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An optimal power mean inequality for the complete elliptic integrals

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ABSTRACT

In this work, we prove that $M_p(\mathcal{K}(r), \mathcal{E}(r)) > \pi/2$ for all $r \in (0, 1)$ if and only if $p \geq -1/2$, where $M_p(x, y)$ denotes the power mean of order p of two positive numbers x and y, and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ denote the complete elliptic integrals of the first and second kinds, respectively.

1. Introduction

Throughout this work, we write $r' = \sqrt{1-r^2}$ for $0 < r < 1$. The well-known complete elliptic integrals of the first and second kinds [1,2] are defined by

$$\begin{cases} \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{-1/2} d\theta, \\ \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2\theta)^{1/2} d\theta, \\ \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1, \end{cases} \tag{1.2}$$

respectively.

In the sequel, we use the symbols $\mathcal{K}$ and $\mathcal{E}$ for $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively.

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and other related fields [2–10].

Recently, the complete elliptic integrals have been the subject of intensive research. In particular, many remarkable properties and inequalities can be found in the literature [3,10–15].

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For $p \in \mathbb{R}$, the power mean $M_p(x, y)$ of order p of two positive numbers x and y is defined by

$$M_p(x, y) = \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{xy}, & p = 0. \end{cases} \tag{1.3}$$

The main properties of the power mean are given in [16].

In [4, Theorem 3.31], Anderson et al. studied the monotonicity and convexity of $\mathcal{K}(r)\mathcal{E}(r)$ in $(0, 1)$, and obtained the following inequality:

$$M_0(\mathcal{K}(r), \mathcal{E}(r)) > \pi/2 \tag{1.4}$$

for all $r \in (0, 1)$.

It is natural to ask what is the least value p such that $M_p(\mathcal{K}(r), \mathcal{E}(r)) > \pi/2$ for all $r \in (0, 1)$. The main purpose of this work is to answer this question. Our main result is the following Theorem 1.1.

Theorem 1.1. *Inequality*

$$M_p(\mathcal{K}(r), \mathcal{E}(r)) > \pi/2 \tag{1.5}$$

holds for all $r \in (0, 1)$ if and only if $p \geq -1/2$.

2. Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [4, Appendix E, pp. 474–475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}.$$

Lemma 2.1 *([4, Theorem 1.25]).* *For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b), and let $g(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. (1) $(\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})$ *is strictly increasing from* $(0, 1)$ *onto* $(1/2, 1)$*;*
(2) $\mathcal{E}/r'^{\frac{1}{2}}$ *is strictly increasing from* $(0, 1)$ *onto* $(\pi/2, +\infty)$*;*
(3) $r'^{1/2}\mathcal{K}$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, \pi/2)$*;*
(4) $(\mathcal{E} - r'^2\mathcal{K})/r^2$ *is strictly increasing from* $(0, 1)$ *onto* $(\pi/4, 1)$*;*
(5) $r'(\mathcal{K} - \mathcal{E})/r^2$ *is strictly decreasing from* $(0, 1)$ *onto* $(0, \pi/4)$*;*
(6) $\mathcal{E}(\mathcal{E} - r'^2\mathcal{K})/[r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})]$ *is strictly increasing from* $(0, 1)$ *onto* $(1, +\infty)$.

Proof. Parts (1)–(4) follow from [4, Exercise 3.43(32), Theorem 3.21(1), (7) and (8)].

For part (5), let $F(r) = r'(\mathcal{K} - \mathcal{E})/r^2$. Then

$$F'(r) = \frac{\mathcal{K}}{rr'}\left(1 - 2\frac{\mathcal{K} - \mathcal{E}}{r^2\mathcal{K}}\right). \tag{2.1}$$

It follows from (2.1), and parts (1) and (3) that $F'(r) < 0$ for $r \in (0, 1)$ and $F(r)$ is strictly decreasing in $(0, 1)$. Moreover, making use of (1.1) and (1.2) together with part (3) and l'Hôpital's rule we get $\lim_{r\to 1} F(r) = 0$ and $\lim_{r\to 0} F(r) = \pi/4$.

