On stable domains

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Abstract


In denotational semantics of programming languages, various categories of domains, with continuous functions as morphisms, and their closure properties under operations like taking products or function space have been intensively studied. However, classes of domains which, like bifinite domains, are also closed under the Plotkin powerdomain operation are rare. Here we investigate stable domains. They naturally generalize the concept of dI-domains studied by Berry and others and satisfy a strong finiteness condition for compact elements, but in general no distributivity assumption. These classes recently were shown (in joint work with R. Göbel) to contain universal objects. We first derive an order-theoretic characterization of stability and then show that the class of all stable domains is closed under countable cartesian products, stable function space and the Plotkin powerdomain operation. As a consequence, we also obtain that the categories of all stable L-domains and of all distributive stable L-domains, with stable functions as morphisms, are cartesian-closed.

1. Introduction

In the theory of denotational semantics of programming languages, various categories of domains have been intensively studied. Scott [17, 18] investigated the classes of all ω-algebraic lattices and of all consistently complete ω-algebraic cpo’s. Plotkin [14] introduced the class of bifinite domains. Coquand [4] and Jung [11] studied L-domains and bifinite L-domains. In each of these cases, the morphisms are continuous functions and the resulting categories are cartesian-closed. As is well known, this closure property is not necessary, but very useful to obtain models of the untyped λ-calculus; see, e.g., [18, 1, 12]. Moreover, the class of bifinite domains is closed under the Plotkin powerdomain operation [14].

However, as Plotkin [15] and Milner [13] showed, continuous function models do not capture all operational properties of ALGOL-like sequential languages, like, e.g., PCF. This led Berry [2, 3] to investigate the category of dI-domains, with stable
functions as morphisms, in order to obtain models of typed $\lambda$-calculi. Intuitively, these functions reflect not only the continuity of computations, but also that a definite information is needed from the argument in order to obtain a given approximation of the result. DI-domains were used in [5] to obtain a model of the polymorphic calculus. Recently, Droste and Göbel [9] introduced stable domains as generalizations of DI-domains and showed that several categories of stable domains contain universal objects.

In this paper, we will study further order-theoretic properties of stable domains. Let CPO* denote the category of all cpo’s, with stable embedding–projection pairs as morphisms. As is well known, CPO* is closed under colimits of $\omega$-chains. Then let $\omega B_\omega$, the class of all $\omega$-stable domains, comprise precisely all colimits in CPO* of $\omega$-chains of finite cpo’s. Here we will first give an order-theoretic characterization of when a domain $(D, \leq)$ is $\omega$-stable in terms of properties of the set of stable projections of $(D, \leq)$. Then we use this characterization to show the following.

**Theorem 1.1.** The class $\omega B_\omega$ is closed under the formation of countable cartesian products, stable function space and Plotkin powerdomain.

As far as we know, the $\omega$-bifinite domains (SFP-domains) studied in Plotkin [14] provide the only class of domains in the literature to date which is closed under taking products, function space and the Plotkin powerdomain operation. By Theorem 1, we obtain another class of domains with such strong closure properties. However, if we endow $\omega B_\omega$ with stable functions as morphisms, the resulting category unfortunately is not cartesian-closed (it already lacks finite products). It remains open whether a different choice of morphisms for $\omega B_\omega$ than stable functions would therefore be more appropriate.

Next let $\omega BL^\omega$ ($\omega DBL^\omega$) denote the categories of all L-domains (distributive L-domains) belonging to $\omega B_\omega$, respectively, and let $\omega DI^\omega$ denote the category of all $\omega$-DI-domains, in each case with stable morphisms (the precise technical definitions are given in Section 3). Then we have $\omega DI^\omega \subseteq \omega DBL^\omega \subseteq \omega BL^\omega$, and an internal order-theoretic characterization of the domains belonging to $\omega BL^\omega$ was given in [9]; a similar result also holds for the category $\omega DBL^\omega$ introduced here. Then from Theorem 1 and the results of Taylor [21] we obtain the following.

**Corollary 1.2.** The categories $\omega BL^\omega$ and $\omega DBL^\omega$ have countable products and are cartesian-closed.

In fact, given two domains $(D, \leq), (E, \leq)$ from $\omega BL^\omega$ or $\omega DBL^\omega$, their exponential is the set of all stable functions from $(D, \leq)$ to $(E, \leq)$, ordered by Berry’s stable ordering for functions. We obtain Berry’s result that $\omega DI^\omega$ is cartesian-closed as a consequence of our present considerations. Here we refer the reader to a forthcoming work of Taylor, cf. [20], for a general category-theoretic result on cartesian-closed categories, which also contains Corollary 1.2 and part of Theorem 1.1 as a consequence. The present proofs are order-theoretic.
As shown in [9], $\omega$BL contains a universal domain, and the same arguments also yield a universal domain for $\omega$DBL. Hence, by Theorem 1.1 and standard techniques (cf. [1, 12]), these universal domains can be used to obtain weakly extensional models of the untyped $\lambda$-calculus. In this context, we note that universal domains for the category $\omega$DI have already been constructed in [7, 9].

