MATHEMATICS

APPROXIMATION METHODS BY POLYNOMIALS AND POWER-SERIES

BY

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1. *Introduction*

Recently a large number of papers have appeared dealing with extensions and generalizations of Bernstein polynomials in different directions. Thus SZÁSZ [9], MEYER-KÖNIG and ZELLER [7] deal with power series which have approximation properties similar to those of Bernstein polynomials. CHENEY and SHARMA [1] generalize the Meyer-König Zeller-operators further while JAKIMOVSKI and LEVIATAN [3], JAKIMOVSKI and RAMANUJAN [4] deal with families of generalized Bernstein polynomials and their relation to the moment problem. The recent results of SHAH and SURYNARAYANA [8] are also closely related.

Our object here is to introduce (§ 2) a family of linear polynomial operators which generalize (and include as special case) the classical Bernstein polynomials and (§ 3) another family of linear power series operators which generalize the Szász operator. We prove the approximation properties of these operators and indicate an application to summability of series.

2. *The Operators* $B_n^{(\lambda,\alpha)}$ $(\lambda \leqslant 0,\ \alpha > -1)$

Let $L_\nu^{(\alpha)}(t)$ denote the Laguerre polynomials of degree ν, $\alpha > -1$. For a given function $f(x)$ defined on $[0, 1]$ and for $\lambda \leqslant 0$, we define

$$(2.1) \qquad B_n^{(\lambda,\alpha)}(f;x) = \frac{1}{L_n^{(\alpha)}(\lambda)} \sum_{\nu=0}^{n} \binom{n+\alpha}{\nu+\alpha} L_\nu^{(\alpha)}\left(\frac{\lambda}{x}\right) x^\nu (1-x)^{n-\nu} f\left(\frac{\nu}{n}\right).$$

The right side is a polynomial in x and for $\alpha > -1$, $\lambda \leqslant 0$ the coefficients of $f\left(\frac{\nu}{n}\right)$ are non-negative, so that $B_n^{(\lambda,\alpha)}$ is a sequence of positive operators. For $\lambda = 0$, they reduce to the Bernstein polynomials. We formulate

THEOREM 1. *If* $f \in C[0, 1]$, *then* $B_n^{(\lambda,\alpha)}(f;x)$ *converges uniformly to* $f(x)$ *in* $[0, 1]$.

For the proof of the theorem we shall need the following

Lemma 1. *For a given $\lambda \leqslant 0$, we have*

$$\frac{L_{n-1}^{(\alpha+1)}(\lambda)}{\sqrt{n}\, L_n^{(\alpha)}(\lambda)} = 0(1) \text{ as } n \to \infty \tag{2.2}$$

$$\frac{L_{n-1}^{(\alpha)}(\lambda)}{L_n^{(\alpha)}(\lambda)} = 1 + 0\left(\frac{1}{\sqrt{n}}\right) \text{as } n \to \infty. \tag{2.3}$$

Proof of Lemma 1. We shall first list some of the known properties of Laguerre polynomials which we need (Szegö [10] p. 100–101)

$$n\, L_n^{(\alpha)}(\lambda) = (-\lambda + 2n + \alpha - 1)\, L_{n-1}^{(\alpha)}(\lambda) - (n + \alpha - 1)\, L_{n-2}^{(\alpha)}(\lambda) \tag{2.4}$$

$$\frac{d}{d\lambda} L_n^{(\alpha)}(\lambda) = -L_{n-1}^{(\alpha+1)}(\lambda) = \lambda^{-1}[n\, L_n^{(\alpha)}(\lambda) - (n+\alpha)\, L_{n-1}^{(\alpha)}(\lambda)]. \tag{2.5}$$

$$L_n^{(\alpha)}(\lambda) = \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-\lambda)^\nu}{\nu!} \tag{2.6}$$

If $\lambda_\nu^{(n)}$ $(1 \leqslant \nu \leqslant n)$ are the zeros of $L_n^{(\alpha)}(\lambda)$ and $\{j_\nu\}_0^\infty$ denotes the zeros of $J_\alpha(x)$, then [Szegö [10] p. 126]

$$\frac{(j_\nu/2)^2}{n + \dfrac{\alpha+1}{2}} < \lambda_\nu^{(n)}, \quad \alpha > -1, \tag{2.7}$$

and [Watson [11] p. 506]

$$j_\nu \sim \left(\nu + \frac{\alpha}{2} - \frac{1}{4}\right) \cdot \pi \tag{2.8}$$