For part (6), note that

$$\frac{\mathcal{E}(\mathcal{E} - r'^2\mathcal{K})}{r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})} = \frac{\mathcal{E}}{r'^{\frac{1}{2}}} \cdot \frac{1}{r'^{\frac{1}{2}}\mathcal{K}} \cdot \frac{(\mathcal{E} - r'^2\mathcal{K})/r^2}{r'(\mathcal{K} - \mathcal{E})/r^2}. \tag{2.2}$$

Therefore, part (6) follows from (2.2) and parts (2)–(5). □

Lemma 2.3. *Let $r \in (0, 1)$. Then the function $f(r) \equiv \frac{\mathcal{E}(\mathcal{E} - r'^2\mathcal{K}) + r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})}{r^2\mathcal{K}\mathcal{E}^2}$ is strictly decreasing from $(0, 1)$ onto $(0, 2/\pi)$.*

Proof. By differentiation, we have

$$f'(r) = \frac{f_1(r)}{r^3 r'^2 \mathcal{K}^2 \mathcal{E}^3}, \tag{2.3}$$

where

$$\begin{aligned} f_1(r) &= 2r^2 r'^2 \mathcal{K}^2 \mathcal{E}(2\mathcal{E} - \mathcal{K}) - [\mathcal{E}(\mathcal{E} - r'^2\mathcal{K}) + r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})] \times (\mathcal{E}^2 + 3r'^2\mathcal{K}\mathcal{E} - 2r'^2\mathcal{K}^2) \\ &= (\mathcal{E} - r'^2\mathcal{K})(-\mathcal{E}^3 - 2r'^2\mathcal{K}\mathcal{E}^2 - 2r'^2\mathcal{K}^3 + 5r'^2\mathcal{K}^2\mathcal{E}) \\ &= -r'^2\mathcal{K}\mathcal{E}(\mathcal{K} - \mathcal{E})(\mathcal{E} - r'^2\mathcal{K})\left[\frac{\mathcal{E}(\mathcal{E} - r'^2\mathcal{K})}{r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})} + \frac{2\mathcal{K}}{\mathcal{E}} - 3\right]. \end{aligned} \tag{2.4}$$

From Lemma 2.2(6) we know that $\frac{\mathcal{E}(\mathcal{E}-r'^2\mathcal{K})}{r'^2\mathcal{K}(\mathcal{K}-\mathcal{E})} + \frac{2\mathcal{K}}{\mathcal{E}} - 3$ is strictly increasing from $(0, 1)$ onto $(0, \infty)$. Hence $f'(r) < 0$ for all $r \in (0, 1)$ follows from (2.3) and (2.4). Moreover, making use of (1.1) and (1.2) together with Lemma 2.2(3)–(5) and l'Hôpital's rule we get $\lim_{r\to 1} f(r) = 0$ and $\lim_{r\to 0} f(r) = 2/\pi$. □

Lemma 2.4. *Let $r \in (0, 1)$. Then the function $g(r) \equiv \frac{\mathcal{K}-\mathcal{E}-(\mathcal{E}-r'^2\mathcal{K})}{(\mathcal{E}-r'^2\mathcal{K})(\mathcal{K}-\mathcal{E})}$ is strictly increasing from $(0, 1)$ onto $(1/\pi, 1)$.*