2. Stable domains

This section is devoted to an order-theoretic characterization of stable domains and the proof of Theorem 1.1. Let us introduce our notation (which is mostly standard). Let $(D, \leq)$ be a partially ordered set (a poset). A nonempty subset $A \subseteq D$ is called directed if for any $a, b \in A$ there exists $c \in A$ with $a \leq c$ and $b \leq c$. Then $(D, \leq)$ is a cpo if it contains a smallest element and each directed subset of $D$ has a supremum in $D$. An element $x \in D$ is compact if whenever $A \subseteq D$ is directed, $\sup A \in D$ exists and $x \leq \sup A$, then $x \leq a$ for some $a \in A$. We write $D^0$ for the set of all compact elements of $D$, and we use the prefix $\omega$ to denote that $D^0$ is countable. $(D, \leq)$ is algebraic if for each $x \in D$ the set $\{d \in D^0 : d \leq x\}$ is directed and has $x$ as supremum. An algebraic cpo is called a domain. A function $f : P \to Q$ between two posets $(P, \leq), (Q, \leq)$ is called continuous if $f$ preserves all (existing) suprema of directed subsets of $(P, \leq)$.

Next we recall from Berry [2, 31 the important notion of stable functions.

Definition 2.1 (cf. Berry [2, 3] and Curien [6]). Let $(D, \leq)$, $(E, \leq)$ be two posets.

(a) A continuous function $f : D \to E$ is called stable if for all $x \in D$ and $y \in E$, with $y \leq f(x)$, there exists $m = m(f, x, y) \in D$ with the following property:

$$m \leq x, y \leq f(m), \text{ and whenever } d \in D, \text{ with } d \leq x \text{ and } y \leq f(d), \text{ then } m \leq d.$$  

(b) Let $f, g : D \to E$ be functions. We put $f \leq g$ if $f(x) \leq g(x)$ for each $x \in D$. If $f, g$ are stable, we put $f \leq_s g$ if $f \leq g$ and whenever $x \in D$, $y \in E$, with $y \leq f(x)$, then $m(f, x, y) = m(g, x, y)$. We let

$$[D \to E] = \{f : (D, \leq) \to (E, \leq) : f \text{ is stable}\}.$$  

Here, intuitively, $m(f, x, y)$ represents the smallest amount of information from $x$ needed to obtain, via the computation $f$, at least $y$. It is easy to check that compositions of stable functions are again stable.

Next let $(P, \leq)$, $(Q, \leq)$ be two posets and $f : P \to Q$, $g : Q \to P$ continuous functions. Then $(f, g)$ is called a stable embedding–projection pair (SEPP) from $(P, \leq)$ into $(Q, \leq)$ if the following two conditions are satisfied:

1. $g \circ f(x) = x$ and $f \circ g(y) \leq y$ for each $x \in P$, $y \in Q$.
2. Whenever $x \in P$ and $y \in Q$ with $y \leq f(x)$, then $f \circ g(y) = y$.

Clearly, $(f, g)$ is a SEPP from $P$ into $Q$ iff $f$ and $g$ are stable, $g \circ f = \text{id}_P$ and $f \circ g \leq_s \text{id}_Q$.

Now we introduce a few further notions. A function $h : P \to P$ will be called a stable projection if $h$ is stable and $h \leq_s \text{id}_P$. Observe that then, in particular, $h \circ h = h$. A stable
projection \( h \) is called a stable deflation if the image \( \text{im}(h) \) of \( h \) is finite. For \( A \subseteq P \) and \( x \in P \), we let \( A \leq x \) abbreviate that \( a \leq x \) for each \( a \in A \). A subset \( S \) of \( P \) is called an ideal, denoted as \( S \triangleleft P \), if the following conditions hold:

(a) Whenever \( A \subseteq S \), \( x \in P \) and \( A \leq x \), then there exists \( s \in S \) with \( A \leq s \leq x \).

(b) Whenever \( x \in P \), \( s \in S \) and \( x \leq s \), then \( x \in S \).

If \( S \triangleleft P \), we let \( p_S: P \to S \) be the projection of \( P \) onto \( S \), defined by \( p_S(x) = \sup\{s \in S : s \leq x\} \) (\( x \in P \)). Stable embedding-projection pairs, stable projections and ideals are closely related, as is well known:

**Proposition 2.2.** Let \((D, \leq)\) be a cpo, let \( p: D \to D \) be a function, and let \( S = \text{im}(p) \). The following are equivalent:

1. \( p \) is a stable projection.
2. \((\text{id}_S, p)\) is a SEPP from \((S, \leq)\) into \((D, \leq)\).
3. \( S \triangleleft D \), \( p = p_S \) and \( p \) is continuous.

Moreover, in this case \((S, \leq)\) is a cpo and \((S, \leq)^0 = (D, \leq)^0 \cap S \). Furthermore, if \( S \triangleleft D \) and \( s = \sup\{x \in D^0 : x \leq s\} \) for each \( s \in S \), then \( p = p_S \) is continuous and, hence, a stable projection.

**Proof.** (1)\(\Rightarrow\)(3): If \( x, y \in D \), with \( y \leq p(x) \), then \( m(p(x), y) = m(\text{id}_D, x, y) = y \); so, \( y = p(y) \in S \). This also shows (letting \( y = p(x) \)) that \( s = p(s) \) for each \( s \in S \). If \( x \in D \), then \( p(x) \in S \) and \( s = p(s) \leq p(x) \leq x \) for any \( s \in S \) with \( s \leq x \). Hence, \( S \triangleleft D \) and \( p = p_S \).

(3)\(\Rightarrow\)(2): Straightforward.

(2)\(\Rightarrow\)(1): If \( x, y \in D \), with \( y \leq p(x) \), then \( m(p(x), y) = y \). Hence, \( p \) is stable and \( p \leq \text{id}_D \).

