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Order-Preserving Representations of the Partitions on the Finite Set

TONY T. LEE*

Bell Laboratories,
Holmdel, New Jersey 07733

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A partial ordering is defined for monotone projections $f: N \to N$, $N = \{1, 2,..., n\}$, such that the class of these mappings is a lattice which is isomorphic to the partition lattice. Thus a partition can be uniquely represented by an element of this class of mappings and the partial ordering of partitions is preserved. Algorithms for computing the join and meet of given partitions are presented.

1. INTRODUCTION

A partition π on a finite set $N = \{1, 2,..., n\}$ is a collection of mutually disjoint nonempty subsets of N whose union is N. The members of π are called "blocks" or "equivalence classes." If the elements $a, b \in N$ are in the same block of π, this is indicated by $a = b(\pi)$. The set of all partitions on N is denoted by $\Pi(N)$. It is well known that $\Pi(N)$ is a lattice with the partial ordering $\leqslant$, such that for $\forall a, b \in N$, if $a \equiv b(\pi)$ implies $a \equiv b(\tau)$ then $\pi \leqslant \tau$ [3]. This ordering is by "refinement," so that blocks of π are obtained by further partitioning of blocks in τ.

A partition $\pi \in \Pi(N)$ can be represented by a mapping $f: N \to N$ in the sense that $a \equiv b(\pi)$ if and only if $f(a) = f(b)$. Certainly, there are many different mappings that can represent the same partition, that is, a representation is by no means unique. Hutchinson [2] has given a set of rules for representing the partitions on a finite set of n elements by an n-tuple integer array, which can be considered as a mapping from N into itself, and an algorithm for generating these partitions. A loopless algorithm for generating the partitions is also developed by Ehrlich [1]. We define an alternate set of rules for the representation of partitions which preserves the partial ordering.

* This work was done while the author was at the Polytechnic Institute of New York, Brooklyn, New York.

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That is, we define a class of mappings $f: N \to N$ which have a partial ordering that is isomorphic to the partition lattice. Also, the representations by these mappings are unique.

For an arbitrary mapping $f: N \to N$, we can always find a partition defined by

$$\pi = \{\bar{N}_k \mid \bar{N}_k = f^{-1}(i), i \in \text{Range } f\}.$$

This partition induced by f will be denoted by N/f. In other words, if two elements i and j yield the same values of f, they are in the same class of N/f, otherwise they are in different classes of N/f. As we discussed above we may have $N/f \equiv N/g$ for $f \neq g$. Thus in order to represent the partitions uniquely, additional criteria have to be imposed on the mappings. These criteria are given in the next section. Algorithms for calculating the join and meet of given partitions are presented in Section 4.

2. Preliminary Results

Let $F(N)$ be the collection of all mappings from N into itself which satisfy the following criteria:

$$\textit{Contraction}: f(i) \leqslant i, \qquad \forall i \in N, \tag{1}$$

$$\textit{Idempotent}: f^2(i) = f(i), \qquad \forall i \in N. \tag{2}$$

Definition 1. For $f_1, f_2 \in F(N)$, the binary relation $f_1 \leqslant f_2$ is defined by

$$f_1(i) = f_1(j) \text{ implies } f_2(i) = f_2(j). \qquad \forall i, j \in N. \tag{3}$$

Theorem 1.

$$\text{For } f_1, f_2 \in F(N),\ f_1 \leqslant f_2 \text{ iff } f_2 f_1 = f_2. \tag{4}$$

Proof. First we show $f_1 \leqslant f_2$ implies $f_2 f_1 = f_2$. Since f_1 satisfies the idempotent criterion (2), $f_1(f_1(i)) = f_1(i)$, $\forall i \in N$. It follows from (3) that $f_2(f_1(i)) = f_2(i)$, $\forall i \in N$. Therefore, $f_2 f_1 = f_2$. Next we prove $f_2 f_1 = f_2$ implies $f_1 \leqslant f_2$. Suppose $f_1(i) = f_1(j)$, then $f_2(i) = f_2(f_1(i)) = f_2(f_1(j)) = f_2(j)$. From (3) we know $f_1 \leqslant f_2$.

