Another realization of the category of modules over the small quantum group

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Abstract

Let $\mathfrak{g}$ be a semi-simple simply-connected Lie algebra and let U_ℓ be the corresponding quantum group with divided powers, where ℓ is an even order root of unity. Let in addition $u_\ell \subset U_\ell$ be the corresponding "small" quantum group. In this paper we establish the following relation between the categories of representations of U_ℓ and u_ℓ. We show that the category of u_ℓ-modules is naturally equivalent to the category of U_ℓ-modules, which have a *Hecke eigen-property* with respect to representations lifted by means of the quantum Frobenius map $U_\ell \to U(\check{\mathfrak{g}})$, where $\check{\mathfrak{g}}$ is the Langlands dual Lie algebra. This description allows to express the regular linkage class in the category u_ℓ-mod in terms of perverse sheaves on the affine flag variety with a Hecke eigen-property. Moreover, it can serve as a basis to the program to understand the connection between the category u_ℓ-mod and the category of representations of the corresponding affine algebra at the critical level.
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0. Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra. Given a root of unity (cf. Section 1.2), one can consider two remarkable algebras, $\mathbf{U}_\ell$ and $\mathfrak{u}_\ell$, called the *big* and the *small* quantum group, respectively. Let $\mathbf{U}_\ell$- and $\mathfrak{u}_\ell$-mod denote the corresponding categories of modules. It is explained in [1,15] that the former is an analog in characteristic 0 of the category of algebraic representations of the corresponding group G over a field of

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positive characteristic, and the latter is an analog of the category of representations of its first Frobenius kernel.

It is a fact of crucial importance, that although $\mathbf{U}_\ell$ is introduced as an algebra defined by an explicit set of generators and relations, the category $\mathbf{U}_\ell$-mod (or, rather, its regular block, cf. Section 5.1) can be described in purely geometric terms, as perverse sheaves on the (enhanced) affine flag variety $\widetilde{\mathrm{F1}}$, cf. Section 6.5. This is obtained by combining the Kazhdan–Lusztig equivalence between quantum groups and affine algebras and the Kashiwara–Tanisaki localization of modules over the affine algebra on $\widetilde{\mathrm{F1}}$. This paper is a first step in the project of finding a geometric realization of the category $\mathfrak{u}_\ell$-mod. We should say right away that one such realization already exists, and is a subject of [6]. However, we would like to investigate other directions.

We were motivated by a set of conjectures proposed by B. Feigin, E. Frenkel and G. Lusztig, which, on the one hand, tie the category $\mathfrak{u}_\ell$-mod to the (still hypothetical) category of perverse sheaves on the semi-infinite flag variety (cf. [7,8]), and on the other hand, relate the latter to the category of modules over the affine algebra at the critical level.

Since we already know the geometric interpretation for modules over the big quantum group, it is a natural idea to first express $\mathfrak{u}_\ell$-mod entirely in terms of $\mathbf{U}_\ell$-mod. This is exactly what we do in this paper.

According to [13], there is a functor Fr* from the category of finite-dimensional representations of the Langlands dual group to $\mathbf{U}_\ell$-mod. In particular, we obtain a bi-functor: $\check{G}$-mod$\times\mathbf{U}_\ell$-mod $\to\mathbf{U}_\ell$-mod: $V, M \to \mathrm{Fr}^*(V) \otimes M$. We introduce the category $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ to have as objects $\mathbf{U}_\ell$-modules M, which satisfy the *Hecke eigen-condition*, in the sense of [5].

In other words, an object of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ consists of $M \in \mathbf{U}_\ell$-mod and a collection of maps $\alpha_V: \mathrm{Fr}^*(V) \otimes M \to \underline{V} \otimes M$, where $\underline{V}$ is the vector space underlying the representation V. The main result of this paper is Theorem 2.4, which states that there is a natural equivalence between $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ and $\mathfrak{u}_\ell$-mod.

As the reader will notice, the proof of Theorem 2.4 is extremely simple. However, it allows one to give the desired description of the regular block $\mathfrak{u}_\ell$-mod$_0$ of the category of $\mathfrak{u}_\ell$-modules in terms of perverse sheaves on the enhanced affine flag variety satisfying the Hecke eigen-condition, cf. Section 6.4.

In a future publication, we will explain how Theorem 6.4 can be used to define a functor from $\mathfrak{u}_\ell$-mod$_0$ to the category of perverse sheaves on the semi-infinite flag variety and to other interesting categories that arise in representation theory. In particular, $\mathfrak{u}_\ell$-mod$_0$ obtains an interpretation in terms of the geometric Langlands correspondence: it can be thought of as a categorical counterpart of the space of Iwahori-invariant vectors in a spherical representation.

In another direction, Theorem 2.4 has as a consequence the theorem that $\mathfrak{u}_\ell$-mod is equivalent to the category of $G[[t]]$-integrable representations of the chiral Hecke algebra, introduced by Beilinson and Drinfeld. (We do not state this theorem explicitly, because the definition of the chiral Hecke algebra is still unavailable in the published literature.)

Let us briefly describe the contents of the paper. In Section 1 we recall the basic definitions concerning quantum groups. In Section 2 we state our main theorem and its generalization for pairs of bi-algebras $(\mathscr{A}, \mathfrak{a})$. In Section 3 we prove Theorem 2.4 in the general setting. In Section 4 we discuss several categorical interpretations of Theorem 2.4 and, in particular, its variant that concerns the graded version $\dot{\mathfrak{u}}_\ell$ of $\mathfrak{u}_\ell$. In Section 5 we discuss the relation between the block decompositions of $\mathbf{U}_\ell$ and $\mathfrak{u}_\ell$.

Finally, in Section 6 we prove Theorem 6.4, which provides a geometric interpretation for the category $\mathfrak{u}_\ell$-mod$_0$.

In this paper, we consider quantum groups at a root of unity of an even order, in order to be able to apply the Kazhdan–Lusztig equivalence. However, the main result, i.e., Theorem 2.4 holds and can be proved in exactly the same way in the case of a root of unity of an odd order, with the difference that in the definition of the quantum Frobenius, the Langlands dual group $\check{G}$ must be replaced by G.

1. Quantum groups

1.1. Root data. Let G be a semi-simple simply connected group. Let T be the Cartan group of G and let (I, X, Y) be the corresponding root data, where I is the set of vertices of the Dynkin diagram, X is the set of characters $T \to \mathbb{G}_m$ (i.e. the weight lattice of G) and Y is the set of co-characters $\mathbb{G}_m \to T$ (i.e. the coroot lattice of G). We will denote by $\langle \, , \, \rangle$ the canonical pairing $Y \times X \to \mathbb{Z}$. For every $i \in I$, $\alpha_i \in X$ (resp., $\check{\alpha}_i \in Y$) will denote the corresponding simple root (resp., coroot); for $i, j \in I$ we will denote by $a_{i,j}$ the corresponding entry of the Cartan matrix, i.e. $a_{i,j} = \langle \check{\alpha}_i, \alpha_j \rangle$.

Let $(\cdot, \cdot): X \times X \to \mathbb{Q}$ be the *canonical* inner form. In other words, $||\alpha_i||^2 = 2d_i$, where $d_i \in \{1, 2, 3\}$ is the minimal set of integers such that the matrix $(\alpha_i, \alpha_j) := d_i \cdot a_{i,j}$ is symmetric.

1.2. The big quantum group. Given the root data (I, Y, X) Drinfeld and Jimbo constructed a Hopf algebra $\mathbf{U}_v$ over the field $\mathbb{C}(v)$ of rational functions in v. Namely, $\mathbf{U}_v$ has as generators the elements[1] $E_i, F_i, i \in I, K_t, t \in T$ and the relations are:

$$K_{t_1} \cdot K_{t_2} = K_{t_1 \cdot t_2},$$

$$K_t \cdot E_i \cdot K_t^{-1} = \alpha_i(t) \cdot E_i, K_t \cdot F_i \cdot K_t^{-1} = \alpha_i(t^{-1}) \cdot F_i,$$

$$E_i \cdot F_j - F_j \cdot E_i = \delta_{i,j} \cdot \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \quad \text{where } K_i = K_{d_i \cdot \check{\alpha}_i(v)},$$

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[1] We are using a slightly non-standard version of $\mathbf{U}$, in which the toric part coincides with the group-algebra of the classical torus T.

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r \cdot E_j \cdot E_i^s = 0 \quad \text{if } i \neq j,$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r \cdot F_j \cdot F_i^s = 0 \quad \text{if } i \neq j,$$

where

$$\begin{bmatrix} m \\ t \end{bmatrix}_d := \prod_{s=1}^t \frac{v^{d \cdot (m-s+1)} - v^{-d \cdot (m-s+1)}}{v^{d \cdot s} - v^{-d \cdot s}} \quad \text{for } m \in \mathbb{Z}.$$

The co-product is given by the formulae:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta(K_t) = K_t \otimes K_t,$$

and the co-unit and antipode maps are

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_t) = 1,$$

$$\tau(K_t) = K_{t^{-1}}, \quad \tau(E_i) = -K_i^{-1} \cdot E_i, \quad \tau(F_i) = -F_i \cdot K_i.$$

Let now ℓ be a sufficiently large even natural number, which divides all the d_i's. We set $\ell_i = \ell/d_i$ and let us fix a primitive 2ℓth root of unity ζ. Let $R \subset \mathbb{C}(v)$ denote the localization of the algebra $\mathbb{C}[v, v^{-1}]$ at the ideal corresponding to $v - \zeta$.