The final statement can be derived as in \[8, \text{Proposition 4.5}\]. \(\blacksquare\)

Now let \((D, \leq)\) be a domain and \( A \subseteq D \). Let \( \text{Mub}(A) \) denote the set of all minimal upper bounds of \( A \) in \( D \). We say that \( \text{Mub}(A) \) is complete if for any \( y \in D \), with \( A \leq y \), there exists \( x \in \text{Mub}(A) \) with \( A \leq x \leq y \). Inductively, we put \( U^0(A) = A \), \( U^{n+1}(A) = \bigcup \{ \text{Mub}(X) : X \subseteq U^n(A), X \text{ finite} \} \) (\( n \in \omega \)), and \( U^\omega(A) = \bigcup_{n \in \omega} U^n(A) \). Then \((D, \leq)\) is called bifinite if for each finite subset \( A \subseteq D^0 \) \( \text{Mub}(A) \) is complete and \( U^\omega(A) \) is finite. The \( \omega \)-bifinite domains are precisely the SFP-objects studied in Plotkin \[14\]. With continuous functions as morphisms, they form a cartesian-closed category; also, the class of all \( \omega \)-bifinite domains is closed under the Plotkin powerdomain operation (see \[14\]).

Let \((D, \leq)\) again be a domain and \( A \subseteq D \). Inductively, we put \( V^0(A) = A \), \( V^{n+1}(A) = U^\omega(\{x \in D : x \leq y \text{ for some } y \in V^n(A)\}) \) (\( n \in \omega \)), and \( V^\omega(A) = \bigcup_{n \in \omega} V^n(A) \). As in \[9\], we say that \((D, \leq)\) is stable if for each finite subset \( A \subseteq D^0 \) \( \text{Mub}(A) \) is complete and \( V^\omega(A) \) is finite. In particular, \((D, \leq)\) is bifinite. Now we have the following first characterization of the objects of the class \( \omega B_n \) defined in the introduction.

**Proposition 2.3.** Let \((D, \leq)\) be any poset. The following are equivalent:

1. \((D, \leq) \in \omega B_n \).
(2) \((D, \leq)\) is an \(\omega\)-stable domain.

(3) \((D, \leq)\) is an \(\omega\)-domain and for each finite subset \(A \subseteq D^0\) there exists a finite ideal \(S \triangleleft D\) such that \(A \subseteq S\).

**Proof.** (1)\(\rightarrow\)(2): See [9, Proposition 4.2].

(2)\(\rightarrow\)(3): Immediate, observing that if \(A \subseteq D^0\), then \(A \subseteq V^\omega(A) \triangleleft D\). Conversely, if \(A \subseteq S \triangleleft D\), then \(V^\omega(A) \subseteq S\). \(\square\)

Proposition 2.3 is basic for all of the following and could indeed also be used as a definition of the objects of the class \(\omega B_{st}\). We will use this result subsequently without mentioning it again.

If \((P, \leq), (Q, \preceq)\) are posets and \(f, f_i : P \rightarrow Q\) functions \((i \in I)\), we say that \(f\) is the **pointwise supremum** of \((f_i)_{i \in I}\) if \(f(x) = \sup \{f_i(x) : i \in I\}\) for each \(x \in P\). **Pointwise infima** are defined analogously. Now we have the following useful characterization of \(\omega\)-stable domains, which is analogous to a characterization of bifinite domains given in [11, Theorem 1.26].

**Theorem 2.4.** Let \((D, \leq)\) be a cpo. The following are equivalent:

1. \((D, \leq) \in \omega B_{st}\)
2. The set of all stable deflations on \(D\) is countable and directed in \([D \rightarrow_{s} D], \leq_{s}\) and has \(\text{id}_D\) as pointwise supremum.
3. \([D \rightarrow_{s} D], \leq_{s}\) contains a countable directed set of stable deflations on \(D\) with pointwise supremum \(\text{id}_D\).

**Proof.** (1)\(\rightarrow\)(2): Let \(\text{SD}(D)\) denote the set of all stable deflations on \(D\). By Proposition 2.2, a continuous function \(p : D \rightarrow D\) belongs to \(\text{SD}(D)\) iff \(p = p_A\) for some finite ideal \(A\) of \(D\), and then \(A \subseteq D^0\). Thus, \(\text{SD}(D)\) is countable, as \(D^0\) is countable. For any two finite ideals \(A, B\) of \(D\) we have \(p_A \leq_{s} p_B\) iff \(A \preceq B\) and, as \(D\) is stable, \(C = V^\omega(A \cup B)\) is a finite ideal of \(D\) containing \(A\) and \(B\); thus, \(\text{SD}(D)\) is \(\leq_{s}\)-directed. Furthermore, any \(x \in D^0\) belongs to some finite ideal \(A\) of \(D\); then \(p_A(x) = x\). Hence, \(\text{SD}(D)\) has \(\text{id}_D\) as pointwise supremum.

(2)\(\rightarrow\)(3): Trivial.

(3)\(\rightarrow\)(1): Let \(P\) be a countable, \(\leq_{s}\)-directed set of stable deflations on \(D\) with \(\text{id}_D\) as pointwise supremum. Then \(d = \sup \{p(d) : p \in P\}\) for each \(d \in D\). Thus, by Proposition 2.2, \(\text{im}(p) \subseteq D^0\) for each \(p \in P\), and \((D, \leq)\) is algebraic. For any \(x \in D^0\) there exists \(p \in P\) with \(x = p(x) \in \text{im}(p)\). Hence, \(D^0 = \bigcup \{\text{im}(p) : p \in P\}\) is countable. Moreover, if \(A \subseteq D^0\) is finite, there is \(p \in P\) with \(A \subseteq \text{im}(p) \triangleleft D\), and \(\text{im}(p)\) is finite. So, \((D, \leq)\) is stable. \(\square\)

As an application of Theorem 2.4, we obtain the following corollary.

