Define inductively $J_r, \ r = 1, 2, \ldots$ so that $J_r$ is a dyadic father of $J_{r-1}$, that is, a unique dyadic cube satisfying $J_{r-1} \subset J_r$ and $2^n |J_{r-1}| = |J_r|$ for every $r \in \mathbb{N}$. Then we have
\[
\sum_{J \in B} w(J)^{-\frac{q}{p}} \chi_J(x) \leq \sum_{r=0}^{\infty} w(J_r)^{-\frac{q}{p}}. \tag{24}
\]
By Lemma 17, there exists a constant $1 < d < \infty$ such that for all $r \in \mathbb{N}$,
\[
w(J_r) \geq d w(J_{r-1}) \geq \cdots \geq d^r w(J_0) = d^r w(J(x)),
\]
which gives us the right inequality in (23).

By virtue of (23) and (24), we obtain
\[
\int \bigcup_{J^\prime \in B} J^\prime \left( \sum_{J \in B} \left| w(J) \right|^q \chi_J(x) \right)^{\frac{p}{q}} w(x)dx \simeq \int \bigcup_{J^\prime \in B} \frac{w(x)dx}{w(J(x))}, \tag{25}
\]
where the implicit constants appearing in $\simeq$ do not depend on $A$ and $B$. To analyze more the right-hand side, we define $J := \{ x \in \bigcup_{J^\prime \in B} J^\prime : J(x) = J \}$ for each $J \in B$. Since
\( \bigcup_{J' \in B} J' = \bigcup_{J \in B} \tilde{J}, \) we obtain
\[
\int_{\bigcup_{J' \in B} J'} \frac{w(x)dx}{w(J(x))} = \int_{\bigcup_{J \in B} \tilde{J}} \frac{w(x)dx}{w(J(x))}.
\]

Then, since \( \tilde{J} \subset J, \) it follows that
\[
\int_{\bigcup_{J' \in B} J'} \frac{w(x)dx}{w(J(x))} \leq \sum_{J' \in B} \int_{j} \frac{w(x)dx}{w(J(x))} = \sum_{J \in B} \int_{J} \frac{w(x)dx}{w(J)} = \frac{1}{2} \mu B.
\]

Let \( B' := \{J \in B : J(x) = J\}. \) Then a simple geometric observation shows us that \( \frac{1}{2} \mu B \leq \mu B' \leq \mu B. \) Therefore, if we use \( \bigcup_{J' \in B} J' = \bigcup_{J \in B} \tilde{J}, \) then we obtain
\[
\int_{\bigcup_{J' \in B} J'} \frac{w(x)dx}{w(J(x))} = \sum_{J \in B} \int_{J} \frac{w(x)dx}{w(J)} \geq \frac{1}{2} \mu B.
\]

As a result, in view of (26) and (27), it follows that \( \int_{\bigcup_{J' \in B} J'} \frac{w(x)dx}{w(J(x))} \simeq \mu B. \)

Putting our observations all together, we have
\[
\|g : F_{p,q}^{s,w}\| \simeq (\mu A + \frac{1}{2} \mu B)^{\frac{1}{p}}.
\]

Consequently we have proved that the sequence
\[
\{\tilde{\varphi}_{0,k}, \tilde{\psi}_{j,k}^\varepsilon : \varepsilon \in E, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z}^n\}
\]
is democratic. \( \square \)

A similar calculation shows the following.

**Proposition 19.** Consider the normalization of \( \varphi_{0,k} \) and \( \psi_{j,k}^\varepsilon \) in \( B_{p,q}^{s,w}. \) Redefine
\[
\tilde{\varphi}_{0,k} := \frac{\varphi_{0,k}}{\|\varphi_{0,k} : B_{p,q}^{s,w}\|}, \quad \tilde{\psi}_{j,k}^\varepsilon := \frac{\psi_{j,k}^\varepsilon}{\|\psi_{j,k}^\varepsilon : B_{p,q}^{s,w}\|}.
\]
\( \{\tilde{\varphi}_{0,k}, \tilde{\psi}_{j,k}^\varepsilon : \varepsilon \in E, \ j \in \mathbb{N}_0, \ k \in \mathbb{Z}^n\} \) is greedy for \( B_{p,q}^{s,w} \) precisely when \( p = q < \infty. \)

**Proof.** Reconsider \( g \) in the above proof. Let us decompose
\[
B = \prod_{j=0}^\infty E \times \{(j, k) : k \in G_j\},
\]
where each \( G_j \) is a subset of \( \mathbb{Z}^n. \) Let us define
\[
h := \sum_{J \in \Lambda} \tilde{\varphi}_J, \quad g_j := \sum_{(\varepsilon, J) \in E \times \{(j, k) : k \in G_j\}} \tilde{\psi}_j.
\]

Then, re-examining the proof above and utilizing **Theorem 14**, we have
\[
\|g : B_{p,q}^{s,w}\| \simeq \|h : B_{p,q}^{s,w}\| + \left( \sum_{j=0}^\infty \|g_j : B_{p,q}^{s,w}\|\right)^{\frac{1}{q}}.
\]
\[ \approx \|h : F_{p,q}^{s,w}\| + \left(\sum_{j=0}^{\infty} \|g_j : F_{p,q}^{s,w}\|^q\right)^{\frac{1}{q}} \]

\[ \approx (\#A)^{\frac{1}{p}} + \left(\sum_{j=0}^{\infty} (\#G_j)^{\frac{q}{p}}\right)^{\frac{1}{q}}. \]

With this in mind, let us set

\[ g_l := \sum_{k=1}^{l} \varphi(0, (k, k, \ldots, k)), \quad h_l := \sum_{k=1}^{l} \psi(1, 1, \ldots, 1). \]

Then we obtain

\[ \|g_l : B_{p,q}^{s,w}\| \approx \|h_l : B_{p,q}^{s,w}\| \approx l^{\frac{1}{2}}. \]

If \( \{\varphi_0, k, \psi_{j,k}^\varepsilon : \varepsilon \in E, j \in \mathbb{N}_0, k \in \mathbb{Z}^n\} \) is greedy for \( B_{p,q}^{s,w} \), then we have the following norm equivalence:

\[ \|h_l : B_{p,q}^{s,w}\| \approx \|g_l : B_{p,q}^{s,w}\|. \]

This yields \( l^{\frac{1}{2}} \approx l^{\frac{1}{p}} \). Therefore, we conclude that \( p = q \). \( \Box \)

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