Corollary 1.

$$\textit{For } f_1, f_2 \in F(N), f_1 \leqslant f_2 \textit{ implies } f_2(i) \leqslant f_1(i), \qquad \forall i \in N. \tag{5}$$

Proof. From (4) we know that $f_1 \leqslant f_2$ implies $f_2(f_1(i)) = f_2(i)$, $\forall i \in N$. From the contraction criterion (1) we have $f_2(f_1(i)) \leqslant f_i(i)$, $\forall i \in N$; therefore $f_2(i) \leqslant f_1(i)$, $\forall i \in N$ and the corollary is proved.

A binary relation $\leqslant$ on a set X is called a "partial order" of X when it is reflexive, transitive, and antisymmetric. A set X with a partial order $\leqslant$ is called a "poset" $\{X, \leqslant\}$ [3]. The binary relation $\leqslant$ on the set $F(N)$ defined by (3) is obviously reflexive and transitive. From (5) of Corollary 1, we can verify tat the binary relation $\leqslant$ on $F(N)$ is antisymmetric. (It is a consequence of the fact that the binary relation "less than or equal to," $\leqslant$, on the set of natural numbers is antisymmetric.) Thus the set $F(N)$ with the partial ordering $\leqslant$ is a poset.

DEFINITION 2. A "join" (or, a "least upper bound") of $f_1, f_2 \in F(N)$ is denoted by $f_1 \vee f_2$, and has the property that for $f = f_1 \vee f_2$, $f_1 \leqslant f$, $f_2 \leqslant f$ and for any $g \in F(N)$, $f_1 \leqslant g$, $f_2 \leqslant g$ implies $f \leqslant g$.

DEFINITION 3. A "meet" (or, a "greatest lower bound") of $f_1, f_2 \in F(N)$ is denoted by $f_1 \wedge f_2$, and has the property that for $f = f_1 \wedge f_2$, $f \leqslant f_1$, $f \leqslant f_2$ and for any $g \in F(N)$, $g \leqslant f_1$, $g \leqslant f_2$ implies $g \leqslant f$.

In what follows we show how to construct a join and a meet for any f_1, $f_2 \in F(N)$. Thus the poset $\{F(N), \leqslant\}$ is a lattice $\{F(N), \vee, \wedge\}$. Furthermore, we show it is isomorphic to the partition lattice. In order to make our constructions, we need the following definition.

DEFINITION 4. For f_1, $f_2 \in F(N)$, let h_0 be the identity mapping, $h_0(i) = i$, $\forall i \in N$. For $j = 1, 2, \ldots, n$, define $h_{j,1} = h_{j-1}$, and for $k > 1$, define

$$h_{j,k}(i) = \begin{cases} h_{j,k-1}(j) & \text{if } h_{j,k-1}(i) > j \text{ and } h_{j,k-1}(j) \text{ satisfies either condition } (a) \text{ or } (b), \\ h_{j,k-1}(i) & \text{otherwise,} \end{cases} \tag{6}$$

for $i = 1, 2, \ldots, n$, where conditions (a) and (b) are:

$$\text{(a)} \quad h_{j,k-1}(f_1(i)) = h_{j,k-1}(j) \text{ or } h_{j,k-1}(f_2(i)) = h_{j,k-1}(j). \tag{7}$$

$$\text{(b)} \quad h_{j,k-1}(m) = h_{j,k-1}(j) \text{ and either } f_1(m) = i \text{ or } f_2(m) = i, \text{ for some } m \geqslant i. \tag{8}$$

If k_j is the smallest integer such that $h_{j,k_j} = h_{j,k_j+1}$ then define $h_j = h_{j,k_j}$, $j = 1, 2, \ldots, n$.

THEOREM 2. *The mappings $h_{j,k}$ have the following properties:*

(i) $h_{j,1}(j) = h_{j,2}(j) = \cdots = h_{j,k_j}(j)$, *for* $j = 1, 2, \ldots, n$. (9)

(ii) *If* $h_{j,k-1}(i) > j$, *then* $h_{j,k-1}(i) = \cdots = h_{j,1}(i) = h_{j-1,k_{j-1}}(i) = \cdots = h_{1,1}(i) = i$, *for* $j = 1, 2, \ldots, n$, *and* $k > 1$. (10)

(iii) $h_{j,k} \in F(N)$, *for* $j = 1, 2, \ldots, n$, *and* $k = 1, \ldots, k_j$.