In his book [13], Lusztig defined an R-lattice $\mathbf{U}_R$ inside $\mathbf{U}_v$. Namely, $\mathbf{U}_R$ is an R-subalgebra of $\mathbf{U}_v$ generated by E_i, F_i, K_t and the following additional elements:

$$E_i^{\ell_i} := \frac{E_i^{\ell_i}}{[\ell_i]_{d_i}!}, F_i^{\ell_i} := \frac{F_i^{\ell_i}}{[\ell_i]_{d_i}!}, \quad \text{where } [m]_d! = \prod_{s=1}^m \frac{v^{d \cdot s} - v^{-d \cdot s}}{v^d - v^{-d}}$$

and

$$\begin{bmatrix} K_i; m \\ t \end{bmatrix}_{d_i} := \prod_{s=1}^t \frac{K_i \cdot v^{d_i \cdot (m-s+1)} - K_i^{-1} \cdot v^{-d_i \cdot (m-s+1)}}{v^{d_i \cdot s} - v^{-d_i \cdot s}} \quad \text{for } m \in \mathbb{Z}.$$

It is shown in [13] that $\mathbf{U}_R$ is a Hopf subalgebra of $\mathbf{U}_v$. Finally, following Lusztig we set $\mathbf{U}_\ell$ to be the reduction of $\mathbf{U}_R$ modulo the ideal $(v - \zeta) \subset R$. By construction, $\mathbf{U}_\ell$ is a Hopf algebra over $\mathbb{C}$.

The main object of study of this paper is not so much the algebra $\mathbf{U}_\ell$ itself, but rather certain categories of its representations. We introduce the category $\mathbf{U}_\ell$-mod as follows: its objects are finite-dimensional representations M of $\mathbf{U}_\ell$, for which the

action of the K_t's comes from an algebraic action of the torus T on M, and such that for $\lambda \in X$, the action of $\begin{bmatrix} K_i; m \\ t \end{bmatrix}_{d_i}$ on the subspace of M of weight $\lambda \in X$ is given by the scalar $\begin{bmatrix} \langle \check{\alpha}_i, \lambda \rangle + m \\ t \end{bmatrix}_{d_i}$. (Note that the elements $\begin{bmatrix} m \\ t \end{bmatrix}_{d_i} \in \mathbb{C}(v)$ all belong to R, and hence they are well defined in $\mathbb{C} = R/(v - \zeta)$.)

The category $\mathbf{U}_\ell$-mod is a monoidal category endowed with a forgetful functor to the category of finite-dimensional $\mathbb{C}$-vector spaces. Hence, there exists a Hopf algebra, such that the category $\mathbf{U}_\ell$-mod is equivalent to the category of finite-dimensional co-modules over it. We will denote this Hopf algebra by $\mathscr{A}_G$.

One should think of $\mathscr{A}_G$ as of a quantization of the algebra of regular functions on the group G. It is known that $\mathscr{A}_G$ is finitely generated as an associative algebra. Moreover, we will see that $\mathscr{A}_G$ contains a large commutative subalgebra, over which it is finitely generated as a module.

1.3. Quantum Frobenius homomorphism. Let (I, X^*, Y^*) be the Langlands dual root data. In other words, $X^* := Y$ and $Y^* := X$ are the weight and the coweight lattices of the Langlands dual torus $\check{T}$. The corresponding semi-simple group $\check{G}$ is by definition of the adjoint type. Let $\check{\mathfrak{g}}$ denote the Lie algebra of $\check{G}$. Let $\check{G}$-mod denote the category of finite-dimensional $\check{G}$-modules and let $\mathcal{O}_{\check{G}}$ be the algebra of functions on $\check{G}$. We will denote by $\mathbf{U}(\check{\mathfrak{g}})$ the usual universal enveloping algebra of $\check{\mathfrak{g}}$.

The canonical inner form $(\cdot, \cdot)$ on X gives rise to the inner form on Y, which is not necessarily integral-valued, since $||\check{\alpha}_i|| = \frac{2}{d_i}$. However, if we multiply the latter by ℓ, we obtain an integral valued form $(\cdot, \cdot)_\ell : Y \times Y \to \mathbb{Z}$. By construction,

$$||\check{\alpha}_i||_\ell^2 = 2 \cdot \ell_i \text{ and } (\check{\mu}, \check{\alpha}_i)_\ell = \ell_i \cdot \langle \check{\mu}, \alpha_i \rangle.$$

Using the pairing $(\cdot, \cdot)_\ell$ we obtain the map $\phi : Y \to X$ given by $\check{\mu} \mapsto (\check{\mu}, \cdot)_\ell$ and the map $\phi_T : T \to \check{T}$.

Following Lusztig [13, Theorem 35.1.9] one defines the *quantum Frobenius morphism*. For us, this will be a functor

$$\text{Fr}^* : \check{G}\text{-mod} \to \mathbf{U}_\ell\text{-mod},$$

constructed as follows:

Starting with a $\check{G}$-module V, we define a $\mathbf{U}_\ell$-action on it by letting the torus T act via

$$T \xrightarrow{\phi_T} \check{T} \hookrightarrow \check{G},$$

which defines the action of the K_t's and the $\begin{bmatrix} K_i; m \\ t \end{bmatrix}_{d_i}$'s.

The generators E_i, F_i will act by 0, and $E_i^{(\ell_i)}, F_i^{(\ell_i)}$ will act as the corresponding Chevalley generators e_i and f_i of $\mathbf{U}(\check{\mathfrak{g}})$.

It is essentially a theorem of Lusztig [13, Theorem 35.1.9] that the above formulae indeed define an action of $\mathbf{U}_\ell$ on V. Moreover, from [13] it follows that the functor

Fr* preserves the tensor structure and is full. Hence, we obtain an injective homomorphism of Hopf algebras $\phi_G : \mathcal{O}_{\check{G}} \to \mathscr{A}_G$.

Let $\check{G}_{sc}$ be the simply connected cover of the group $\check{G}$ and let X^*_{sc}, Y^*_{sc} and $\check{T}_{sc}$ be the corresponding objects for $\check{G}_{sc}$. In particular, Y identifies with the coroot lattice inside the coweight lattice X^*_{sc}, and $Y^*_{sc} = \text{Span}(\alpha_i)$.

Since $\phi(\check{\alpha}_i) = \ell_i \cdot \alpha_i$, we obtain that the map ϕ gives rise to a map $\phi_{sc} : X^*_{sc} \to X$.

Therefore, we have a map $\phi_{T,sc} : T \to \check{T}_{sc}$ and the functor $\text{Fr}^* : \check{G}\text{-mod} \to \mathbf{U}_\ell\text{-mod}$ can be extended to a tensor functor $\text{Fr}^*_{sc} : \check{G}_{sc}\text{-mod} \to \mathbf{U}_\ell\text{-mod}$ by the same formula.

1.4. The small quantum group. Following Lusztig, we first define the graded version of the small quantum group, denoted $\overset{\bullet}{\mathfrak{u}}_\ell$.

By definition, this is a sub-algebra of $\mathbf{U}_\ell$ generated by $E_i, F_i, i \in I$ and all the K_t's.

From the formula for coproduct of the above generators, it follows that $\overset{\bullet}{\mathfrak{u}}_\ell$ is in fact a Hopf subalgebra of $\mathbf{U}_\ell$.

We define the category $\overset{\bullet}{\mathfrak{u}}_\ell\text{-mod}$ to consist of all finite-dimensional $\mathfrak{u}_\ell$-modules M, on which the action of the K_t's comes from an algebraic action of T on M.

The restriction functor $\mathbf{U}_\ell\text{-mod} \to \overset{\bullet}{\mathfrak{u}}_\ell\text{-mod}$ corresponds to a map of Hopf algebras $\mathscr{A}_G \to \overset{\bullet}{\mathfrak{a}}_G$.

Finally, we are ready to introduce our main object of study–the *small quantum group*, $\mathfrak{u}_\ell$. One would want it to be a Hopf subalgebra of $\mathbf{U}_\ell$, universal with the property that it acts trivially on representations of the form $\text{Fr}^*(V)$.

When one works with a root of unity of an odd order, the corresponding subalgebra is just generated by K_i, E_i and $F_i, i \in I$. However, in the case of a root of unity of an even order considered in the present paper, it appears that a *Hopf subalgebra* with such properties does not exist.

In our definition, $\mathfrak{u}_\ell$ will be just an associative subalgebra of $\mathbf{U}_\ell$, generated by $K_i E_i, F_i, i \in I$ and K_t for $K_t \in \ker(\phi_T)$. It is easy to see that $\mathfrak{u}_\ell$ is finite-dimensional.

We define the category $\mathfrak{u}_\ell\text{-mod}$ to have as objects all finite-dimensional $\mathfrak{u}_\ell$-modules.

By construction, we have a restriction functor Res : $\mathbf{U}_\ell\text{-mod} \to \mathfrak{u}_\ell\text{-mod}$. It corresponds to a homomorphism of co-algebras $\mathscr{A}_G \to \mathfrak{a}_G$.

Note that although the co-product on $\mathbf{U}_\ell$ does not preserve $\mathfrak{u}_\ell$, it maps it to $\mathfrak{u}_\ell \otimes \mathbf{U}_\ell$. This means that $\mathfrak{a}_G$ has a structure of a right $\mathscr{A}_G$-module. In categorical terms, we have a well-defined functor $(M \in \mathfrak{u}_\ell\text{-mod}, N \in \mathbf{U}_\ell\text{-mod}) \mapsto M \otimes \text{Res}(N) \in \mathfrak{u}_\ell\text{-mod}$.