In order to prove the lemma, we have from (2.5)

$$\left|\frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)}\right| = \left|\frac{L_n^{(\alpha)\prime}(\lambda)}{L_n^{(\alpha)}(\lambda)}\right| = \sum_{\nu=1}^{n} \frac{1}{|\lambda| + \lambda_\nu^{(n)}}$$

$$\leqslant \frac{\sqrt{n}}{|\lambda|} + \sum_{\nu > \sqrt{n}} \left\{\frac{1}{\lambda_\nu^{(n)}}\right\}$$

$$\leqslant \frac{\sqrt{n}}{|\lambda|} + C_1 n \sum_{\nu > \sqrt{n}} \frac{1}{\nu^2} = 0(\sqrt{n})$$

This proves (2.2). Dividing (2.5) by $n\, L_n^{(\alpha)}(\lambda)$ and using (2.2), we obtain (2.3).

Proof of Theorem 1. Since for $\lambda \leqslant 0$, $B_n^{(\lambda,\alpha)}$ is a positive linear operator on the interval [0, 1], we may apply the well-known theorem of Korovkin [5] and verify the uniform convergence for the test functions 1, t and t^2.

By straightforward computation on using (2.4), (2.5) and (2.6), we can easily derive the following:

$$B_n^{(\lambda,\alpha)}(1;x) \equiv 1. \tag{2.9}$$

$$B_n^{(\lambda,\alpha)}(t;x) = x + \frac{\lambda(x-1)}{n}\,\frac{L_{n-1}^{(\alpha+1)}(\lambda)}{L_n^{(\alpha)}(\lambda)} \tag{2.10}$$

$$(2.11)\left\{\begin{aligned} B_n^{(\lambda,\alpha)}(t^2;x) &= \frac{1}{n^2 L_n^{(\alpha)}(\lambda)}\left[x^2(n+\alpha)(n+\alpha-1)L_{n-2}^{(\alpha)}(\lambda)\right.\\ &\quad \left.+x\{nL_n^{(\alpha)}(\lambda)-2\lambda(n+\alpha)L_{n-2}^{(\alpha+1)}(\lambda)\}+\lambda^2 L_{n-2}^{(\alpha+2)}(\lambda)\right]\end{aligned}\right.$$

From (2.2), $B_n^{(\lambda,\alpha)}(t;x)=x+0\left(\frac{1}{\sqrt{n}}\right)$ uniformly in [0, 1] and using (2.3) and (2.2) we have

$$\begin{aligned} B_n^{(\lambda,\alpha)}(t^2;x) &= x^2\left(1+0\left(\frac{1}{\sqrt{n}}\right)\right)\\ &\quad + x\left(\frac{1}{n}-2\lambda\cdot 0\left(\frac{1}{\sqrt{n}}\right)\right)\\ &\quad + \frac{\lambda^2}{n}0(1) = x^2+0\left(\frac{1}{\sqrt{n}}\right). \end{aligned}$$

uniformly in [0, 1]. This completes the proof of the theorem.

3. *The Operator* S_n^λ (λ real)

Let $H_\nu(u)$ denote the Hermite polynomials of degree ν given by the formula

$$\frac{H_\nu(u)}{\nu!} = \sum_{k=0}^{[\nu/2]} \frac{(-1)^k}{k!}\,\frac{(2u)^{\nu-2k}}{(\nu-2k)!} \tag{3.1}$$

Then it is known that

$$H_\nu(u) = 2u\,H_{\nu-1}(u) - 2(\nu-1)H_{\nu-2}(u) \tag{3.2}$$

From (3.1) we see immediately that for any real λ, $(-1)^\nu H_{2\nu}(i\lambda) \geqslant 0$, and it is easily seen that

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}(i\lambda)}{(2k)!}(nx)^k = e^{nx}\cosh(2\lambda\sqrt{nx}). \tag{3.3}$$

For $\lambda=0$, (3.3) reduces to the identity

$$\sum_{k=0}^{\infty}\frac{(nx)^k}{k!} = e^{nx}.$$

Setting

$$\alpha_{nk}^{\lambda}(x) = e^{-nx}\,\mathrm{sech}\,2\lambda\sqrt{nx}\,.\,\frac{(-1)^k H_{2k}(i\lambda)}{(2k)!}(nx)^k \tag{3.4}$$

we define the operator S_n^λ by

$$S_n^\lambda(f;x) = \sum_{k=0}^{\infty} \alpha_{nk}^\lambda(x) f\left(\frac{k}{n}\right) \tag{3.5}$$

for functions $f(t)$ for which the right side of (3.5) converges. It will follow from the proof of our Theorem 2 that S_n^λ is well defined when $f(t)$ is of exponential growth.