(iv) $h_{j,k}$ *form an ascending chain such that*

$$\begin{aligned} h_0 = h_{1,1} \leqslant \cdots \leqslant h_{1,k_1} = h_1 = h_{2,1} \leqslant \cdots \leqslant h_{j,k_j} \\ = h_j = h_{j+1,1} \leqslant \cdots \leqslant h_{n,k_n} = h_n. \end{aligned} \tag{11}$$

Proof. (i) Since $h_{j,1}(j) \leqslant j$, it follows from (6) that $h_{j,2}(j) = h_{j,1}(j)$. Similarly, $h_{j,k}(j) = h_{j,k-1}(j)$, for $k > 2$.

(ii) We prove (10) by contradiction. Suppose $h_{j,k-1}(i) = \cdots = h_{j,1}(i) = h_{j-1,k_{j-1}}(i) = \cdots = h_{l,p}(i) \neq h_{l,p-1}(i)$. According to (6), we must have $h_{l,p}(i) = h_{l,p-1}(l) \leqslant 1 \leqslant j$. It follows that $h_{j,k-1}(i) = h_{l,p}(i) \leqslant j$. But $h_{j,k-1}(i) > j$ is given and therefore (10) must be true.

(iii) First, we prove that $h_{j,k}$ satisfies the contraction criterion by mathematical induction. We have $h_{1,1}(i) = h_0(i) = i \leqslant i$, $\forall i \in N$. If $h_{j,k-1}(i) \leqslant i$, $\forall i \in N$, then from (7) we know either $h_{j,k}(i) = h_{j,k-1}(j) \leqslant j < h_{j,k-1}(i) \leqslant i$, or $h_{j,k}(i) = h_{j,k-1}(i) \leqslant i$. Therefore $h_{j,k}(i) \leqslant i$, $\forall i \in N$ and $h_{j,k}$ satisfies the contraction criterion (1).

Next, we prove $h_{j,k}$ satisfies the idempotent criterion. Suppose $h_{j,k-1}^2 = h_{j,k-1}$. We show $h_{j,k}^2 = h_{j,k}$ by considering the two possibilities:

Case 1. $h_{j,k}(i) = h_{j,k-1}(j)$. Since $h_{j,k-1}(j) \leqslant j$, then $h_{j,k-1}(h_{j,k-1}(j)) \leqslant h_{j,k-1}(j) \leqslant j$. It follows from (6) that $h_{j,k}(h_{j,k}(i)) = h_{j,k}(h_{j,k-1}(j)) = h_{j,k-1}(h_{j,k-1}(j)) = h_{j,k-1}(j) = h_{j,k}(i)$.

Case 2. $h_{j,k}(i) = h_{j,k-1}(i)$. If $h_{j,k-1}(i) \leqslant j$, then $h_{j,k-1}(h_{j,k-1}(i)) \leqslant h_{j,k-1}(i) \leqslant j$. It follows from (6) that $h_{j,k}(h_{j,k}(i)) = h_{j,k}(h_{j,k-1}(i)) = h_{j,k-1}(h_{j,k-1}(i)) = h_{j,k-1}(i) = h_{j,k}(i)$. If $h_{j,k-1}(i) > j$, it follows from (10) that $h_{j,k-1}(i) = i$. Consequently, $h_{j,k}(h_{j,k}(i)) = h_{j,k}(h_{j,k-1}(i)) = h_{j,k}(i)$. Now we have $h_{j,k}(h_{j,k}(i)) = h_{j,k}(i)$, $\forall i \in N$. Therefore $h_{j,k}$ satisfies the idempotent criterion (2).

(iv) From Theorem 1, we have to prove that $h_{j,k}(h_{j,k-1}(i)) = h_{j,k}(i)$, $\forall i \in N$. There are two possibilities:

Case 1. $h_{j,k-1}(i) \leqslant j$. In this case, we have $h_{j,k-1}(h_{j,k-1}(i)) \leqslant h_{j,k-1}(i) \leqslant j$. It follows from (6) that $h_{j,k}(i) = h_{j,k-1}(i)$, and $h_{j,k}(h_{j,k-1}(i)) = h_{j,k-1}(h_{j,k-1}(i)) = h_{j,k-1}(i)$. Hence $h_{j,k}(h_{j,k-1}(i)) = h_{j,k}(i)$.