Remark. As we shall see later, although $\mathfrak{a}_G$ is not a Hopf algebra, the category $\mathfrak{u}_\ell$-mod will be in fact a monoidal category. The "paradox" is explained as follows: the tautological forgetful functor $\mathfrak{u}_\ell\text{-mod} \to \{\text{Vector spaces}\}$ cannot be made into a tensor functor so that the composition

$$\mathbf{U}_\ell\text{-mod} \overset{\text{Res}}{\to} \mathfrak{u}_\ell\text{-mod} \to \{\text{Vector spaces}\}$$

is the standard fiber functor on $\mathbf{U}_\ell$-mod.

Consider now the restriction of the quantum Frobenius to $\mathfrak{u}_\ell$, i.e. the composition

$$\check{G}\text{-mod} \overset{\mathrm{Fr}^*}{\to} \mathbf{U}_\ell\text{-mod} \overset{\mathrm{Res}}{\to} \mathfrak{u}_\ell\text{-mod}.$$

From the formula for Fr^* it is easy to see that it factors through the forgetful functor $V \mapsto \underline{V}$ from $\check{G}$-mod to vector spaces, i.e.

$$\check{G}\text{-mod} \to \{\text{Vector spaces}\} \overset{\text{co-unit}}{\to} \mathfrak{u}_\ell\text{-mod}.$$

Moreover, we have the following assertion [13, Theorem 35.1.9]:

Proposition 1.5. *Let M be an object of $\mathbf{U}_\ell$-mod. Then*

  (1) *The subspace of $\mathfrak{u}_\ell$-invariants $M^{\mathfrak{u}_\ell} \subset M$ (i.e. $m \in M^{\mathfrak{u}_\ell}$ if $u \cdot m = \varepsilon(u) \cdot m$ for $u \in \mathfrak{u}_\ell$) is $\mathbf{U}_\ell$-stable.*
  (2) *If the $\mathfrak{u}_\ell$-action on M is trivial, there exists a (unique up to a unique isomorphism) $\check{G}$-module V such that $M \simeq \mathrm{Fr}^*(V)$.*

 For completeness, let us sketch the proof of the second part of this proposition.

Proof. From the short exact sequence $1 \to \ker(\phi_T) \to T \to \check{T} \to 1$ we obtain the T-action on M comes from a $\check{T}$-action.

  In particular, the element $\begin{bmatrix} K_i ; 0 \\ \ell_i \end{bmatrix}_{d_i}$ acts on M as the Lie algebra element $h_i \in \check{\mathfrak{g}}$.

  We define the action of e_i and f_i as $E_i^{(\ell_i)}$ and $F_i^{(\ell_i)}$, respectively, and we need just to check that the relation $[e_i, f_i] = h_i$ holds. But this follows from the formula

$$[E_i^{(\ell_i)}, F_i^{(\ell_i)}] = \sum_{0 \leqslant k < \ell_i} \left(\frac{1}{[k]_{d_i}!}\right)^2 \cdot (E_i)^k \cdot \begin{bmatrix} K_i ; 2k \\ \ell_i - k \end{bmatrix}_{d_i} \cdot (F_i)^k,$$

and all the terms but $\begin{bmatrix} K_i ; 0 \\ \ell_i \end{bmatrix}_{d_i}$ belong to the two-sided ideal generated by the E_i's and the F_i's. $\square$

  In a similar fashion one defines the "simply connected" version of $\mathfrak{u}_\ell$, which we will denote by $\mathfrak{u}_{\ell,\mathrm{sc}}$. By definition, this is an associative subalgebra of $\mathfrak{u}_\ell$ generated by $K_i E_i$, F_i, $i \in I$ and K_t for $K_t \in \ker(\phi_{T,\mathrm{sc}})$.

  The category $\mathfrak{u}_{\ell,\mathrm{sc}}$-mod and the co-algebra $\mathfrak{a}_{G\mathrm{sc}}$ are defined in a similar way. The analog of Proposition 1.5 holds for $\mathfrak{u}_\ell$ replaced by $\mathfrak{u}_{\ell,\mathrm{sc}}$ and $\check{G}$ replaced by $\check{G}_{\mathrm{sc}}$, respectively.

2. The main result

2.1. The category $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$. We now come to the definition crucial for this paper. To avoid redundant repetitions, we will work with $\mathfrak{u}_\ell$ (resp., $\check{G}$), while the case of $\mathfrak{u}_{\ell,\mathrm{sc}}$ (resp., $\check{G}_{\mathrm{sc}}$) can be treated similarly.

Let us consider the ind-completions of the categories $\mathbf{U}_\ell$-, $\mathfrak{u}_\ell$- and $\check{G}$-mod. Each of these categories consists of all co-modules over the corresponding co-algebra, i.e. $\mathscr{A}_G$, $\mathfrak{a}_G$ or $\mathcal{O}_{\check{G}}$, respectively.

We define the category $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ to have as objects vector spaces M endowed with an action of the *algebra* $\mathcal{O}_{\check{G}}$ and with a co-action of the *co-algebra* $\mathscr{A}_G$ compatible in the following natural way:

$$\text{co-ac}(f \cdot m) = \Delta(f) \cdot (\text{co-ac}(m)).$$

Here $f \in \mathcal{O}_{\check{G}}$, $m \in M$, co-ac $: M \to \mathscr{A}_G \otimes M$ denotes the co-action map, the element $\Delta(f)$ belongs to $\mathcal{O}_{\check{G}} \otimes \mathcal{O}_{\check{G}} \subset \mathscr{A}_G \otimes \mathscr{A}_G$ and acts on $\mathscr{A}_G \otimes M$. Morphisms in this category are the ones preserving both the action and the co-action.

In other words, we need that the action map $\mathcal{O}_{\check{G}} \otimes M \to M$ is a map of $\mathscr{A}_G$-comodules, or equivalently, that the co-action map $M \to \mathscr{A}_G \otimes M$ is the map of $\mathcal{O}_{\check{G}}$-modules.

An example of an object of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ is $M = \mathcal{O}_{\check{G}}$, with the natural $\mathscr{A}_G$-co-action (coming from the fact that $\mathcal{O}_{\check{G}}$ is a Hopf subalgebra in $\mathscr{A}_G$) and the $\mathcal{O}_{\check{G}}$-action. Another basic example is $M = \mathscr{A}_G$.

2.2. A reformulation. Here is a more "geometric" way to formulate this definition. We claim that the category $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ is equivalent to the category of pairs

$$(M \in \mathscr{A}_G\text{-comod}, \{\alpha_V, \forall V \in \check{G}\text{-mod}\}),$$

where each α_V is a map of $\mathscr{A}_G$-comodules (i.e., of $\mathbf{U}_\ell$-modules)

$$\alpha_V : \mathrm{Fr}^*(V) \otimes M \to \underline{V} \otimes M$$

(recall that for $V \in \check{G}$-mod, the notation $\underline{V}$ stands for the underlying vector space), such that

- For $V = \mathbb{C}$, $\alpha_V : M \to M$ is the identity map.
- For a map $V_1 \to V_2$, the diagram

$$
\begin{array}{ccc}
\mathrm{Fr}^*(V_1) \otimes M & \xrightarrow{\alpha_{V_1}} & \underline{V_1} \otimes M \\
\downarrow & & \downarrow \\
\mathrm{Fr}^*(V_2) \otimes M & \xrightarrow{\alpha_{V_2}} & \underline{V_2} \otimes M
\end{array}
$$

commutes.

• A compatibility with tensor products holds in the sense that the map

$$\mathrm{Fr}^*(V_1)\otimes\mathrm{Fr}^*(V_2)\otimes M\to\mathrm{Fr}^*(V_1\otimes V_2)\otimes M\overset{\alpha_{V_1\otimes V_2}}{\to}\underline{V_1\otimes V_2}\otimes M\to\underline{V_2}\otimes\underline{V_1}\otimes M$$

equals $\mathrm{Fr}^*(V_1)\otimes\mathrm{Fr}^*(V_2)\otimes M\overset{\mathrm{id}\otimes\alpha_{V_2}}{\to}\mathrm{Fr}^*(V_1)\otimes\underline{V_2}\otimes M$

$$\simeq\underline{V_2}\otimes\mathrm{Fr}^*(V_1)\otimes M\overset{\mathrm{id}\otimes\alpha_{V_1}}{\to}\underline{V_2}\otimes\underline{V_1}\otimes M.$$

Morphisms in this category between (M,α_V) and (M',α'_V) are $\mathbf{U}_\ell$-module maps $M\to M'$, such that each square

$$
\begin{array}{ccc}
\mathrm{Fr}^*(V)\otimes M & \overset{\alpha_V}{\longrightarrow} & \underline{V}\otimes M \\
\downarrow & & \downarrow \\
\mathrm{Fr}^*(V)\otimes M' & \overset{\alpha'_V}{\longrightarrow} & \underline{V}\otimes M'
\end{array}
$$

commutes.