**Corollary 2.5.** The class \(\omega B_{st}\) is closed under countable cartesian products.
Proof. Let \( \{(D_i, \leq) : i \in I\} \) be a collection of countably many \( \omega \)-stable domains. Let \( (D, \leq) = \prod_{i \in I} (D_i, \leq) \), the usual cartesian product of partial orders. Clearly, \( (D, \leq) \) is a cpo. Now we construct stable deflations \( g : D \to D \) as follows. Let \( J \) be a finite subset of \( I \), and let \( g_i : D_i \to D_i \) be any fixed stable deflation on \( D_i \) if \( i \in J \), and \( g_i(x) = x \) for each \( x \in D_i \) if \( i \in I \setminus J \). Put \( g = \prod_{i \in I} g_i \). Clearly, this way we obtain a countable \( \leq \)-directed set of stable deflations on \( D \) with pointwise supremum \( \text{id}_D \). Hence, \( (D, \leq) \in \omega B_\omega \), by Theorem 2.4.

The following result characterizes the case when a continuous function between two domains is stable.

**Proposition 2.6** (Berry [3, Proposition 4.2.3]). Let \( (D, \leq), (E, \leq) \) be two domains and \( f : D \to E \) a continuous function. Then \( f \) is stable if and only if \( m(f, a, b) \in D \) exists for all \( a \in D^0, b \in E^0 \), with \( b \leq f(a) \). Moreover, if \( f \) is stable, \( x \in D, y \in E, y \leq f(x) \) and \( M_{x,y} = \{ m(f, a, b) : a \in D^0, b \in E^0, a \leq x, b \leq y, b \leq f(a) \} \), then \( M_{x,y} \subseteq D^0 \), \( M_{x,y} \) is directed and \( m(f, x, y) = \sup M_{x,y} \).

Now we show the following.

**Proposition 2.7.** Let \( (D, \leq), (E, \leq) \) be two domains. Then \( ([D \to E], \leq_s) \) is a cpo. Moreover, suprema of directed subsets of \( ([D \to E], \leq_s) \) are determined pointwise.

**Proof.** Clearly, \( ([D \to E], \leq_s) \) contains a smallest element. Let \( F \subseteq [D \to E] \) be \( \leq_s \)-directed. Let \( f^* : D \to E \) be the pointwise supremum of \( F \). We show that \( f^* \) is stable. Let \( x \in D, y \in E^0 \), with \( y \leq f^*(x) \). Put \( F^* = \{ f \in F : y \leq f(x) \} \). Clearly, \( F^* \) is \( \leq_s \)-directed and \( f^* \) is the pointwise supremum of \( F^* \). Hence, \( m(f, x, y) = m(f^*, x, y) \) for any \( f, f' \in F^* \). Let \( m = m(f_0, x, y) \) for some \( f_0 \in F^* \). Clearly, \( m \leq_s x \) and \( f^*(m) \leq f_0(m) \geq y \). If \( d \in D \) with \( d \leq x \) and \( f^*(d) \geq y \), then \( y \leq f(d) \) for some \( f \in F^* \); thus, \( m = m(f^*, x, y) \leq d \). This shows that \( m = m(f^*, x, y) \), and \( f^* \) is stable by Proposition 2.6.

Next we show that \( f^* = \sup F \) in \( ([D \to E], \leq_s) \). Choose any \( f \in F \) and let \( x \in D, y \in E \) with \( y \leq f(x) \). Then \( f \leq f^* \), and we claim that \( m(f, x, y) = m(f^*, x, y) \). Indeed, if \( y \in E^0 \), this was proved above. Hence, our claim follows from Proposition 2.6. Thus, \( f \leq_s f^* \).

Now let \( g \in [D \to E] \), with \( f \leq g \) for each \( f \in F \). Clearly, \( f^* \leq g \). Let \( x \in D, y \in E \), with \( y \leq f^*(x) \). We claim that \( m(f^*, x, y) = m(g, x, y) \). Again by Proposition 2.6, we may assume that \( y \in E^0 \). Choose any \( f \in F \) with \( y \leq f(x) \). Then \( m(f^*, x, y) = m(f, x, y) = m(g, x, y) \). Hence, \( f^* \leq_s g \). The result follows.

Next we use Theorem 2.4 to show the following.

**Theorem 2.8.** Let \( (D, \leq), (E, \leq) \) be two stable \( \omega \)-bifinite domains. Then \( ([D \to E], \leq_s) \) is a stable \( \omega \)-bifinite domain.
Proof. By Proposition 2.7, \( ([D \to E], \leq_s) \) is a cpo. Let \( A \triangleleft D, B \triangleleft E \) be two finite ideals, and let \( p_A, p_B \) denote the projection from \( D \) onto \( A \) (from \( E \) onto \( B \)). We define a function \( F = F_{A,B} \) from \( ([D \to E], \leq_s) \) into itself by letting \( F(g) = p_B \circ g \circ p_A \) \( (g \in [D \to E]) \). Clearly, \( F \) is well-defined and continuous, \( F \circ F = F \) and \( F \leq \text{id}_{[D \to E]} \).