Case 2. $h_{j,k-1}(i) > j$. In this case, from (10) we have $h_{j,k-1}(i) = i$. Hence $h_{j,k}(h_{j,k-1}(i)) = h_{j,k}(i)$. Thus we have proven that $h_{j,k} \leqslant h_{j,k+1}$ and the mappings $h_{j,k}$ form an ascending chain.

It should be noted that part (iv) of Theorem 2 implies there always exists a finite k_j such that $h_{j,k_j} = h_{j,k_j+1}$.

THEOREM 3. *The mapping h_n is the join of f_1 and f_2.*

Proof. There are two parts to be proved. First, we show $f_1 \leqslant h_n$ and $f_2 \leqslant h_n$. Next, we prove that if $g \in F(N)$, $f_1 \leqslant g$ and $f_2 \leqslant g$ imply $h_n \leqslant g$.

(i) We first prove $f_1 \leqslant h_n$ and $f_2 \leqslant h_n$. Let $f_1(i) = j$, then $h_{j,1}(f_1(i)) = h_{j,1}(j)$. If $h_{j,1}(i) > j$, then since $h_{j,1}(j)$ satisfies condition (a) (7), we have $h_{j,2}(i) = h_{j,1}(j)$. From (11) we know $h_{j,2} \leqslant h_n$, $h_{j,1} \leqslant h_n$. It follows that $h_n(i) = h_n(h_{j,2}(i)) = h_n(h_{j,1}(j)) = h_n(j) = h_n(f_1(i))$. On the other hand, if $h_{j,1}(i) \leqslant j$, there are two possibilities:

Case 1. $h_{j,1}(i) = h_{j-1,k_{j-1}}(i) = \cdots = h_{1,1}(i) = i$. In this case, $h_{j,1}(i) = i \leqslant j$, but given $f_1(i) = j \leqslant i$. It follows that $i = j = f_1(i)$ and $h_n(f_1(i)) = h_n(i)$.

Case 2.

$$h_{j,1}(i) = h_{j-1,k_{j-1}}(i) = \cdots = h_{l,k}(i) \begin{cases} = h_{l,k-1}(l). \\ \neq h_{l,k-1}(i). \end{cases}$$

In this case, if $h_{l,k}(j) > l$, we know from (9) that $h_{l,k}(i) = h_{l,k-1}(l) = h_{l,k}(l)$. Since $f_1(i) = j \leqslant i$, therefore $h_{l,k}(l)$ satisfies (8), condition (b). It follows from (6) that $h_{l,k+1}(j) = h_{l,k}(l) = h_{l,k-1}(l) = h_{l,k}(i)$. From (11) we have $h_n(j) = h_n(h_{l,k+1}(j)) = h_n(h_{l,k}(i)) = h_n(i)$. But $h_n(f_1(i)) = h_n(j)$, therefore $h_n(f_1(i)) = h_n(i)$. On the other hand, for $h_{l,k}(j) \leqslant 1$, since $h_{l,k}(j) = \cdots = h_{l,1}(j) = h_{l-1,k_{l-1}}(j) = \cdots = h_{1,1}(j) = j \leqslant l$ is impossible, therefore it must be

$$h_{l,k}(j) = \cdots = h_{l,1}(j) = \cdots = h_{m,r}(j) \begin{cases} = h_{m,r-1}(m). \\ \neq h_{m,r-1}(j). \end{cases}$$

From (6), we know that $h_{l,k-1}(i) > l$. It follows from (10) that $h_{l,k-1}(i) = \cdots = h_{l,1}(i) = \cdots = h_{m,r}(i) = h_{m,r-1}(i) = \cdots = h_{l,1}(i) = i$. Since $f_1(i) = j$ is given we have $h_{m,r}(f_1(i)) = h_{m,r}(j) = h_{m,r-1}(m) = h_{m,r}(m)$, which implies $h_{m,r}(m)$ satisfies (7), the condition (*a*), and we know that $h_{m,r}(i) = i \geqslant f_1(i) = j > l \geqslant m$. From (6), we obtain $i = h_{m,r+1}(i) = h_{m,r}(m) = h_{m,r}(j) = h_{m,r}(f_1(i))$. Therefore $h_n(i) = h_n(h_{m,r+1}(i)) = h_n(h_{m,r}(f_1(i)) = h_n(f_1(i))$. From the above discussion we have $h_n(f_1(i)) = h_n(i)$, $\forall i \in N$ and $f_1 \leqslant h_n$. Similarly we have $f_2 \leqslant h_n$. This completes the first part of the proof.