Indeed, given M as above we define the action of $\mathcal{O}_{\check{G}}$ on it as the composition map

$$\mathrm{Fr}^*(\mathcal{O}_{\check{G}})\otimes M\overset{\alpha_{\mathcal{O}_{\check{G}}}}{\to}\underline{\mathcal{O}_{\check{G}}}\otimes M\overset{\varepsilon\otimes\mathrm{id}}{\to}M,$$

where ε is the co-unit $f\mapsto f(1)$ in $\mathcal{O}_{\check{G}}$. Conversely, given $M\in\mathfrak{C}(\mathscr{A}_G,\mathcal{O}_{\check{G}})$, the map α_V comes by adjunction from the map

$$\mathrm{Fr}^*(V)\otimes\underline{V^*}\otimes M\overset{\mathrm{matr.coef.}\otimes\mathrm{id}}{\to}\mathrm{Fr}^*(\mathcal{O}_{\check{G}})\otimes M\to M.$$

Let us make the following observation:

Proposition 2.3. *For $(M,\alpha_V)\in\mathfrak{C}(\mathscr{A}_G,\mathcal{O}_{\check{G}})$, the maps α_V are automatically isomorphisms.*

Proof. Let N be the kernel of the map $\mathrm{Fr}^*(V)\otimes M\to\underline{V}\otimes M$ and let $V^*\in\check{G}$-mod be the dual of V. From the axioms on the α_V's, we obtain that the composition

$$\mathrm{Fr}^*(V^*)\otimes N\to\mathrm{Fr}^*(V^*\otimes V)\otimes M\to\underline{V^*\otimes V}\otimes M\to M$$

is on the one hand zero, and on the other hand equals the natural map $\mathrm{Fr}^*(V^*)\otimes N\to M$, which is a contradiction. The surjectivity of α_V is proved in the same way.　□

Our main result is the following theorem:

Theorem 2.4. *The category* $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ *is naturally equivalent to the category of* $\mathfrak{a}_G$-*comodules. Objects in* $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$, *which are finitely generated over* $\mathcal{O}_{\check{G}}$, *correspond under this equivalence to finite-dimensional* $\mathfrak{a}_G$-*comodules.*

This theorem has the following interesting corollary:

Corollary 2.5. *The Langlands dual group* $\check{G}$ *acts on the category* $\mathfrak{u}_\ell$-mod *by endofunctors. In other words*:

(1) *for every* $\gamma \in \check{G}$ *there is a functor* $T_\gamma : \mathfrak{u}_\ell$-mod $\to \mathfrak{u}_\ell$-mod,
(2) *for each pair* $\gamma_1, \gamma_2 \in \check{G}$ *there is an isomorphism of functors* $T_{\gamma_1} \circ T_{\gamma_2} \Rightarrow T_{\gamma_1 \cdot \gamma_2}$,
(3) *for each triple* $\gamma_1, \gamma_2, \gamma_3$ *the two natural transformations* $T_{\gamma_1} \circ T_{\gamma_2} \circ T_{\gamma_3} \Rightarrow T_{\gamma_1 \cdot \gamma_2 \cdot \gamma_3}$ *coincide.*

Proof. Let us view $\mathfrak{u}_\ell$-modules as objects of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ via Theorem 2.4.

Given an object $(M, \alpha_V) \in \mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ and an element $\gamma \in \check{G}$ we define a new object $T_\gamma(M, \alpha_V)$ as follows:

The underlying $\mathbf{U}_\ell$-module is the same, i.e. M. However, the corresponding morphism

$$\mathrm{Fr}^*(V) \otimes M \to \underline{V} \otimes M$$

is the old α_V composed with $\underline{V} \otimes M \overset{\gamma \otimes \mathrm{id}}{\to} \underline{V} \otimes M$, where $\gamma \in \check{G}$ is viewed as an automorphism of the vector space $\underline{V}$.

It is clear that in this way we indeed obtain an action of $\check{G}$ on $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$, and hence on $\mathfrak{a}_G$-comod, by endo-functors. $\square$

Another corollary of Theorem 2.4 is as follows:

Corollary 2.6. *The category* $\mathfrak{u}_\ell$-mod *has a natural monoidal structure.*

Proof. Given two objects (M, α_V) and (M', α'_V) in $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ we have to define their tensor product (M'', α''_V) as a new object of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$.

Consider first their naive tensor product $M \otimes M'$ as a $\mathbf{U}_\ell$-module. We claim that the algebra $\mathcal{O}_{\check{G}}$ acts on it by endomorphisms. Indeed, to define such an action, it is enough to define $\mathbf{U}_\ell$-module maps

$$\underline{V} \otimes (M \otimes M') \to \underline{V} \otimes (M \otimes M')$$

for every $V \in \check{G}$-mod, compatible with the tensor structure on $\check{G}$-mod in the same sense as in the definition of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$.

The sought-for maps are defined as follows:

$$\underline{V} \otimes M \otimes M' \overset{\alpha_V}{\simeq} (\mathrm{Fr}^*(V) \otimes M) \otimes M' \simeq M \otimes (\mathrm{Fr}^*(V) \otimes M')$$

$$\overset{\alpha'_V}{\simeq} \underline{V} \otimes M \otimes M',$$

where the second arrow comes from the braiding on the category $\mathbf{U}_\ell$-mod.

The $\mathbf{U}_\ell$-module M'' is defined as the fiber at $1 \in \check{G}$ of $M \otimes M'$ viewed as a quasi-coherent sheaf on $\check{G}$. It comes equipped with a data of α'' by construction.

It is easy to see that the functor $(M, \alpha_V), (M', \alpha') \mapsto (M'', \alpha''_V)$ admits a natural associativity constraint, which makes $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ into a monoidal category. Moreover, if both M and M' are finitely generated as $\mathcal{O}_{\check{G}}$-modules, then so is M''. Hence, this monoidal structure preserves the sub-category of finite-dimensional $\mathfrak{a}_G$-comodules, which is the same as $\mathfrak{u}_\ell$-mod. $\square$

2.7. The general setting.

It will be convenient to generalize our setting as follows. Let $\mathcal{O}, \mathscr{A}$ be two Hopf algebras and let $\mathcal{O} \to \mathscr{A}$ be an embedding.

In addition, let $\mathfrak{a}$ be a co-algebra and a right $\mathscr{A}$-module, and let $\mathscr{A} \to \mathfrak{a}$ be a surjection respecting both structures.

We impose the following conditions on our data:

(i) The composition $\mathcal{O} \to \mathscr{A} \to \mathfrak{a}$ factors as $\mathcal{O} \overset{\text{co-unit}}{\to} \mathbb{C} \overset{\text{unit}}{\to} \mathfrak{a}$.
(ii) The inclusion $\mathcal{O} \subset \mathscr{A}^{\mathfrak{a}}$ is an equality.[2]
(iii) The inclusion $\mathfrak{m} \cdot \mathscr{A} \subset \mathrm{Ker}(\mathscr{A} \to \mathfrak{a})$ is an equality, where $\mathfrak{m}$ is the augmentation ideal in $\mathcal{O}$.

In addition, we impose the following technical condition, that one of the following two properties is satisfied (compare with [16, Section 3.4]):

(iva) Either $\mathscr{A}$ is faithfully-flat as an $\mathcal{O}$-module,
(ivb) or the induction functor Ind : $\mathfrak{a}$-comod $\to \mathscr{A}$-comod (cf. Section 3.1 for the definition of Ind) is exact and faithful.

Of course, we will prove that our triple $(\mathcal{O}_{\check{G}}, \mathscr{A}_G, \mathfrak{a}_G)$ satisfies conditions (i)–(iv).

We define the category $\mathfrak{C}(\mathscr{A}, \mathcal{O})$ to have as objects vector spaces M endowed with a left action of the *algebra* $\mathcal{O}$ and a left co-action of the *co-algebra* $\mathscr{A}$ which are compatible in the same sense as in the definition of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$. The following is a generalization of Theorem 2.4:

---

[2] For a co-algebra $\mathscr{B}$ co-acting on M, the notation $M^{\mathscr{B}}$ will mean "invariants", i.e.

$$M^{\mathscr{B}} = \mathrm{Hom}_{\mathscr{B}\text{-comod}}(\mathbb{C}, M) = \mathrm{Ker}(M \overset{\text{co-ac}-1 \otimes \mathrm{id}}{\to} \mathscr{B} \otimes M).$$

For example, for a $\mathfrak{u}_\ell$-module M (which is the same as an $\mathfrak{a}_G$-comodule) $M^{\mathfrak{u}_\ell} = M^{\mathfrak{a}_G}$.

Theorem 2.8. *The categories* $\mathfrak{C}(\mathscr{A}, \mathscr{O})$ *and* $\mathfrak{a}$-comod *are naturally equivalent.*

2.9. The "classical" case. The general Theorem 2.8 models the following familiar situation. Let

$$1 \to H' \to H'' \to H \to 1$$

be a short exact sequence of linear algebraic groups. Take $\mathscr{O} = \mathscr{O}_H$, $\mathscr{A} = \mathscr{O}_{H''}$ and $\mathfrak{a} = \mathscr{O}_{H'}$. Conditions (i)–(iv) obviously hold and we claim, that the assertion of Theorem 2.8 in this case is following well-known phenomenon:

First, by definition, the category $\mathfrak{C}(\mathscr{A}, \mathscr{O})$ is naturally equivalent to the category $\mathrm{QCoh}^{H''}(H)$ of H''-equivariant quasi-coherent sheaves on H. By taking the fiber of a sheaf at $1 \in H$ we obtain a functor

$$\mathrm{QCoh}^{H''}(H) \to \mathrm{QCoh}^{H'}(\mathrm{pt}),$$

which is known to be an equivalence of categories. However,

$$\mathrm{QCoh}^{H'}(\mathrm{pt}) \simeq H'\text{-mod} \simeq \mathfrak{a}\text{-comod}.$$

The proof of Theorem 2.8 in the general case will be essentially a translation of the above two-line proof into the language of Hopf algebras.

Remark. Suppose that in the setting of Theorem 2.8, $\mathscr{O}$ is in fact commutative, i.e. $\mathscr{O}$ is the algebra of functions on an affine group-scheme Γ.