Now let \( g, h \in [D \to E] \), with \( h \leq_s F(g) \). We claim that \( F(h) = h \). Let \( x \in D \). Note that \( h(x) \in B \) by \( h \leq_s F(g) \). Put \( m = m(h, x, h(x)) \). Clearly, \( F(g)(p_A(x)) = F(g)(x) \geq h(x) \); so, \( m \leq p_A(x) \). Thus, \( h \circ p_A(x) \geq h(m) \geq h(x) \), showing that \( h(x) = h \circ p_A(x) = F(h)(x) \) and our claim. Hence, \( F \) is stable and \( F \leq \text{id}_{[D \to E]} \). As there are only finitely many functions from \( A \) to \( B \), \( F = F_{A,B} \) has a finite image and is, thus, a stable deflation.

Clearly, the set of all such functions \( F_{A,B} \), where \( A \triangleleft D, B \triangleleft E \) are finite ideals, is countable and \( \leq_s \)-directed in \( ([D \to E] \to_s [D \to E]) \), with pointwise supremum \( \text{id}_{[D \to E]} \). Hence, by Theorem 2.4, \( ([D \to E], \leq_s) \) is a stable \( \omega \)-bifinite domain. □

As a side-observation and consequence of the above argument, we characterize the compact elements of \( ([D \to E], \leq_s) \) if \( (D, \leq) \) and \( (E, \leq) \) are \( \omega \)-stable domains. First let \( (D, \leq), (E, \leq) \) be arbitrary cpo’s and \( h : D \to E \) a stable function. As in [3, p. 4.69] we say that \( x \in D \) is a minimal point of \( h \) if \( m(h, x, h(x)) = x \); equivalently, \( d \in D, d \leq x \) and \( h(d) = h(x) \) imply \( d = x \). Let \( M(h) \) be the set of all minimal points of \( h \). Note that for any \( d \in D \) there exists a greatest element \( d^* \in M(h) \), with \( d^* \leq d \), namely \( d^* = m(h, d, h(d)) \). As \( h(d) = h(d^*) \), we obtain that \( h \) is uniquely determined by its restriction to \( M(h) \). The following result generalizes [3, Proposition 4.6.12].

Corollary 2.9. Let \( (D, \leq), (E, \leq) \) be \( \omega \)-stable domains, and let \( h : D \to E \) be a stable function. Then \( h \) is compact in \( ([D \to E], \leq_s) \) if and only if \( M(h) \) is finite, \( M(h) \subseteq D^0 \), and \( h(M(h)) \subseteq E^0 \).

Proof. First let \( h \) be compact. As shown in the proof of Theorem 2.8, the functions \( F_{A,B} \), where \( A \triangleleft D, B \triangleleft E \) are finite ideals, form a \( \leq_s \)-directed subset of \( ([D \to E] \to_s [D \to E]) \) with pointwise supremum \( \text{id}_{[D \to E]} \). Hence, there exist finite ideals \( A \triangleleft D, B \triangleleft E \), with \( h = F_{A,B}(h) = p_B \circ h \circ p_A \). Then \( M(h) \triangleleft A \subseteq D^0 \), \( M(h) \) is finite, and \( h(M(h)) \subseteq B \subseteq E^0 \).

Conversely, assume that \( M(h) \) is finite, with \( M(h) \subseteq D^0 \) and \( h(M(h)) \subseteq E^0 \). Choose finite ideals \( A \triangleleft D, B \triangleleft E \), with \( M(h) \subseteq A, h(M(h)) \subseteq B \). Then \( h = F_{A,B}(h) \in \text{im}(F_{A,B}) \), and \( F_{A,B} \) is a stable deflation on \( ([D \to E], \leq_s) \). Hence, \( h \) is compact in \( ([D \to E], \leq_s) \). □

Next we turn to powerdomains. We first fix our notation: we refer the reader to [14] for further background. We let \( \leq_M \) denote the Egli–Milner ordering of the power set of \( D \), that is, for \( A, B \subseteq D \) we put

\[ A \leq_M B \text{ iff } (\forall a \in A \exists b \in B. \ a < b) \text{ and } (\forall b \in B \exists a \in A. \ a < b). \]

We write \( A =_M B \) if \( A \subseteq_M B \) and \( B \subseteq_M A \).
If $A \subseteq D$, we put $\text{Con}(A) = \{ d \in D : a \leq d \leq a' \text{ for some } a, a' \in A \}$, the \textit{convexification} of $A$ in $D$. Let $\text{Conv}(D) = \{ \text{Con}(A) : \emptyset \neq A \subseteq D, A \text{ finite} \}$. Then we regard the canonical completion of the poset $(\text{Conv}(D), \subseteq_m)$ to a cpo as the powerdomain $\mathcal{P}[D]$ of $(D, \leq)$. Thus, clearly, $\mathcal{P}[D]$ is a domain. Moreover, if $(D, \leq)$ is $\omega$-bifinite, then so is $\mathcal{P}[D]$. Now we give a direct order-theoretic argument to show the following.