We observe from (3.1) and (3.3) that

$$\alpha_{nk}^\lambda(x) \geqslant 0, \quad n, k = 0, 1, 2, \ldots \tag{3.6}$$

$$S_n^\lambda(1;x) = \sum_{k=0}^{\infty} \alpha_{nk}^\lambda(x) \equiv 1. \tag{3.7}$$

and that for $\lambda = 0$, S_n^λ reduces to the Szász operator [9].

We now formulate the following theorems.

Theorem 2. *Let $f(x)$ be defined on $[0, \infty)$ and let $A > 0$, $\beta \geqslant 0$, be such that*

$$|f(x)| \leqslant A e^{\beta x} \qquad 0 \leqslant x < \infty \tag{3.8}$$

If $f(x)$ is continuous at $x = \xi$, then

$$\lim_{n \to \infty} S_n^\lambda(f;\xi) = f(\xi). \tag{3.9}$$

If further $f(x)$ is continuous in $[a, b]$, $0 \leqslant a < b < \infty$ then the convergence is uniform in $a \leqslant \xi \leqslant b$.

Theorem 3. *Let $f(x)$ be defined on $[0, \infty)$ and satisfy* (3.8). *If for some $\xi > 0$, $f'(\xi)$ exists then*

$$\lim_{n \to \infty} \sqrt{n}\,[S_n^\lambda(f;\xi) - f(\xi)] = \lambda \sqrt{\xi}\, f'(\xi). \tag{3.10}$$

Remark. For $\lambda = 0$, Theorem 3 includes Theorem 5 of Szász [9]. If $f''(\xi)$ exists for some $\xi > 0$, then by a similar analysis one can show that

$$\lim_{n \to \infty} n\left[S_n^\lambda(f;\xi) - f(\xi) - \frac{\lambda\sqrt{\xi}}{\sqrt{n}} f'(\xi)\right] = \frac{\lambda^2+1}{2} \xi \cdot f''(\xi). \tag{3.11}$$

For $\lambda = 0$, this reduces to the statement of Theorem 6 of Szász [9], which is a Voronovskaja-type of result [Lorentz [6] p. 22].

For the proof we shall require the following lemmas.

Lemma 2. *For all real λ, and $n, k = 0, 1, 2, \ldots$, we have*

$$\alpha_{nk}^\lambda(x) \leqslant e^{-nx + 2|\lambda|\sqrt{k}} \frac{(nx)^k}{k!} \tag{3.12}$$

Proof. Since from (3.1),

$$\frac{H_{2k}(i\lambda)}{(2k)!} = \sum_{\nu=0}^{k} \frac{(2\lambda)^{2k-2\nu}}{\nu!(2k-2\nu)!}$$

$$= \sum_{\nu=0}^{k} \frac{(2\lambda)^{2\nu} k^{\nu}}{(2\nu)!\,k!} \leqslant \frac{\cosh 2\lambda\sqrt{k}}{k!} \leqslant \frac{e^{2|\lambda|\sqrt{k}}}{k!}$$

and since sech $2\lambda\sqrt{nx} \leqslant 1$ for all real λ the lemma follows.

Lemma 3. (Hardy [2]) *Suppose* $u>0$, $0<\delta<1$. *Then*

$$(3.13) \qquad e^{-u} \sum_{|m-u|>\delta u} \frac{u^m}{m!} = 0(e^{-\gamma u}),\ \gamma = \frac{1}{3}\delta^2.$$

Lemma 4. *For* λ *real,* $n \geqslant 1$, *we have*

$$(3.14) \qquad S_n^\lambda((t-x;x) = \lambda\sqrt{\frac{x}{n}} \cdot \tanh 2\lambda\sqrt{nx}.$$

$$(3.15) \qquad S_n^\lambda((t-x)^2;x) = \frac{(\lambda^2+1)x}{n} + \frac{\lambda\sqrt{x}}{2n\sqrt{n}} \tanh 2\lambda\sqrt{nx}.$$

On differentiating the identity (3.3) with respect to x we get (3.14) after easy manipulation. Differentiating (3.3) twice, we get (3.15) after straightforward simplification.