Then we have an analog of dual group action, that Γ acts on the category $\mathfrak{a}$-comod by endo-functors. In the above example of $(\mathscr{A} = \mathscr{O}_{H''}, \mathfrak{a} = \mathscr{O}_{H'})$, this action corresponds to the natural map of $\Gamma = H$ to the group of *outer* automorphisms of H'.

3. Proof of the main theorem

We will first prove the general Theorem 2.8. Then we show that conditions (i)–(iv) are satisfied for $\mathscr{A}_G$, $\mathscr{O}_{\check{G}}$ and $\mathfrak{a}_G$.

3.1. The functor of (finite) induction. Now we proceed to the proof of Theorem 2.8 in general.

Let us recall the definition of the (finite) induction functor $\mathfrak{a}$-comod $\mapsto \mathscr{A}$-comod. Recall that if M_1^{r} is a right co-module and M_2^{l} is a left co-module over a co-algebra $\mathfrak{a}$, it makes sense to consider the vector space $(M_1^{\mathrm{r}} \otimes M_2^{\mathrm{l}})^{\mathfrak{a}}$, equal by definition to the equalizer of the two maps

$$\Delta_1 \otimes \mathrm{id}, \mathrm{id} \otimes \Delta_2 : M_1^{\mathrm{r}} \otimes M_2^{\mathrm{l}} \to M_1^{\mathrm{r}} \otimes \mathfrak{a} \otimes M_2^{\mathrm{l}}.$$

For $M \in \mathfrak{a}$-comod consider $\mathscr{A}$ as a left $\mathscr{A}$-comodule and a right $\mathfrak{a}$-comodule, and set $\mathrm{Ind}(M) := (\mathscr{A} \otimes M)^{\mathfrak{a}}$, which carries a left $\mathscr{A}$-coaction by functoriality.

By construction, this functor is left exact, and it is the right adjoint of the natural restriction functor Res $: \mathscr{A}$-comod $\to \mathfrak{a}$-comod.

Note now that since $\mathcal{O} \to \mathscr{A} \to \mathfrak{a}$ factors through the augmentation map, the action of $\mathcal{O}$ on $\mathscr{A} \otimes M$ by left multiplication maps the subspace $(\mathscr{A} \otimes M)^{\mathfrak{a}}$ to itself. Moreover, this action $\mathcal{O}$-action on $\mathrm{Ind}(M)$ is compatible with the $\mathscr{A}$-coaction. Therefore, the functor Ind can be extended to a functor from $\mathfrak{a}$-comod to $\mathfrak{C}(\mathscr{A}, \mathcal{O})$, which we will denote by $\underline{\mathrm{Ind}}$.

Let us consider two examples. First, it is easy to see that $\underline{\mathrm{Ind}}(\mathfrak{a}) \simeq \mathscr{A}$. Secondly, $\underline{\mathrm{Ind}}(\mathbb{C}) \simeq \mathcal{O}$, and more generally, for M is of the form $\mathrm{Res}(N)$ for $N \in \mathscr{A}$-comod, we have: $\underline{\mathrm{Ind}}(\mathrm{Res}(N)) \simeq \mathcal{O} \otimes N$, with the diagonal $\mathscr{A}$-coaction and the $\mathcal{O}$-action on the first factor.

3.2. The adjoint functor. Now we will define a functor $\mathfrak{C}(\mathscr{A}, \mathcal{O}) \to \mathfrak{a}$-comod.

Given an object $N \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$, consider the vector space $\Psi(N) := \mathbb{C} \otimes_{\mathcal{O}} N$, where $\mathcal{O} \to \mathbb{C}$ is the co-unit map.

Since the $\mathcal{O}$-action on N commutes with the $\mathfrak{a}$-coaction, $\Psi(N)$ carries a natural co-action of $\mathfrak{a}$.

Thus, we obtain a functor, denoted $\underline{\mathrm{Res}} : \mathfrak{C}(\mathscr{A}, \mathcal{O}) \to \mathfrak{a}$-comod. By construction, this functor is right exact.

By definition, for $\mathcal{O}$ viewed as an object of $\mathfrak{C}(\mathscr{A}, \mathcal{O})$, $\underline{\mathrm{Res}}(\mathcal{O}) \simeq \mathbb{C}$. Property (iii) of Section 2.7 implies that $\underline{\mathrm{Res}}(\mathscr{A}) \simeq \mathfrak{a}$. More generally, for objects of $\mathfrak{C}(\mathscr{A}, \mathcal{O})$ of the form $\mathcal{O} \otimes \mathrm{Res}(N)$, we have: $\underline{\mathrm{Res}}(\mathcal{O} \otimes \mathrm{Res}(N)) \simeq \mathrm{Res}(N)$.

Proposition 3.3. *The functor* $\underline{\mathrm{Res}}$ *is the left adjoint to* $\underline{\mathrm{Ind}}$.

Proof. We need to construct the adjunction maps

$$\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(M) \to M \text{ and } N \to \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N)$$

for M and N in $\mathfrak{a}$-mod and $\mathfrak{C}(\mathscr{A}, \mathcal{O})$, respectively.

Let M be as above. Consider the composition $(\mathscr{A} \otimes M)^{\mathfrak{a}} \hookrightarrow \mathscr{A} \otimes M \overset{\varepsilon \otimes \mathrm{id}}{\to} M$. By construction, this is a map of $\mathfrak{a}$-comodules and it obviously factors through

$$(\mathscr{A} \otimes M)^{\mathfrak{a}} \to \Psi((\mathscr{A} \otimes M)^{\mathfrak{a}}) \to M.$$

Therefore, we obtain a map $\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(M) \simeq \Psi((\mathscr{A} \otimes M)^{\mathfrak{a}}) \to M$.

For $N \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$, consider the map

$$N \overset{\Delta}{\to} (\mathscr{A} \otimes N)^{\mathscr{A}} \hookrightarrow (\mathscr{A} \otimes N)^{\mathfrak{a}} \to (\mathscr{A} \otimes \Psi(N))^{\mathfrak{a}}.$$

This map respects the $\mathscr{A}$-coaction and the $\mathcal{O}$-action by construction.

Thus, we obtain a map

$$N \to (\mathscr{A} \otimes \Psi(N))^{\mathfrak{a}} \simeq \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N). \qquad \square$$

Now we are ready to prove Theorem 2.8. We will give two proofs corresponding to the two variants of condition (iv).

3.4. Proof 1. Let us first prove Theorem 2.8 under the assumption that $(\mathcal{O}, \mathscr{A}, \mathfrak{a})$ satisfies condition (ivb) of Section 2.1, i.e. that the induction functor Ind is exact and faithful.

We claim that the adjunction map

$$N \to \underline{\mathrm{Ind}}(\underline{\mathrm{Res}}(N))$$

is an isomorphism for any $N \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$.

Since the functor $\underline{\mathrm{Res}}$ is right-exact and Ind is exact, the composition $\underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}$ is also right exact. Hence, it suffices to show that for any N as above there exists another object $N' \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$ with a surjection $N' \twoheadrightarrow N$, for which the map $N' \to \underline{\mathrm{Ind}}(\underline{\mathrm{Res}}(N'))$ is an isomorphism.

We set $N' = \mathcal{O} \otimes N$, where $\mathcal{O}$ *acts on* $\mathcal{O} \otimes N$ via

$$a' \cdot (a \otimes n) \mapsto a' \cdot a \otimes n,$$

and the $\mathscr{A}$-coaction is the diagonal one. The map $\mathcal{O} \otimes N \to N$ is given by the original $\mathcal{O}$-action on N.

Now, $\underline{\mathrm{Res}}(N') \simeq \underline{\mathrm{Res}}(N)$, and $\underline{\mathrm{Ind}}(\underline{\mathrm{Res}}(N')) \simeq \mathcal{O} \otimes N$, such that the above adjunction map for N' becomes the identity map on $\mathcal{O} \otimes N$.

Thus, to prove Theorem 2.8, it suffices to check that the other adjunction map $\underline{\mathrm{Res}}(\underline{\mathrm{Ind}}(M)) \to M$ is an isomorphism for any $M \in \mathfrak{a}$-comod. However, since the functor Ind (and hence $\underline{\mathrm{Ind}}$) is faithful, it suffice to check that

$$\underline{\mathrm{Ind}}(\underline{\mathrm{Res}}(\underline{\mathrm{Ind}}(M))) \to \underline{\mathrm{Ind}}(M)$$

is an isomorphism. However, we know that the composition

$$\underline{\mathrm{Ind}}(M) \to \underline{\mathrm{Ind}}(\underline{\mathrm{Res}}(\underline{\mathrm{Ind}}(M))) \to \underline{\mathrm{Ind}}(M)$$

is the identity map on $\underline{\mathrm{Ind}}(M)$, and the first arrow is an isomorphism by what we have proved above. Hence, the second arrow is an isomorphism as well.

3.5. Proof 2. Now let us prove Theorem 2.8 under the assumption that $(\mathcal{O}, \mathscr{A}, \mathfrak{a})$ satisfies condition (iva) of Section 2.7.

Proposition 3.6. *The functor* $\underline{\mathrm{Res}} : \mathfrak{C}(\mathscr{A}, \mathcal{O}) \to \mathfrak{a}$-comod *is exact and faithful.*

Proof. For an object $N \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$, consider the tensor product $\mathscr{A} \otimes_{\mathcal{O}} N$. This is a left $\mathscr{A}$-module and a left $\mathscr{A}$-comodule via the diagonal co-action.