(ii) We next prove that if $g \in F(N)$, $f_1 \leqslant g$ and $f_2 \leqslant g$ imply $h_n \leqslant g$. From Theorem 1, we want to show $gf_1 = g$ and $gf_2 = g$ imply $gh_n = g$. From (11), we know that it is sufficient to prove that $gf_1 = g$ and $gf_2 = g$ imply $gh_{j,k} = g$, for $j = 1, 2,..., n$, $k = 1,..., k_j$. Since $h_{1,1} = h_0$ is the identity mapping, we have $gh_{1,1} = g$. Suppose $gh_{j,k-1} = g$; we show $gh_{j,k} = g$ by considering two possibilities:

Case 1. $h_{j,k}(i) = h_{j,k-1}(j)$ and $h_{j,k-1}(j)$ satisfies either condition (a) or condition (b). In this case, it follows that $g(h_{j,k}(i)) = g(h_{j,k-1}(j)) = g(j)$. If $h_{j,k-1}(j)$ satisfies condition (a), then from (7) we have $h_{j,k-1}(f_s(i)) = h_{j,k-1}(j)$, where f_s is either f_1 or f_2. It follows from the assumptions that $g(h_{j,k-1}(f_s(i))) = g(f_s(i)) = g(i)$ and $g(h_{j,k-1}(j)) = g(j)$. Thus, $g(i) = g(j)$. Similarly, it can be shown that this is also true if $h_{j,k-1}(j)$ satisfies condition (b). Thus, we have $g(h_{j,k}(i)) = g(i)$ for Case 1.

Case 2. $h_{j,k}(i) = h_{j,k-1}(i)$. In this case, we have $g(h_{j,k}(i)) = g(h_{j,k-1}(i)) = g(i)$. Hence, $g(h_{j,k}(i)) = g(i)$, $\forall i \in N$, and $gh_{j,k} = g$. In particular, $gh_n = g$. Therefore h_n is the join of f_1 and f_2.

THEOREM 4. *For $f_1, f_2 \in F(N)$, let $f(1) = 1$ and for $i \geqslant 2$,*

$$f(i) = f(k) \qquad \text{if } f_1(i) = f_1(k) \text{ and } f_2(i) = f_2(k), \text{ for some } k = 1, 2,..., i-1. \tag{12}$$

$$= i \qquad \text{otherwise,}$$

then $f \in F(N)$ is the meet of f_1 and f_2.

Proof. We begin by proving $f \in F(N)$.

(i) First we show that f satisfies the contraction criterion (1). The proof is by induction. Given $f(1) = 1 \leqslant 1$, suppose $f(i) \leqslant i$, for $i = 1, 2,..., m < n$. From (7), the definition of f, we know that either $f(m+1) = m+1$, or $f(m+1) = f(k) \leqslant k$ for some $k = 1, 2,..., m$. It follows that $f(m+1) \leqslant m+1$. By induction $f(i) \leqslant i, \forall i \in N$. Thus f satisfies the contraction criterion (1).

(ii) Next we show that f satisfies idempotent criterion (2). The proof is also by induction. Given $f(f(1)) = f(1)$, suppose $f(f(i)) = i$, for $i = 1, 2,..., m < n$. From (7), we know that either $f(m+1) = m+1$ implies $f(f(m+1)) = f(m+1)$, or $f(m+1) = f(k)$ for some $k = 1, 2,..., m$. From the assumption, $f(f(m+1)) = f(f(k)) = f(k) = f(m+1)$. Therefore $f(f(i)) = f(i)$, $\forall i \in N$ by induction. Consequently f satisfies the idempotent criterion (2) and $f \in F(N)$.

(iii) Now we prove f is the meet of f_1 and f_2. First we show $f \leqslant f_1$ and $f \leqslant f_2$. The proof is by induction. Given $f_1(f(1)) = f_1(1)$, suppose

$f_1(f(i)) = f_1(i)$, for $i = 1, 2,..., m < n$. From (12), we know that either $f(m+1) = m+1$ implies $f_1(f(m+1)) = f_1(m+1)$, or $f(m+1) = f(k)$ for some $k = 1, 2,..., m$, such that $f_1(k) = f_1(m+1)$ and $f_2(k) = f_2(m+1)$, but then $f_1(f(m+1)) = f_1((f(k)) = f_1(k) = f_1(m+1)$. It follows that $f_1(f(i)) = f_1(i)$, $\forall i \in N$ by induction. Then we have $f \leqslant f_1$ from (4). Similarly $f \leqslant f_2$. Thus f is a lower bound of f_1 and f_2.