Lemma 5. *For* $x \geqslant 0$, *and for any fixed* $\delta > 0$, *we have*

$$(3.16) \qquad \sigma_n(x) \equiv \sum_{|(k/n)-x|>\delta} \alpha_{nk}^\lambda(x)\, e^{\beta k/n} = 0(n^{-\frac{1}{2}})$$

uniformly in $0 \leqslant x \leqslant b < \infty$.

Proof. If $x \geqslant \delta$, let η be a positive number to be specified later. By lemma 2,

$$\sigma_n(x) \leqslant \sum_{|(k/n)-x|>\delta} e^{-nx} \frac{(nx\, e^{\beta/n} \cdot e^{2|\lambda|/\sqrt{k}})^k}{k!} = \sum_{k\leqslant k_1} + \sum_{k_1<k<nx-n\delta} + \sum_{k>nx+n\delta}$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

where k_1 is so chosen that $\exp \dfrac{2|\lambda|}{\sqrt{k_1}} < \sqrt{1+\eta}$.

Let now $\exp \dfrac{\beta}{n_1} < \sqrt{1+\eta}$. Then for $n \geqslant n_1$, we have

$$\sum\nolimits_2 + \sum\nolimits_3 \leqslant e^{-nx} \sum_{|(k/n)-x|>\delta} \frac{\{nx(1+\eta)\}^k}{k!}$$

$$\leqslant e^{nx\eta} \cdot e^{-nx(1+\eta)} \cdot \sum_{|k-nx(1+\eta)|>nx(1+\eta)\delta'} \frac{(nx(1+\eta))^k}{k!}$$

where $\delta' = \dfrac{\delta-\eta x}{x(1+\eta)}$ and for sufficiently small η, $0<\delta'<1$.

Using now Lemma 3, we get

$$\sum_2+\sum_3=e^{nx\eta}\,.\;0\left(\exp\left\{-\frac{1}{3}\delta'^2\,nx(1+\eta)\right\}\right)$$

$$=0\left(\exp\left\{-\frac{1}{3}nx(\delta'^2\,(1+\eta)-3\eta)\right\}\right).$$

For sufficiently small η, $\delta'^2(1+\eta)-3\eta\equiv\eta_1>0$ and thus

$$\sum_2+\sum_3=o\left(\exp\left(-\frac{1}{3}n\delta\eta_1\right)\right)=o(n^{-\frac{1}{2}}).$$

It is obvious from Lemma 3 that $\sum_1=o(n^{-\frac{1}{2}})$ as $n\to\infty$. Thus $\sigma_n(x)=$ $=o(n^{-\frac{1}{2}})$ for $x\geqslant\delta$.

If $0\leqslant x<\delta$, $x=\theta\delta$, $0\leqslant\theta<1$ then for sufficiently large n,

$$\sigma_n(x)=\sum_{k>nx+n\delta}\alpha^{\lambda}_{n;\,k}(x)\,e^{\beta k/n}$$

$$\leqslant e^{-nx}\sum_{k>nx+n\delta}\frac{(nx(1+\eta))^k}{k!}$$

$$<e^{-nx}\frac{(nx(1+\eta))^{M+1}}{M!}\;\frac{1}{M-nx(1+\eta)}\,,\quad M=[nx+n\delta]$$

$$\leqslant\frac{C}{\sqrt{M}}\,e^{-n\delta\theta}\left(\frac{en\delta\theta(1+\eta)}{M}\right)^M\frac{1}{n\delta(1-\theta\eta)}$$

since $M!\sim\sqrt{2\pi M}\,(Me^{-1})^M$.