Thus, we obtain a functor $\mathrm{Coind}_{\mathcal{O}}^{\mathscr{A}} : \mathfrak{C}(\mathscr{A}, \mathcal{O}) \to \mathfrak{C}(\mathscr{A}, \mathscr{A})$, which is exact and faithful, since $\mathscr{A}$ was assumed faithfully flat over $\mathcal{O}$.

Now, the functor $N \mapsto \Psi(N)$ considered as a functor from $\mathfrak{C}(\mathscr{A}, \mathcal{O})$ to the category of vector spaces can be factored as

$$\Psi \simeq \Psi_{\mathscr{A}} \circ \mathrm{Coind}_{\mathcal{O}}^{\mathscr{A}},$$

where $\Psi_{\mathscr{A}}$ is the corresponding functor for $\mathfrak{C}(\mathscr{A}, \mathscr{A})$. Therefore, it suffices to show that $\Psi_{\mathscr{A}}$ is exact and faithful.

However, the triple $(\mathscr{A}, \mathscr{A}, \mathbb{C})$ satisfies assumption (iiib), and we already know that $\Psi_{\mathscr{A}}$ induces an equivalence between $\mathfrak{C}(\mathscr{A}, \mathscr{A})$ and the category of vector spaces. In particular, Ψ is exact and faithful. $\square$

The rest of the proof proceeds very much in the same way as Proof 1 above.

First, we claim that Proposition 3.6 above implies that the adjunction morphism

$$\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(M) \to M$$

is an isomorphism for every $M \in \mathfrak{a}$-comod. Indeed, every object in $\mathfrak{a}$-comod can be embedded into a direct sum of several copies of $\mathfrak{a}$, viewed as a co-module over itself. Hence, every $M \in \mathfrak{a}$-comod admits a resolution of the form:

$$M \to \mathfrak{a} \otimes W_0 \to \mathfrak{a} \otimes W_1 \to, \ldots,$$

where W_i are some vector spaces. Since the composition $\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(M)$ is left-exact, it is enough to prove that $\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(\mathfrak{a}) \to \mathfrak{a}$ is an isomorphism. However, this is obvious, since this map is the composition $\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}}(\mathfrak{a}) \simeq \underline{\mathrm{Res}}(\mathscr{A}) \simeq \mathfrak{a}$.

Thus, it remains to show that the adjunction map $N \to \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N)$ is an isomorphism. However, since the functor $\underline{\mathrm{Res}}$ is faithful, it is enough to show that

$$\underline{\mathrm{Res}}(N) \to \underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N)$$

is an isomorphism. But we already know that $\underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N) \to \underline{\mathrm{Res}}(N)$ is an isomorphism and the composition

$$\underline{\mathrm{Res}}(N) \to \underline{\mathrm{Res}} \circ \underline{\mathrm{Ind}} \circ \underline{\mathrm{Res}}(N) \to \underline{\mathrm{Res}}(N)$$

is the identity map.

Remark. Note that Theorem 2.8 implies that under the assumption that $\mathscr{A}$ is faithfully flat over $\mathcal{O}$, the induction functor $\mathrm{Ind} : \mathfrak{a}$-comod $\to \mathscr{A}$-comod is automatically exact. I.e., condition (iva) in fact implies condition (ivb).

3.7. Finiteness properties of $\mathscr{A}$. Thus, Theorem 2.8 is proved. Let us now describe the image of the category of finite-dimensional $\mathfrak{a}$-comodules under our equivalence of categories.

Proposition 3.8. *An $\mathfrak{a}$-comodule M is finite-dimensional if and only of $\underline{\mathrm{Ind}}(M)$ is finitely generated as an $\mathcal{O}$-module.*

Proof. One direction is clear: if $N \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$ is finite as an $\mathcal{O}$-module, then $\underline{\mathrm{Res}}(N) = \Psi(N)$ is finite-dimensional.

Conversely, assume that M is finite-dimensional, and let $M' \subset \underline{\mathrm{Ind}}(M)$ be a finite-dimensional $\mathscr{A}$-subcomodule, which surjects onto M under

$$\mathrm{Res}(\underline{\mathrm{Ind}}(M)) \to \underline{\mathrm{Res}}(\underline{\mathrm{Ind}}(M)) \simeq M.$$

Then the $\mathcal{O}$-submodule N' in $\underline{\mathrm{Ind}}(M)$ generated by M' is stable under both the $\mathcal{O}$-action and $\mathscr{A}$-coaction, and $\underline{\mathrm{Res}}(N')$ surjects onto M. Hence, $N' = \underline{\mathrm{Ind}}(M)$. $\square$

This proposition implies among the rest, that if $\mathfrak{a}$ is finite-dimensional, then $\mathscr{A} \simeq \underline{\mathrm{Ind}}(\mathfrak{a})$ is finitely generated as a module over $\mathcal{O}$. In particular, we obtain that in the quantum group setting, $\mathscr{A}_G$ is a finite $\mathcal{O}_{\check{G}}$-module.

3.9. Verification of properties (i)–(iv) for quantum groups. First, the map $\mathcal{O}_{\check{G}} \to \mathscr{A}_G$ is injective because the quantum Frobenius homomorphism is surjective.

To show that $\mathscr{A}_G \to \mathfrak{a}_G$ is surjective is equivalent to showing that every object $M \in \mathfrak{u}_\ell\text{-mod}$ appears as a sub-quotient of one of the form $\mathrm{Res}(N)$ for some $N \in \mathbf{U}_\ell\text{-}$mod. We will prove a stronger assertion, namely, that any M as above is in fact a quotient of some $\mathrm{Res}(N)$:

Proposition 3.10. *For any $M \in \mathfrak{u}_\ell$-mod, the canonical map $\mathrm{Res}(\mathrm{Ind}(M)) \to M$ is surjective.*

Proof. First, it is known (cf. [4, Theorem 4.8] or [2, Proposition 3.15]) that the functor $\mathrm{Ind} : \mathfrak{u}_\ell\text{-mod} \to \mathbf{U}_\ell\text{-mod}$ is exact. Therefore, it is sufficient to show that the map $\mathrm{Res}(\mathrm{Ind}(M)) \to M$ is surjective when M is an irreducible $\mathfrak{u}_\ell$-module.

However, it is known (cf. [14, Proposition 5.11]) that every irreducible $\mathfrak{u}_\ell$-module is of the form $\mathrm{Res}(N)$ for an irreducible $N \in \mathbf{U}_\ell$-mod.

Now, for N as above, the co-action map defines a map $N \to \mathrm{Ind}(\mathrm{Res}(N))$, and the composition

$$\mathrm{Res}(N) \to \mathrm{Res}(\mathrm{Ind}(\mathrm{Res}(N))) \to \mathrm{Res}(N)$$

is the identity map.

Hence, $\mathrm{Res}(\mathrm{Ind}(M)) \simeq \mathrm{Res}(\mathrm{Ind}(\mathrm{Res}(N))) \to \mathrm{Res}(N) \simeq M$ is a surjection. $\square$

Condition (i) of Section 2.7 follows immediately from the fact that $\mathfrak{u}_\ell$ acts trivially on any module of the form $\mathrm{Fr}^*(V)$ for $V \in \check{G}$-mod. To verify condition (ii) we will use Proposition 1.5:

Indeed, we know that if M is a $\mathscr{A}_G$-comodule, then the co-action map restricted to $M^{\mathfrak{a}_G}$ factors as

$$M^{\mathfrak{a}_G} \to \mathcal{O}_{\check{G}} \otimes M^{\mathfrak{a}_G} \hookrightarrow A_G \otimes M.$$

By taking $M = \mathscr{A}_G$ and evaluating $(\mathrm{id} \otimes \varepsilon) \circ \varDelta$ on $a \in \mathscr{A}_G^{\mathfrak{a}_G}$, we obtain that

$$a = (\mathrm{id} \otimes \varepsilon) \circ \varDelta(a) \in \mathcal{O}_{\check{G}}.$$

Let us now verify (iii). This follows from the next proposition:

Proposition 3.11. *Let $(\mathscr{A}, \mathcal{O}, a)$ be satisfying properties* (i), (ii). *Suppose that the adjunction map* $\mathrm{Res}(\mathrm{Ind}(M)) \to M$ *is surjective for any $\mathfrak{a}$-comodule M. Then* $\mathfrak{m} \cdot \mathscr{A} = \mathrm{Ker}(\mathscr{A} \to \mathfrak{a})$.

Proof. Set $I' = \mathfrak{m} \cdot \mathscr{A}$, $I = \mathrm{Ker}(\mathscr{A} \to \mathfrak{a})$. Set $\mathfrak{a}' := \mathscr{A}/I'$. Then $\mathfrak{a}'$ is a co-algebra and a right $\mathscr{A}$-module, and we have a sequence of epimorphisms $\mathscr{A} \to \mathfrak{a}' \to \mathfrak{a}$ respecting both structures.

We must show that the inclusion $I' \subset I$ is an equality. For this, it is enough to show that I is $\mathfrak{a}'$-stable, i.e. that the composition

$$I \to \mathscr{A} \xrightarrow{\varDelta} \mathscr{A} \otimes \mathscr{A} \to \mathfrak{a}' \otimes \mathscr{A}$$

maps to $\mathfrak{a}' \otimes I$. Indeed, by applying $(\mathrm{id} \otimes \varepsilon) \circ \varDelta$ to $a \in I$ we then obtain that $a = (\mathrm{id} \otimes \varepsilon) \circ \varDelta(a)$ projects to the 0 element in $\mathfrak{a}'$, i.e. belongs to I'.