**Theorem 2.10.** Let $(D, \leq)$ be an $\omega$-stable domain. Then $\mathcal{P}[D]$ is also an $\omega$-stable domain.

**Proof.** By construction, $\mathcal{P}[D]$ is a domain. We may assume that $\mathcal{P}[D]^0 = \text{Conv}(D)$. Let $\mathcal{A} \subseteq \mathcal{P}[D]^0$ be a finite subset. Put $A = \bigcup \{ X : X \in \mathcal{A} \}$. As $(D, \leq)$ is stable, there exists a finite ideal $S \subseteq D$, with $A \subseteq S \subseteq D^0$. Let $\mathcal{F} = \text{Conv}(S)$. Then $\mathcal{F}$ is finite, $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{P}[D]$, and we claim that $\mathcal{F} < \mathcal{P}[D]$. Indeed, if $X \in \mathcal{P}[D]$, $Y \in \mathcal{F}$ and $X \subseteq Y$, then $X \in \mathcal{F}$ by $\mathcal{F} \subseteq \mathcal{P}[D]$. Now let $\mathcal{A} \subseteq \mathcal{P}[D]$ and $X \in \text{Conv}(D)$ such that $Y \subseteq X$ for each $Y \in \mathcal{A}$. For each pair $(x, W)$, with $x \in X$, $W \subseteq \bigcup \{ Y : Y \in \mathcal{A} \}$, and $\forall Y' \in \mathcal{A}$ with $Y' \neq Y$, there exists $s(x, W) \in S$ such that $W \subseteq s(x, W) \subseteq x$. Let $Z$ be the set of all such elements $s(x, W) \in S$. Clearly, $Z$ is nonempty and finite and $Z \subseteq \mathcal{F}$, hence $Y \subseteq Z$. For each $Y \in \mathcal{A}$ with $Y' \neq Y$, select $y' \in Y'$ with $y' \neq x$. Let $W$ be the set comprising all these elements $y'$. Then $y \leq s(x, W) \leq x$ and $s(x, W) \in Z$. Hence, $Y \subseteq Z$. Thus, $Y \subseteq \text{Con}(Z) \subseteq X$ for each $Y \in \mathcal{A}$ and $\text{Con}(Z) \subseteq \mathcal{F}$. $\square$

We note that an argument very similar to the one given above shows that if $(D, \leq)$ is $\omega$-bifinite, then so is $\mathcal{P}[D]$. It may be useful to consider also another argument for Theorem 2.10. As shown in [14], we can naturally make $\mathcal{P}[ ]$ a functor on the category of all $\omega$-bifinite domains with continuous functions as morphisms. Here, if $(D, \leq), (E, \leq)$ are finite domains and $f : D \to E$ is continuous, we may put $\mathcal{P}[f] = \hat{f}$, where $\hat{f} : \mathcal{P}[D] \to \mathcal{P}[E]$ is given by $\hat{f}(X) = \text{Con}(f(X))$ ($X \in \text{Conv}(D)$). If $(f, g)$ is an embedding-projection pair from $(D, \leq)$ to $(E, \leq)$ (i.e., $g \circ f = \text{id}_D$ and $f \circ g \leq \text{id}_E$), then so is $(\hat{f}, \hat{g})$ from $\mathcal{P}[D]$ to $\mathcal{P}[E]$. Now if $(D, \leq)$ is a colimit of an $\omega$-chain $(D_i, (f_i, g_i))_{i \in \omega}$ with finite domains $D_i$ and EPPs $(f_i, g_i)$ from $D_i$ to $D_{i+1}$, then $\mathcal{P}[D]$ is isomorphic to the colimit of the $\omega$-chain $(\mathcal{P}[D_i], (\hat{f}_i, \hat{g}_i))_{i \in \omega}$. Hence, for Theorem 2.10 it suffices to prove the following.

**Proposition 2.11.** Let $(D, \leq), (E, \leq)$ be two finite domains and $(f, g) : (D, \leq) \to (E, \leq)$ a SEPP. Then $(\hat{f}, \hat{g}) : \mathcal{P}[D] \to \mathcal{P}[E]$ is also a SEPP.

**Proof.** As noted before, $(\hat{f}, \hat{g})$ is an EPP. Now let $X \in \text{Conv}(D)$, $Y \in \text{Conv}(E)$, with $Y \subseteq f(X)$. For any $y \in Y$ there exists $x \in X$ with $y \leq f(x)$; hence, $f \circ g(y) = y$. Thus, $\hat{f} \circ \hat{g}(Y) = \text{Con}(f \circ g(Y)) = Y$ and, so, by convexity, $\hat{f} \circ \hat{g}(Y) = Y$. Hence, $(\hat{f}, \hat{g})$ is a SEPP. $\square$
Now the proof of Theorem 1.1 is immediate by Corollary 2.5 and Theorems 2.8 and 2.10.

Finally, we note still another closure property of the class of all stable domains.

**Proposition 2.12.** Let \((E, \leq)\) be a stable domain and \((D, \leq)\) a domain for which there exists an embedding–projection pair \((f, g)\) from \((D, \leq)\) to \((E, \leq)\). Then \((D, \leq)\) is stable.

**Proof.** As is well known, \(f\) is an isomorphism from \(D\) to \(f(D)\), \(f(D^0) \subseteq E^0\) and for any \(X \subseteq f(D), \text{Mub}_{f(D)}(X) = \text{Mub}_{f(E)}(X)\). Now let \(A \subseteq (f(D))^0 = f(D^0)\) be finite. By induction, we obtain \(A \subseteq V_f\)(\(A\)) = \(V_{f(E)}(A) \cap f(D)\), which is finite. Thus, \((D, \leq)\) is stable. \(\square\)

Let \(T\) be the 3-element truth value cpo. The countable cartesian product \(T^\omega\) of countably many copies of \(T\) was studied in [16]. As a consequence of Corollary 2.5, Theorem 2.10 and Proposition 2.12, we see, for example, that any domain \((D, \leq)\) for which there exists an embedding–projection pair from \((D, \leq)\) to \(\mathcal{P}[T^\omega]\) is stable.