(iv) Next we show $f = f_1 \wedge f_2$. For $g \in F(N)$, let $g \leqslant f_1$ and $g \leqslant f_2$ which imply $f_1 g = f_1$ and $f_2 g = f_2$ due to (4). We want to show $g \leqslant f$, that is $fg = f$. The proof is again by induction. Given $f(g(1)) = f(1)$, suppose $f(g(i)) = f(i)$, for $i = 1, 2,..., m < n$. Then either $g(m+1) = m+1$ implies $f(g(m+1)) = f(m+1)$, or $g(m+1) < m+1$. Since we know $f_1(g(m+1)) = f_1(m+1)$ and $f_2(g(m+1)) = f_2(m+1)$, it follows from (7) directly that $f(m+1) = f(g(m+1))$. By induction $f(g(i)) = f(i)$, $\forall i \in N$. Therefore, $g \leqslant f$ and $f = f_1 \wedge f_2$.

The above two theorems also state that $\{F(N), \vee, \wedge\}$ is a lattice. An example of how to calculate the join and meet for f_1, $f_2 \in F(N)$ is given in the next section. The algorithms are discussed in Section 4.

Theorems 3 and 4 have given the join and meet of any two mappings f_1, $f_2 \in F(N)$. Neverheless, if more than two mappings $f_1,..., f_s \in F(N)$, $s \geqslant 2$, are given, these theorems can be extended with appropriate modifications to find their join and meet as well.

3. Order-Preserving Representations

In this section we show that every partition in $\Pi(N)$ can be uniquely represented by a mapping in $F(N)$. Furthermore, the lattices $F(N)$ and $\Pi(N)$ are shown to be isomorphic, which is to say the partial ordering in $\Pi(N)$ is preserved in $F(N)$.

Definition 5. For a given $\pi = \{\bar{N}_1, \bar{N}_2,..., \bar{N}_m\} \in \Pi(N)$, the mapping $f: N \to N$ defined by

$$f(i) = \min \bar{N}_k, \qquad \text{for} \quad i \in \bar{N}_k, i = 1, 2,..., n, \tag{13}$$

is called the "representation" of π.

Clearly, not every mapping $f: N \to N$ can be a legitimate partition representation according to the rule given in (13). The following theorem shows that only mappings in $F(N)$ can represent the partitions on N.

Theorem 5. *The necessary and sufficient conditions for $f: N \to N$ to be a partition representation is $f \in F(N)$.*

We want to prove the following:

(i) *For a given $f: N \to N$ which satisfies the contraction and idempotent criteria, let $\pi = N/f = \{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_m\}$ then* $f(i) = \min \bar{N}_k$, for $i \in \bar{N}_k$ and $\forall i \in N$.

(ii) *For a given partition $\pi \in \Pi(N)$, let f be the partition representation of π given by* (13). *Then f satisfies the contraction and idempotent criteria.*

Proof. (i) Let $\bar{N}_k = \{k_1, k_2, \ldots, k_j\}$, then $f(k_1) = f(k_2) = \cdots = f(k_j) = m_k$. From (1), we have $m_k \leqslant \min\{k_1, k_2, \ldots, k_j\} = \min \bar{N}_k$. From (2), we have $f(m_k) = f^2(k_1) = f^2(k_2) = \cdots = f^2(k_j) = f(k_1) = f(k_2) = \cdots = f(k_j) = m_k$. It follows that $m_k \in \bar{N}_k$ and $m_k = \min \bar{N}_k$. Thus we have $f(k_1) = f(k_2) = \cdots = f(k_j) = m_k = \min \bar{N}_k$.

(ii) For given $\pi = \{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_m\} \in \Pi(N)$, let f be defined by (13), then $f(i) = \min \bar{N}_k$ for $i \in \bar{N}_k$. It follows that $f(i) = \min \bar{N}_k \leqslant i$. Thus f satisfies the contraction criterion (1). Also, $f(f(i)) = f(\min \bar{N}_k) = \min \bar{N}_k = f(i)$, for $i \in \bar{N}_k$. We know that f satisfies the idempotent criterion (2). Consequently, $f \in F(N)$.