Proof. Let $g_1(r) = \mathcal{K} - \mathcal{E} - (\mathcal{E} - r'^2\mathcal{K})$ and $g_2(r) = (\mathcal{E} - r'^2\mathcal{K})(\mathcal{K} - \mathcal{E})$; then we clearly see that $g_1(0) = g_2(0) = 0$, and

$$\begin{aligned} \frac{g_1{}'(r)}{g_2{}'(r)} &= \frac{r(\mathcal{E} - r'^2\mathcal{K})}{r\mathcal{E}(\mathcal{E} - r'^2\mathcal{K}) + rr'^2\mathcal{K}(\mathcal{K} - \mathcal{E})} \\ &= \frac{1/\mathcal{E}}{1 + r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})/[(\mathcal{E} - r'^2\mathcal{K})\mathcal{E}]}. \end{aligned} \tag{2.5}$$

It follows from (2.5) and Lemma 2.2(6) together with Lemma 2.1 that $g(r)$ is strictly increasing in $(0, 1)$ and $\lim_{r\to 0} g(r) = 1/\pi$. Moreover, the limiting value of $g(r)$ at $r = 1$ can be obtained from (1.1) and (1.2) together with Lemma 2.2(3). □

Lemma 2.5. *Let $p \in \mathbb{R}$. Then the function $h(r) \equiv \left(\frac{\mathcal{K}}{\mathcal{E}}\right)^{p-1} \frac{\mathcal{E}-r'^2\mathcal{K}}{r'^2(\mathcal{K}-\mathcal{E})}$ is strictly increasing in $(0, 1)$ if and only if $p \geq -1/2$, and there exists $\delta = \delta(p) \in (0, 1)$ such that $h(r) < 1$ for $r \in (0, \delta)$ and $h(r) > 1$ for $r \in (\delta, 1)$ if $p < -1/2$.*

Proof. Simple computations lead to

$$\lim_{r\to 0} h(r) = 1 \tag{2.6}$$

and

$$\begin{aligned} \frac{h'(r)}{h(r)} &= (p-1)\left(\frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2\mathcal{K}} + \frac{\mathcal{K} - \mathcal{E}}{r\mathcal{E}}\right) + \frac{r\mathcal{K}}{\mathcal{E} - r'^2\mathcal{K}} + \frac{2r}{r'^2} - \frac{r\mathcal{E}}{r'^2(\mathcal{K} - \mathcal{E})} \\ &= \frac{\mathcal{E}(\mathcal{E} - r'^2\mathcal{K}) + r'^2\mathcal{K}(\mathcal{K} - \mathcal{E})}{rr'^2\mathcal{K}\mathcal{E}}\left[p - 1 + \frac{1/\mathcal{E} + g(r)}{f(r)}\right], \end{aligned} \tag{2.7}$$

where $f(r)$ and $g(r)$ are defined as in Lemmas 2.3 and 2.4, respectively.

From Lemmas 2.3, 2.4 and (2.7) we clearly see that the function $(1/\mathcal{E} + g(r))/f(r)$ is strictly increasing from $(0, 1)$ onto $(3/2, \infty)$, so $h(r)$ is strictly increasing in $(0, 1)$ if and only if $p \geq -1/2$. Moreover, if $p < -1/2$, then it follows from (2.7) that there exists $r_0 \in (0, 1)$ such that $h'(r) < 0$ for $r \in (0, r_0)$, $h'(r) > 0$ for $r \in (r_0, 1)$, and $h(r)$ is strictly decreasing in $(0, r_0)$ and strictly increasing in $(r_0, 1)$. Therefore, Lemma 2.5 follows from (2.6) and

$$\begin{aligned} \lim_{r\to 1} h(r) &= \lim_{r\to 1} \frac{\mathcal{E}^{1-p}(\mathcal{E} - r'^2\mathcal{K})}{r'^2 r^2 \mathcal{K}^{2-p}(\mathcal{K} - \mathcal{E})/(r^2\mathcal{K})} = \lim_{r\to 1}\left(\frac{(1/r'^2)^{1/(2-p)}}{\mathcal{K}}\right)^{2-p} \\ &= \lim_{r\to 1} \frac{\left(\frac{2r^2}{(2-p)(\mathcal{E}-r'^2\mathcal{K})}\right)^{2-p}}{r'^2} = +\infty \end{aligned}$$

together with the monotonicity of $h(r)$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. If $p = 0$, then inequality (1.5) reduces to inequality (1.4). Thus, we only need to prove inequality (1.5) for $p \neq 0$. Let

$$G(r) = \frac{1}{p}\log\frac{\mathcal{K}(r)^p + \mathcal{E}(r)^p}{2} - \log\frac{\pi}{2}. \tag{3.1}$$