### 3. Stable L-domains

In this section, we wish to prove Corollary 1.2. A poset \((D, \leq)\) is called L-complete, if each nonempty upper-bounded subset of \(D\) has an infimum in \(D\); equivalently, for each \(x \in D\) the set \(\downarrow x = \{d \in D : d \leq x\}\) is a complete lattice [11, (2.9)]. An L-complete domain is called an L-domain. An L-domain \((D, \leq)\) is called distributive if for each \(x \in D\) the set \(\downarrow x, \leq\) is a distributive lattice, i.e., \(a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\) in \(\downarrow x, \leq\) for any \(a, b, c \in \downarrow x\). Finally, \((D, \leq)\) is a Scott domain if each nonempty subset of \(D\) has an infimum in \(D\). The poset \((D, \leq)\) shown in Fig. 1 is the simplest example of a distributive L-domain which is not a Scott domain.

L-domains have been investigated by Coquand [4], Jung [11] and Taylor [21]. With continuous functions as morphisms, they form a cartesian-closed category (see [4, 10, 11]). Clearly, Proposition 2.3 also provides an order-theoretic characterization of stable L-domains. Distributive L-domains with stable functions as morphisms have been studied by Lamarche [22]. Next we note that stability is easy to check for distributive bifinite L-domains.

![Fig 1](image-url)
Proposition 3.1. Let \((D, \leq)\) be a distributive bifinite \(L\)-domain. Then \((D, \leq)\) is stable iff for each \(x \in D^0\) the set \(\downarrow x\) is finite.

Proof. We claim that \(V^x(A) = V^1(A)\) for each subset \(A \subseteq D\). Indeed, let \(A \subseteq D, x \in V^1(A)\) and \(y \in D\), with \(y \leq x\). By [11, Theorem 2.10], we have \(U^x(S) = U^1(S)\) for each \(S \subseteq D\). Hence, there are elements \(a_i \in A, b_i \in D\) (\(i \in I\), \(I\) finite) with \(b_i \leq a_i\) for each \(i \in I\) and \(x \in \text{Mub}(\{b_i : i \in I\})\). Then in \((\downarrow x, \leq)\) we have \(x = \sup \{b_i : i \in I\}\) and \(y = y \wedge x = \sup \{y \wedge b_i : i \in I\}\) by distributivity; hence, \(y \in \text{Mub}(\{y \wedge b_i : i \in I\}) \subseteq V^1(A)\) in \((D, \leq)\). This implies our claim, and the result follows. □

A Scott domain \((D, \leq)\) is called a \(dI\)-domain if \((D, \leq)\) is distributive (as an \(L\)-domain) and for each \(x \in D^0\) \(\downarrow x\) is finite. Then by Proposition 3.1, \((D, \leq)\) is stable. Let \(\omega S^\omega\) be the category of all \(w\)-stable Scott domains. Then the category \(\omega \text{dI}^\omega\) of all \(\omega\)-\(dI\)-domains is the intersection of \(\omega \text{DBL}^\omega\) and \(\omega S^\omega\).

Next we recall some elementary properties of stable functions. First, stable functions preserve infima of upper-bounded subsets.

Proposition 3.2 (cf. [6, proof of Proposition 2.4.2]). Let \((D, \leq), (E, \leq)\) be two cpo’s and \(f : (D, \leq) \rightarrow (E, \leq)\) a continuous function. Consider the following two conditions:

1. \(f\) is stable.
2. Whenever \(X \subseteq D\) is nonempty and upper-bounded such that \(\inf X \in D\) exists, then \(f(\inf X) = \inf f(X)\).

Then \((1) \Rightarrow (2)\). Moreover, if \((D, \leq)\) is \(L\)-complete, we have \((1) \iff (2)\).

Proof. \((1) \Rightarrow (2)\): Let \(\emptyset \neq X \subseteq D\) and \(z \in D\), with \(X \subseteq z\). Let \(y \in E\), with \(y \leq f(X)\). Put \(m = m(f, z, y)\). For each \(x \in X\) we have \(y \leq f(x)\); thus, \(m \leq y\). Hence, \(m \leq \inf X\) and, so, \(y \leq f(\inf X)\). The result follows.

\((2) \Rightarrow (1)\): If \((D, \leq)\) is \(L\)-complete: Let \(x \in D, y \in E\), with \(y \leq f(x)\). Put \(M = \{d \in D : d \leq x, y \leq f(d)\}\) and \(m = \inf M\) in \((\downarrow x, \leq)\). Then \(m \leq x, f(m) = \inf f(M) \geq y\) and \(m = m(f, x, y)\). □

Next we have the following proposition.

Proposition 3.3 (Taylor [21]). Let \((D, \leq), (E, \leq), (F, \leq)\) be three \(L\)-domains.

(a) \([(D \to E), \leq_s]\) is a cpo in which the infimum of any nonempty upper-bounded subset of \([D \to E]\) exists and is determined pointwise.

(b) The evaluation mapping

\[\text{ev} : ([D \to E], \leq_s) \times (D, \leq) \rightarrow (E, \leq)\]

\[(h, d) \mapsto h(d)\]

is stable.
(c) Let \( f : (D \times E, \leq) \to (F, \leq) \) be a stable function. For each \( d \in D \), define \( f_d : (E, \leq) \to (F, \leq) \) by \( f_d(e) = f(d, e) \). Then \( f_d \) is stable, and the function

\[
\tilde{f} : (D, \leq) \to ([E \to F], \leq)
\]

\[ d \mapsto f_d \]

is stable.