A morphism Φ of two posets $\{L, \leqslant\}$ and $\{M, \leqslant\}$ is a function $\Phi: L \to M$ such that

$$a \leqslant b \text{ implies } \Phi(a) \leqslant \Phi(b), \forall a, b \in L. \tag{14}$$

An isomorphism Φ of two posets L and M is a bijection $\Phi: L \to M$ such that both Φ and Φ^{-1} are morphisms of posets. A morphism of lattices is a function $\Phi: L \to M$ on a lattice L to a lattice M such that $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$ and $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$ for all $a, b \in L$. An isomorphism $\Phi : L \to M$ of lattices is a bijection which is also a morphism of lattices; its inverse is then automatically also a morphism of lattices. It should be noted that an isomorphism of posets which are lattices necessarily preserves join and meet, hence it is an isomorphism of lattices [3]. We prove an important theorem:

THEOREM 6. *The lattices $\{F(N), \vee, \wedge\}$ and $\{\Pi(N), \vee, \wedge\}$ are isomorphic.*

Proof. Let $\Phi: F(N) \to \Pi(N)$ be defined by $\Phi(f) = N/f$, $\forall f \in F(N)$. We must show:

(i) Φ is bijection, (ii) Φ satisfies (14).

(i) For given $f, g \in F(N)$, and $f \neq g$, suppose $\Phi(f) = N/f = \pi$, $\Phi(g) = N/g = \tau$, and $\pi \equiv \tau = \{\bar{N}_1, \ldots, \bar{N}_m\}$. We want to prove this is a contradiction. From Theorem 5, we know that $f(i) = \min \bar{N}_k$, $g(i) = \min \bar{N}_k$ for $i \in \bar{N}_k$; then $f(i) = g(i)$, $\forall i \in N$. But this is impossible for given $f \neq g$. It follows that $\Phi(f) \neq \Phi(g)$ and thus Φ is one-to-one.

For $\pi = \{\bar{N}_1, ..., \bar{N}_m\} \in \Pi(N)$, let f be given by (13). Then $f(i) = \min \bar{N}_k$ for $i \in \bar{N}_k$. It follows that $f(i) = \min \bar{N}_k \leqslant i$ and f satisfies (1). Also, $f^2(i) = f(\min \bar{N}_k) = \min \bar{N}_k = f(i)$ for $\min \bar{N}_k \in \bar{N}_k$. Therefore f satisfies (2). Hence $f \in F(N)$ and Φ is onto.

(ii) Now we show that Φ preserves the partial ordering. Let $f_1, f_2 \in F(N)$ and $f_1 \leqslant f_2$. Suppose $\Phi(f_1) = \pi$ and $\Phi(f_2) = \tau$. Then $i \equiv j(\pi)$ implies $f_1(i) = f_1(j)$. It follows that $f_2(f_1(i)) = f_2(f_1(j))$. Since $f_1 \leqslant f_2$ by assumption, from (4) we have $f_2(i) = f_2(j)$. Hence $i \equiv j(\tau)$. Consequently, $\Phi(f_1) \leqslant \Phi(f_2)$ and Φ is a morphism of $F(N)$ and $\Pi(N)$.

Theorems 3, 4, 5, and 6 are illustrated by the following example, where the notation

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix} \text{ is used for } f \in F(N).$$