Then simple computation leads to

$$\begin{aligned} G'(r) &= \frac{\mathcal{K}^{p-1}(\mathcal{E} - r'^2\mathcal{K})/(rr'^2) - \mathcal{E}^{p-1}(\mathcal{K} - \mathcal{E})/r}{\mathcal{K}^p + \mathcal{E}^p} \\ &= \frac{\mathcal{E}^{p-1}(\mathcal{K} - \mathcal{E})/r}{\mathcal{K}^p + \mathcal{E}^p}\left[\left(\frac{\mathcal{K}}{\mathcal{E}}\right)^{p-1}\frac{\mathcal{E} - r'^2\mathcal{K}}{r'^2(\mathcal{K} - \mathcal{E})} - 1\right]. \end{aligned} \tag{3.2}$$

It follows from Eq. (3.2) and Lemma 2.5 that $G'(r) > 0$ for all $r \in (0, 1)$ if and only if $p \geq -1/2$. If $p \geq -1/2$, then $G(r)$ is strictly increasing in $(0, 1)$ and $G(r) > G(0) = 0$. Then from (3.1) we know that inequality (1.5) holds for all $r \in (0, 1)$ and $p \in [-1/2, \infty)$.

Finally, we prove that $p = -1/2$ is the best possible parameter such that inequality (1.5) holds for all $r \in (0, 1)$. If $p < -1/2$, then from (3.2) and Lemma 2.5 we know that there exist $\delta = \delta(p) \in (0, 1)$ and $\lambda \in (0, \delta)$ such that $G'(r) < 0$ and $G(r) < G(0) = 0$ for $r \in (0, \lambda)$. Then (3.1) implies that $M_p(\mathcal{K}(r), \mathcal{E}(r)) < \pi/2$ for $r \in (0, \lambda)$. □

Remark 3.1. For all $r \in (0, 1)$, we have

$$M_p(\mathcal{K}(r), \mathcal{E}(r)) < (1/2)^{1/p} \tag{3.3}$$

if $p \in (-\log 2/\log(\pi/2), 0)$, and

$$M_p(\mathcal{K}(r), \mathcal{E}(r)) < \pi/2 \tag{3.4}$$

if $p \in (-\infty, -\log 2/\log(\pi/2)]$.

Proof. We divide the proof into three cases. (1) If $-1/2 \leq p < 0$, then from (3.1) and the monotonicity of $G(r)$ we know that $G(r) < \lim_{r\to 1} G(r) = -\log 2/p - \log(\pi/2)$ and inequality (3.3) holds for all $r \in (0, 1)$. (2) If $p \in (-\log 2/\log(\pi/2), -1/2)$, then from (3.1) and (3.2) together with Lemma 2.5 we clearly see that $\sup_{r\in(0,1)} G(r) = \max\{G(0), \lim_{r\to 1} G(r)\} = \max\{0, -\log 2/p - \log(\pi/2)\} = -\log 2/p - \log(\pi/2)$ and inequality (3.3) again holds. (3) If $p \leq -\log 2/\log(\pi/2)$, then $\sup_{r\in(0,1)} G(r) = \max\{G(0), \lim_{r\to 1} G(r)\} = \max\{0, -\log 2/p - \log(\pi/2)\} = 0$ and inequality (3.4) holds. □

Remark 3.2. Inequalities (3.3) and (3.4) are sharp when $r \to 1$ and $r \to 0$, respectively.

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