Now we can combine the previous results to show the following.

**Corollary 3.4.** The category \( \omega BL'' \) is cartesian-closed. Moreover, for any two stable \( \omega \)-bifinite \( L \)-domains \( (D, \leq), (E, \leq) \), the exponential object of \( (D, \leq) \) and \( (E, \leq) \) is the domain \( ([D + S E], \leq) \).

**Proof.** The terminal object of \( \omega BL'' \) is the one-point domain. The argument that \( \omega BL'' \) has finite (even countable) products is straightforward, using Corollary 2.5, and left to the reader. Now let \( (D, \leq), (E, \leq) \) and \( (F, \leq) \) be three \( \omega \)-stable \( L \)-domains. By Theorem 2.8 and Proposition 3.3(a), \( ([D \to \varepsilon E], \leq) \) belongs to \( \omega BL'' \), and, by Proposition 3.3(b), the evaluation mapping \( \text{ev} : ([D \to \varepsilon E], \leq) \times (D, \leq) \to (E, \leq) \) is stable. If \( f : (D \times D, \leq) \to (E, \leq) \) is a stable function, by Proposition 3.3(c), there exists a stable function \( \tilde{f} : (F, \leq) \to ([D \to \varepsilon E], \leq) \) such that \( \text{ev} \circ (f \times \text{id}_D) = f \). Moreover, \( \tilde{f} \) is uniquely determined by this equation. The result follows. \( \Box \)

Now we turn to the category \( \omega DBL'' \).

**Proposition 3.5.** Let \( (D, \leq), (E, \leq) \) be two \( L \)-domains such that \( (E, \leq) \) is distributive. Let \( f, g, h \in [D \to E] \), with \( \{f, g\} \leq h \). Then the supremum \( f \vee g \) of \( \{f, g\} \) in \( \langle h, \leq \rangle \) exists and is determined pointwise below \( h \), i.e., \( (f \vee g)(x) = f(x) \lor g(x) \) in \( \langle h(x), \leq \rangle \) for each \( x \in D \). Moreover, if \( x \in D, \ y \in E \), with \( y \leq (f \vee g)(x) \), then \( m(f \vee g, x, y) = m(f, x, y \wedge f(x)) \lor m(g, x, y \wedge g(x)) \) in \( \langle x, \leq \rangle \).

**Proof.** Follow the argument of Berry [3, Proposition 4.4.13(2)]. \( \Box \)

Next we have the following corollary.

**Corollary 3.6.** Let \( (D, \leq), (E, \leq) \) be two \( \omega \)-stable \( L \)-domains such that \( (E, \leq) \) is distributive. Then \( ([D \to \varepsilon E], \leq) \) is a distributive \( \omega \)-stable \( L \)-domain. Moreover, if \( (E, \leq) \) is even a dI-domain, then so is \( ([D \to \varepsilon E], \leq) \).

**Proof.** By Theorem 2.8 and Proposition 3.5, \( ([D \to \varepsilon E], \leq) \) is a distributive \( \omega \)-stable \( L \)-domain, as \( (E, \leq) \) is distributive and all the relevant suprema and infima are determined pointwise. Now assume that \( (E, \leq) \) is even a Scott domain. Let \( f, g, h \in [D \to E] \) such that \( h \) is a minimal upper bound of \( \{f, g\} \) in \( ([D \to \varepsilon E], \leq) \). Let
Now the proof of Corollary 1.2 is immediate by Corollaries 3.4 and 3.6.

Now let \( C \) be a class of domains and \((U, \leq) \in C\). We say that \((U, \leq)\) is universal in \( C \) if for each \((D, \leq) \in C\) there exists a SEPP \((f, g)\) from \((D, \leq)\) into \((U, \leq)\). Furthermore, \((U, \leq)\) is homogeneous if whenever \((D, \leq) \in C\) is finite and \((f_i, g_i):(D, \leq) \to (U, \leq)\) \((i = 1, 2)\) are SEPPs, then there exists an automorphism \( h \) of \((D, \leq)\) such that \( h \circ f_1 = f_2 \).

Then we have the following theorem.

**Theorem 3.7.** The category \( \omegaDBL^n \) contains a universal homogeneous domain \((U, \leq)\). Moreover, this domain \((U, \leq)\) is unique up to isomorphism.

**Proof (Sketch).** The argument is completely analogous to the proof given for [9, Theorem 1.3], observing that the amalgamation property for finite distributive \( L \)-domains holds by [9, Lemma 4.4].

As a consequence of Corollary 1.2 and Theorem 3.7, it follows that if \((U, \leq)\) is the universal homogeneous domain of \( \omegaDBL^n \), then there exists a SEPP from \((\omega \rightarrow, \leq)\) into \((U, \leq)\); hence, \((U, \leq)\) becomes a weakly extensional model of the untyped \( \lambda \)-calculus (cf. [1, 12]).

Finally, we just note without proof that the category \( \omegaB^n \) of all \( \omega \)-stable domains, with stable functions as morphisms, does not have finite products. The category \( \omegaS^n \) of all \( \omega \)-stable Scott domains has countable products, but is not cartesian-closed – the exponential would coincide with the stable function spaces, which, however, in general is not again a Scott domain. The counterexamples (with “small” domains of size \( \leq 6 \)) are easy to obtain; we refer the reader to [20] for positive results.

**References**


On stable domains