EXAMPLE. Given $\pi = (\overline{1, 2}, \overline{3, 6, 8}, \overline{4, 5, 7})$, $\tau = (\overline{1, 3, 6}, \overline{2, 8}, \overline{4, 5}, \bar{7}) \in \Pi(8)$. According to (13), the representations of π and τ are: $f_1 = \Phi^{-1}(\pi) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 3 & 4 & 4 & 3 & 4 & 3 \end{smallmatrix}\right)$, $f_2 = \Phi^{-1}(\tau) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 1 & 4 & 4 & 1 & 7 & 2 \end{smallmatrix}\right)$. The join $f_1 \vee f_2$ is calculated according to Theorem 3 as follows. $h_{1,1} = h_0 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{smallmatrix}\right)$. Since $h_{1,1}(f_1(2)) = h_{1,1}(1)$, $h_{1,1}(f_2(3)) = h_{1,1}(1)$, and $h_{1,1}(f_2(6)) = h_{1,1}(1)$, $h_{1,2} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 4 & 5 & 1 & 7 & 1 \end{smallmatrix}\right)$. Since $h_{2,1}(f_1(8)) = h_{2,1}(1)$, $h_{1,3} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 4 & 5 & 1 & 7 & 1 \end{smallmatrix}\right)$, and $h_{1,4} = h_{1,3}$ implies $h_1 = h_{1,3}$. Also we find that $h_2 = h_1$, $h_3 = h_2$, and $h_{4,1} = h_3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 4 & 5 & 1 & 7 & 1 \end{smallmatrix}\right)$. Since $h_{4,1}(f_1(5)) = h_{4,1}(4)$ and $h_{4,1}(f_1(7)) = h_{4,1}(4)$, $h_{4,2} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 4 & 4 & 1 & 4 & 1 \end{smallmatrix}\right)$, and $h_{4,3} = h_{4,2}$ implies $h_4 = h_{4,2}$. Then we have $h_5 = h_4$, $h_6 = h_5$, $h_7 = h_6$, $h_8 = h_7$. Therefore $h_8 = f_1 \vee f_2$. It should be noted that $\Phi(f_1 \vee f_2) = (\overline{1, 2, 3, 6, 8}, \overline{4, 5, 7}) = \pi \vee \tau = \Phi(F_1) \vee \Phi(f_2)$. The meet $f = f_1 \wedge f_2$ is calculated according to Theorem 4 as follows. $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 4$, $f(5) = f(4) = 4$; since $f_1(5) = f_1(4)$ and $f_2(5) = f_2(4)$, $f(6) = f(3) = 3$; since $f_1(6) = f_1(3)$ and $f_2(6) = f_2(3)$, $f(7) = 7$, $f(8) = 8$. Therefore $f = f_1 \wedge f_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 4 & 3 & 7 & 8 \end{smallmatrix}\right)$. We also find that $\Phi(f_1 \wedge f_2) = \Phi(f) = (\bar{1}, \bar{2}, \overline{3, 6}, \overline{4, 5}, \bar{7}, \bar{8}) = \pi \wedge \tau = \Phi(f_1) \wedge \Phi(f_2)$.

4. ALGORITHMS OF JOIN AND MEET

We have proven that $F(N)$ and $\Pi(N)$ are isomorphic. In order to obtain the join and meet of $\pi, \tau \in \Pi(N)$, we can simply calculate the join and the meet of their representations. The algorithms of join and meet are given below:

Algorithm Join

1. Read f_1 and f_2.
2. (Initialization) set $A(i) = i$, $B(i) = i$ for $i = 1, 2, ..., n$. [A and B correspond respectively to $h_{j,k-1}$ and $h_{j,k}$ in (6).]
3. Do steps 4 to 8 for $j = 1, 2, ..., n$. [At this step $A = h_{j-1}$.]
4. Set $A(i) = B(i)$ for $i = 1, 2, ..., n$. [At this step $h_{j,k} \leftarrow h_{j,k-1}$.]
5. Do steps 6 and 7 for $i = 1, 2, ..., n$, as long as $A(i) > j$.
6. If $A(f_1(i)) = A(j)$ or $A(f_2(i)) = A(j)$ then set $B(i) = A(j)$. [Test for condition a (7).]
7. If $A(m) = A(j)$ and $f_1(m) = i$ of $f_2(m) = i$ for some $m = i, i + 1, ..., n$, then set $B(i) = A(j)$. [Test for condition b (8).]
8. If $A(k) \neq B(k)$ for some $k = 1, 2, ..., n$, then go to step 4. [If $A = B$ then $A = h_{j,k_j} = h_{j,k_j+1} = h_{j+1}$.]
9. Print out A. [$A = h_n$, the join of f_1 and f_2]

Algorithm Meet

1. Read f_1 and f_2.
2. (Initialization) Set $f(i) = i$ for $i = 1, 2, ..., n$.
3. Do step 4 for $i = 2, ..., n$.
4. If there is some $k = 1, ..., i - 1$, such that $f_1(i) = f_1(i) = f_1(k)$ and $f_2(i) = f_2(k)$, set $f(i) = f(k)$.
5. Print out f. [The meet of f_1 and f_2]

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