Thus

$$\sigma_n(x)\leqslant\frac{C_1}{n\sqrt{n}}\cdot e^{n\delta}\left(\frac{\theta(1+\eta)}{1+\theta}\right)^{n\delta(1+\theta)}.$$

It is easy to see that for $0\leqslant\theta\leqslant1$,

$$\left(\frac{\theta}{1+\theta}\right)^{1+\theta}\leqslant\frac{1}{4}$$

so for sufficiently small η we have

$$\sigma_n(x)\leqslant\frac{C_1}{n^{3/2}}\left(\frac{e}{3}\right)^{n\delta}=o(n^{-\frac{1}{2}})\text{ as }n\to\infty.$$

For $0\leqslant x\leqslant b<\infty$, the estimates used above hold uniformly so that (3.12) holds uniformly. This completes the proof of lemma 4.

4. *Proof of Theorem* 2

Since $S^{\lambda}_n(f\,;0)=f(0)$ for all n, we may assume $\xi>0$.

Given $\varepsilon>0$, let $0<\delta<\xi$ be such that $|f(x)-f(\xi)|<\varepsilon$ for $|x-\xi|\leqslant\delta$. Then

$$|S_n(f;\xi)-f(\xi)| \leqslant \sum_{k=0}^{\infty} \alpha^{\lambda}_{n,k}(\xi)\left|f\left(\frac{k}{n}\right)-f(\xi)\right|$$

$$\leqslant \varepsilon \sum_{|(k/n)-\xi|\leqslant\delta} \alpha^{\lambda}_{n,k}(\xi)$$

$$+ \{A+|f(\xi)|\} \sum_{|(k/n)-\xi|>\delta} e^{\beta k/n}\, \alpha^{\lambda}_{nk}(\xi) \ .$$

which by (3.6), and (3.16) is less than 2ε for large n.

The uniform convergence on a compact subset of the real half line is a consequence of uniform continuity and the remark in lemma 4.

Proof of Theorem 3. It follows from the hypothesis that

$$f(t)-f(\xi)=(t-\xi)\,f'(\xi)+(t-\xi)\,\phi\,(t) \tag{4.1}$$

where $\phi(t)\to 0$ as $t\to\xi$. Given $\varepsilon>o$, let $0<\delta<\xi$ be such that $|\phi(t)|<\varepsilon$ for $|t-\xi|<\delta$. We obviously have for all t,

$$|(t-\xi)\,\phi(t)|<A_1 e^{\beta t} \quad \text{for some } A_1>0.$$

Applying the operator S^{λ}_n to (4.1) we have

$$(4.2)\quad \left\{\begin{aligned} & S^{\lambda}_n(f;\xi)-f(\xi)-f'(\xi)\ S^{\lambda}_n(t-\xi;\xi)=S^{\lambda}_n((t-\xi)\,\phi(t);\xi)\ \cdot \\ & \quad = \sum_{|(k/n)-\xi|\leqslant\delta} + \sum_{|(k/n)-\xi|>\delta} \alpha^{\lambda}_{nk}(\xi)\left(\frac{k}{n}-\xi\right)\phi\left(\frac{k}{n}\right)\cdot \\ & \quad \equiv \sigma_1+\sigma_2 \end{aligned}\right.$$

Now

$$\sigma_1 < \varepsilon \sum_{|(k/n)-\xi|\leqslant\delta} \alpha^{\lambda}_{nk}(\xi)\left|\frac{k}{n}-\xi\right|$$

$$\leqslant \varepsilon\left\{\sum_{k=0}^{\infty} \alpha^{\lambda}_{nk}(\xi)\left(\frac{k}{n}-\xi\right)^2\right\}^{\frac{1}{2}}$$

$$\leqslant \varepsilon 0(n^{-\frac{1}{2}}) \quad \text{from (3.15)}\ .$$

Also using (4.1) and lemma 5, we see that

$$\sigma_2 \leqslant A_1 \sum_{|(k/n)-\xi|>\delta} e^{\beta k/n}\,\alpha^{\lambda}_{nk}(\xi)=o(n^{-\frac{1}{2}}) \tag{4.4}$$

From (4.2), (4.3), (4.4) and from (3.14), the theorem follows immediately.