Using the fact that $\mathrm{Res}(\mathrm{Ind}(M)) \to M$ i surjective for any $\mathfrak{a}$-comodule, we can find an $\mathscr{A}$-comodule B with a surjection $N \twoheadrightarrow I$.

However, from condition (ii), we obtain that in

$$\mathcal{O} \hookrightarrow \mathscr{A}^{\mathfrak{a}'} \hookrightarrow \mathscr{A}^{\mathfrak{a}}$$

the composition is an isomorphism. Hence $\mathscr{A}^{\mathfrak{a}'} = \mathscr{A}^{\mathfrak{a}}$, which implies that $N^{\mathfrak{a}'} = N^{\mathfrak{a}}$ for any $N \in \mathscr{A}$-comod. In particular, if N_1 and N_2 are two $\mathscr{A}$-comodules, any map $N_1 \to N_2$ respecting the $\mathfrak{a}$-co-action, respects also the $\mathfrak{a}'$-co-action.

Applying this to the composition $N \twoheadrightarrow I \hookrightarrow \mathscr{A}$, we obtain that $I = \mathrm{Im}(N) \subset \mathscr{A}$ is an $\mathfrak{a}'$-subcomodule. $\quad \square$

Finally, as was mentioned above, the functor Ind is exact, hence $(\mathscr{A}_G, \mathcal{O}_{\check{G}}, \mathfrak{a}_G)$ verifies condition (ivb).

For completeness, we will show that in fact $(\mathscr{A}_G, \mathcal{O}_{\check{G}}, \mathfrak{a}_G)$ verifies also condition (iva). More generally, we will prove the following proposition:

Proposition 3.12. *Let $\mathcal{O} \to \mathcal{A}$ be an embedding of Hopf algebras, with $\mathcal{O}$ being commutative. Then $\mathcal{A}$ is faithfully flat as an $\mathcal{O}$-module.*

3.13. Proof of Proposition 3.12. Let us denote $\mathrm{Spec}(\mathcal{O})$ by Γ, and view $\mathcal{A}$ as a quasi-coherent sheaf on Γ.

We have the following lemma:

Lemma 3.14. *For every $\gamma \in \Gamma$, the pull-back $\gamma^*(\mathcal{A})$ of $\mathcal{A}$ under the translation map $\gamma' \to \gamma \cdot \gamma'$ is (non-canonically) isomorphic to $\mathcal{A}$ as a quasi-coherent sheaf.*

Proof. Let $\gamma \in \Gamma$ be a point over which the embedding $\mathcal{O} \to \mathcal{A}$ induces an injection on fibers $\mathcal{O}_\gamma \to \mathcal{A}_\gamma$. We will call such γ's "good". First, we claim that for a "good" γ we do obtain an isomorphism

$$\gamma^*(\mathcal{A}) \simeq \mathcal{A}.$$

Indeed, let ξ_γ be any linear functional $\mathcal{A}_\gamma \to \mathbb{C}$ which extends the evaluation map $\mathcal{O} \to \mathbb{C}$ corresponding to γ. Consider the map

$$\mathcal{A} \overset{\Delta}{\to} \mathcal{A} \otimes \mathcal{A} \overset{\xi_\gamma \otimes \mathrm{id}}{\to} \mathcal{A}.$$

It is easy to see that this map defines the sought-for isomorphism $\gamma^*(\mathcal{A}) \simeq \mathcal{A}$.

Now let us show that all $\gamma \in \Gamma$ are "good". Suppose not. Since $\mathcal{O} \to \mathcal{A}$ is an embedding, there exists a collection $\cup_k Y_k$ of proper sub-schemes of Γ defined over $\mathbb{C}$, such that all points in $\Gamma \backslash \cup_k Y_k$ are "good". By what we proved above, the translation by a "good" γ maps the collection $\cup_k Y_k$ to itself.

Let us make a field extension $\mathbb{C} \mapsto \mathbb{C}(\Gamma)$. Over this field, Γ has the canonical generic point, which is clearly "good". However, this generic point cannot map a proper sub-scheme defined over $\mathbb{C}$ to another proper sub-scheme defined over $\mathbb{C}$, which is a contradiction. $\square$

This lemma implies Proposition 3.12:

To prove that $\mathcal{A}$ is flat over $\mathcal{O}$, we must show that $\mathrm{Tor}^1_{\mathcal{O}}(\mathcal{A}, \mathbb{C}_\gamma) = 0$, for every $\gamma \in \Gamma$. (Here $\mathbb{C}_\gamma$ denotes the sky-scraper sheaf at γ.) As in the above argument, $\mathrm{Tor}^1_{\mathcal{O}}(\mathcal{A}, \mathbb{C}_\gamma) = 0$ for all γ's lying outside $\cup_k Y_k$.

However, by Lemma 3.14, all points of Γ are "the same" with respect to $\mathcal{A}$. Hence, $\mathrm{Tor}^1_{\mathcal{O}}(\mathcal{A}, \mathbb{C}_\gamma) = 0$ everywhere.

To complete the proof of the proposition, we must show that the fiber of $\mathcal{A}$ at any $\gamma \in \Gamma$ is non-zero. But this has been established in the course of the proof of Lemma 3.14.

4. Further properties of the equivalence of categories

4.1. Definition by the universal property. In this section, we will make several additional remarks about the equivalence of categories established in Theorem 2.4.

When our discussion applies to any triple $(\mathcal{O}, \mathcal{A}, \mathfrak{a})$, we will work in this more general context. Let us denote by F^* the natural functor from $\mathcal{O}$-comod to $\mathcal{A}$-comod.

By condition (i) of Section 2.7, we have an isomorphism of functors

$$\mathcal{O}\text{-comod} \times \mathcal{A}\text{-comod} \to \mathfrak{a}\text{-comod} \ : \alpha_V^{\mathrm{can}} : \ \mathrm{Res}(F^*(V) \otimes N) \simeq \underline{V} \otimes \mathrm{Res}(N).$$

Let $\mathfrak{C}$ be an abelian $\mathbb{C}$-linear category and let $\mathscr{R} : \mathcal{A}\text{-comod} \to \mathfrak{C}$ be a $\mathbb{C}$-linear functor with the property that for each $V \in \mathcal{O}$-comod and $N \in \mathcal{A}$-comod there is a natural transformation

$$\alpha_V^{\mathfrak{C}} : \mathscr{R}(F^*(V) \otimes N) \mapsto \underline{V} \otimes \mathscr{R}(F^*(N)),$$

which satisfies the three properties of Section 2.2.

Proposition 4.2. *There exists a functor* $\mathfrak{r} : \mathfrak{a}\text{-comod} \to \mathfrak{C}$ *and an isomorphism of functors* $\mathscr{R} \simeq \mathfrak{r} \circ \mathrm{Res}$, *such that* $\alpha_V^{\mathfrak{C}} = \mathfrak{r}(\alpha_V^{\mathrm{can}})$.

The meaning of this proposition is, of course, that the forgetful functor $\mathrm{Res} : \mathcal{A}\text{-comod} \to \mathfrak{a}\text{-comod}$ is universal with respect to the property that it transforms $F^*(V) \otimes N$ to $\underline{V} \otimes N$.

Proof. Using Theorem 2.8 we will think of $\mathfrak{a}$-comod in terms of $\mathfrak{C}(\mathcal{A}, \mathcal{O})$ and we will construct a functor $\mathfrak{r} : \mathfrak{C}(\mathcal{A}, \mathcal{O}) \to \mathfrak{C}$.

Let $\mathcal{O} \otimes M \overset{\mathrm{act}}{\to} M$ be the action map. By assumption, we obtain the map

$$\underline{\mathcal{O}} \otimes \mathscr{R}(M) \simeq \mathscr{R}(\mathcal{O} \otimes M) \to \mathscr{R}(M).$$

The axioms on the $\alpha_V^{\mathfrak{C}}$'s imply that $\mathcal{O}$ acts on $\mathscr{R}(M)$ as an associative algebra. We set $\mathfrak{r}(M) := \mathbb{C} \otimes_{\mathcal{O}} \mathscr{R}(M)$.

Let us show now that $\mathscr{R}$ is canonically isomorphic to $\mathfrak{r} \circ \mathrm{Res}$. Under the equivalence of Theorem 2.4, the functor Res goes over to $N \mapsto \mathcal{O} \otimes N$. Therefore,

$$\mathfrak{r} \circ \mathrm{Res}(N) \simeq \mathbb{C} \underset{\mathcal{O}}{\otimes} \mathscr{R}(\mathcal{O} \otimes N) \simeq \mathscr{R}(N). \qquad \square$$

4.3. Reconstruction of $\mathcal{A}$-comod from $\mathfrak{a}$-comod. For $(\mathcal{O}, \mathcal{A}, \mathfrak{a})$ with $\mathcal{O}$ being commutative, let us recall from Corollary 2.5 that the group $\Gamma = \mathrm{Spec}(\mathcal{O})$ acts on the category $\mathfrak{a}$-comod by endo-functors.

Thus, it makes sense to introduce the category $(\mathfrak{a}\text{-comod})^{\Gamma}$ of Γ-equivariant objects of $\mathfrak{a}$-comod.

Proposition 4.4. *The category* $(\mathfrak{a}\text{-comod})^{\Gamma}$ *is naturally equivalent to* $\mathcal{A}$-comod.

Proof. Let M be a Γ-equivariant object in $\mathfrak{C}(\mathcal{A}, \mathcal{O})$. By definition, the underlying $\mathcal{A}$-comodule has an additional commuting structure of a Γ-equivariant

quasi-coherent sheaf on Γ. By taking its fiber at the point $1 \in \Gamma$, we obtain an $\mathscr{A}$-comodule.