5. *Related Summability Method*

Just as the Euler summation method is closely connected to the Bernstein polynomials, so the operator $B^{(\lambda,\alpha)}_n(f;x)$ suggests a corresponding summability method. For $0<r\leqslant 1$, $\lambda\leqslant 0$, $\alpha>-1$, let the triangular matrix

(b_{nk}) be defined by

$$(5.1) \qquad b_{nk} \equiv b_{nk}(\lambda, \alpha, r) = \frac{\binom{n+\alpha}{k+\alpha}}{L_n^{(\alpha)}(\lambda)} L_k^{(\alpha)}\left(\frac{\lambda}{r}\right) r^k (1-r)^{n-k}, \quad 0 \leqslant k \leqslant n$$

$$= 0, \ k > n.$$

We shall say that a sequence $\{S_k\}$ is $E(\lambda, \alpha, r)$ – summable to σ if

$$\lim_{n \to \infty} \sum_{k=0}^{n} b_{nk} S_k = \sigma$$

In particular $E(0, \alpha, r) = E(r)$ is the Euler summation method.

In order to prove that $E(\lambda, \alpha, r)$ methods are regular, it is enough to show on using the Silverman-Toeplitz theorem, that $b_{nk} = o(1)$ for fixed k as $n \to \infty$. Since from (2.6), λ being $\leqslant 0, L_n^{(\alpha)}(\lambda) \geqslant \binom{n+\alpha}{\alpha}$ we have from (5.1)

$$b_{nk} = 0\left(\binom{n+\alpha}{k+\alpha}\binom{n+\alpha}{\alpha}^{-1} \cdot (1-r)^n\right)$$

$$= 0\,(n^k\,(1-r)^n) = o(1) \ \text{ as } \ n \to \infty.$$

This proves the regularity of $E(\lambda, \alpha, r)$ methods.

We formulate the following

Theorem 4. *The $E(\lambda, \alpha, r)$ methods sum the geometric series*

$$1 + z + z^2 + \dots \ \text{to} \ \frac{1}{1-z} \ \text{in the disc} \ \left|\frac{1}{r} - 1 + z\right| < \frac{1}{r}.$$

Proof. Set $t_n(z) = \sum_{k=0}^{n} b_{nk}(1 + z + z^2 + \dots + z^k) = \frac{1}{1-z} - z \sum_{k=0}^{n} b_{nk}\, z^k.$

By an easy computation, one verifies that

$$t_n(z) = (1-z)^{-1} - z(1-r+rz)^n \frac{L_n^{(\alpha)}\left(\dfrac{\lambda z}{1-r+rz}\right)}{L_n^{(\alpha)}(\lambda)}$$

Since $L_n^{(\alpha)}(\lambda) = \frac{1}{n!} \prod_{\nu=1}^{n} (\lambda - \lambda_\nu^{(n)})$, we obtain

$$t_n(z) = (1-z)^{-1} - z\pi_n(z)$$

where

$$\pi_n(z) = \prod_{\nu=1}^{n} \left\{ 1 - (1-z) \frac{r \cdot \lambda_\nu^{(n)} + |\lambda|}{\lambda_\nu^{(n)} + |\lambda|} \right\} \equiv \prod_{\nu=1}^{n} \phi_\nu^{(n)}(z)$$

Now if $|1-(1-z)r| < 1$ then for small enough $\varepsilon > 0$, and any $|\varepsilon_\nu| < \varepsilon$ we have

$$|1 - (1-z)(r + \varepsilon_\nu)| < 1 - \varepsilon$$

Putting

$$\varepsilon_\nu = \frac{r\lambda_\nu^{(n)} + |\lambda|}{\lambda_\nu^{(n)} + |\lambda|} - r = \frac{|\lambda|(1-r)}{\lambda_\nu^{(n)} + |\lambda|},$$

we see from (2.7) and (2.8) that for $\nu \geqslant n^{3/4}$ and n large enough ε_ν can be made as small as we wish.

Also, for all ν, $|\phi_\nu^{(m)}(z)| \leqslant 1 + |1-z|$. Thus

$$|\pi_n(z)| \leqslant (1+|1-z|)^{n^{3/4}} \cdot (1-\varepsilon)^{n-n^{3/4}} = (1-\varepsilon)^n \left(\frac{1+|1-z|}{1-\varepsilon}\right)^{n^{3/4}}$$

$$= o(1) \text{ as } n \to \infty.$$

Therefore $\lim\limits_{n\to\infty} t_n(z) = \dfrac{1}{1-z}$ for $|1-(1-z)r| < 1$.

Similarly a summability method analogous to the Borel transformation can be introduced on the basis of (3.4). It is then easy to verify that this yields a regular transformation which sums the geometric series to $(1-z)^{-1}$ in the complex half plane $\operatorname{Re} z < 1$.

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