Thus, we have constructed a functor

$$(\mathfrak{C}(\mathscr{A}, \mathcal{O}))^\Gamma \to \mathscr{A}\text{-comod},$$

and it is easy to see that it is an equivalence. $\square$

Thus, given $\mathscr{A}$, the category of $\mathfrak{a}$-comodules is a "de-equivariantization" of $\mathscr{A}$-comod.

4.5. Other versions of quantum groups. Let us discuss briefly the generalization of Theorem 2.4 in the context of $\mathfrak{u}_{\ell,\mathrm{sc}}$ and $\dot{\mathfrak{u}}_\ell$.

Consider the triple $\mathscr{A} = \mathscr{A}_G$, $\mathcal{O} = \mathcal{O}_{\check{G}_{\mathrm{sc}}}$ and $\mathfrak{a} = \mathfrak{a}_{G_{\mathrm{sc}}}$. In a way completely analogous to what we did in the previous section, one shows that these co-algebras satisfy conditions (i)–(iv) of Section 2.7.

Let $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}_{\mathrm{sc}}})$ denote the corresponding category $\mathfrak{C}(\mathscr{A}, \mathcal{O})$. We have the following version of Theorem 2.4:

Theorem 4.6. *The categories $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}_{\mathrm{sc}}})$ and $\mathfrak{a}_{G_{\mathrm{sc}}}$-comod are naturally equivalent.*

Now let us consider the case of $\dot{\mathfrak{u}}_\ell$. In what follows, for a $\check{G}$-module V, we will regard $\mathrm{Res}_{\check{T}}^{\check{G}}(V)$ as a Y-graded vector space.

We introduce the category $\dot{\mathfrak{C}}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ as follows: its objects are Y-graded $\mathscr{A}_G$-comodules $M = \oplus_{\nu \in Y} M^\nu$, each endowed with a collection of grading-preserving maps α_V, $V \in \check{G}$-mod

$$\mathrm{Fr}^*(V) \otimes M \simeq \mathrm{Res}_{\check{T}}^{\check{G}}(V) \otimes M$$

(as in Section 2.2) where the Y-grading on the left-hand side comes from the grading on M and on the right-hand side the grading is diagonal. Maps in this category are grading preserving $\mathbf{U}_\ell$-module maps, which intertwine the corresponding α_V's.

Theorem 4.7. *The category $\dot{\mathfrak{C}}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ is equivalent to $\dot{\mathfrak{a}}_G$-comod.*

Proof. First, let us observe that if we put $\mathscr{A} = \dot{\mathfrak{a}}_G$, $\mathcal{O} = \mathcal{O}_{\check{T}}$, $\mathfrak{a} = \mathfrak{a}_G$, the corresponding triple would satisfy conditions (i)–(iv) of Section 2.7. Hence, the general Theorem 2.8 is applicable as well as Proposition 1.1.

Therefore, the category $\dot{\mathfrak{C}}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$ is equivalent to the category of $\check{T}$-equivariant objects in $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$. However, the latter is by definition the same as $\dot{\mathfrak{C}}(\mathscr{A}_G, \mathcal{O}_{\check{G}})$. $\square$

Finally, let us characterize the category $\dot{\mathfrak{u}}_\ell$-mod by a universal property. For an $\check{\lambda}$ in $Y = X^*$ (or even in X_{sc}^*) let us denote by $\mathbb{C}^{\check{\lambda}}$ the corresponding one-dimensional module over $\dot{\mathfrak{u}}_\ell$, and by $P_{\check{\lambda}} : \dot{\mathfrak{u}}_\ell\text{-mod} \mapsto \dot{\mathfrak{u}}_\ell\text{-mod}$ the translation functor $M \mapsto \mathbb{C}^{\check{\lambda}} \otimes M$.

Let now $\mathfrak{C}$ be an abelian $\mathbb{C}$-linear category and let $P_{\check{\lambda}}^{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ be an action of Y on $\mathfrak{C}$ by endo-functors. Let $\mathscr{R} : \mathscr{A}_G\text{-comod} \to \mathfrak{C}$ be a $\mathbb{C}$- linear functor with the property that for each $V \in \mathcal{O}_{\check{G}}$-comod there is a natural transformation

$$\alpha_V^{\mathfrak{C}} : \mathscr{R}(\mathrm{Fr}^*(V) \otimes M) \mapsto \bigoplus_v \underline{V}(\check{v}) \otimes P_{\check{v}}^{\mathfrak{C}}(\mathscr{R}(M)),$$

which satisfies the three properties of Section 2.2. (In the above formula, for a $\check{G}$-module V and $\check{v} \in Y$, $\underline{V}(\check{v})$ denotes the corresponding weight subspace.)

Proposition 4.8. *There exists a functor* $\mathfrak{r} : \dot{\mathfrak{a}}_G\text{-comod} \to \mathfrak{C}$ *and an isomorphism of functors*

$$\mathscr{R} \simeq \mathfrak{r} \circ \mathrm{Res}.$$

Moreover, the functor $\mathfrak{r}$ *commutes with the translation functors in the obvious sense.*

We omit the proof, since it is completely analogous to the proof of Proposition 4.2.

5. The regular block

5.1. Blocks in the categories $\mathscr{A}$-comod and $\mathfrak{a}$-comod. Recall that any Artinian abelian category $\mathfrak{C}$ is a direct sum of its indecomposable abelian sub-categories called *blocks* or *linkage classes* of $\mathfrak{C}$. Obviously, a block of a category is completely described by the set of irreducible objects contained in it. We will denote the set of blocks of $\mathfrak{C}$ by $\mathsf{Bl}(\mathfrak{C})$.

Note that the categories of finite-dimensional $\mathscr{A}$- and $\mathfrak{a}$-comodules (denoted below by $\mathscr{A}$-comod^f and $\mathfrak{a}$-comod^f, respectively) are Artinian, therefore, they admit decompositions into blocks. We will use the notation $\mathsf{Bl}(\mathscr{A})$ and $\mathsf{Bl}(\mathfrak{a})$ for the sets of blocks of $\mathscr{A}$-comod^f and $\mathfrak{a}$-comod^f, respectively.

Evidently, we have

$$\mathscr{A}\text{-comod} = \text{ind. comp.}(\mathscr{A}\text{-comod}^f) \text{ and } \mathfrak{a}\text{-comod}$$

$$= \text{ind. comp.}(\mathfrak{a}\text{-comod}^f).$$

For $\alpha \in \mathsf{Bl}(\mathscr{A})$ (resp., $\alpha' \in \mathsf{Bl}(\mathfrak{a})$) let us denote by $\mathscr{A}$-comod$_\alpha$ (resp., $\mathfrak{a}$-comod$_{\alpha'}$) the ind-completion of the corresponding block of $\mathscr{A}$-comod^f (resp., $\mathfrak{a}$-comod^f).

We will call the block of $\mathscr{A}$-comod (resp., $\mathfrak{a}$-comod) which contains the trivial representation $\mathbb{C}$ *the regular block* and will denote it by $\mathscr{A}$-comod$_0$ (resp., $\mathfrak{a}$-comod$_0$).

Assume that the category $\mathscr{A}$-comod has the following additional property with respect to $\mathcal{O}$-comod:

($*$) *For any* $\alpha \in \mathsf{Bl}(\mathscr{A})$ *and* $V \in \mathcal{O}$-comod, *the functor* $F^*(V) \otimes \cdot : \mathscr{A}$-comod $\to \mathscr{A}$-comod *maps* $\mathscr{A}$-comod$_\alpha$ *to itself*.

Let us compare the block decompositions of $\mathscr{A}$-comod and $\mathfrak{a}$-comod.

Proposition 5.2. *There is a one-to-one correspondence between the sets* $\mathsf{Bl}(\mathscr{A})$ *and* $\mathsf{Bl}(\mathfrak{a})$ *determined by the following properties*:

(a) $N \in \mathscr{A}$-comod$_\alpha$ *if and only if* $\mathrm{Res}(N) \in \mathfrak{a}$-comod$_\alpha$.
(b) $M \in \mathfrak{a}$-comod$_\alpha$ *if and only if* $\mathrm{Ind}(M) \in \mathscr{A}$-comod$_\alpha$.

Proof. First, observe that $\mathrm{Ind} \circ \mathrm{Res} : \mathscr{A}$-comod $\to \mathscr{A}$-comod preserves each $\mathscr{A}$-comod$_\alpha$, by assumption, since $\mathrm{Ind} \circ \mathrm{Res}(N) \simeq F^*(\mathcal{O}) \otimes N$.

Secondly, let us show that $\mathrm{Res} \circ \mathrm{Ind}$ maps each $\mathfrak{a}$-comod$_\alpha$ to itself. Indeed, let $M \in \mathfrak{a}$-comod$_\alpha$ and let N' be an $\mathfrak{a}$-stable direct summand of $\mathrm{Ind}(M)$, which belongs to some $\mathfrak{a}$-comod$_\beta$. Then N' is preserved by the $\mathcal{O}$-action, and thus defines a sub-object of $\underline{\mathrm{Ind}}(M) \in \mathfrak{C}(\mathscr{A}, \mathcal{O})$. But then $\underline{\mathrm{Res}}(N') \in \mathfrak{a}$-comod$_\beta$ is a non-zero direct summand of M, which means that $\beta = \alpha$.

Let N be an object of $\mathscr{A}$-comod. Let $\mathrm{Res}(N) = \mathrm{Res}(N)' \oplus \mathrm{Res}(N)''$ be a block decomposition in $\mathfrak{a}$-comod. Let us show that $\mathrm{Res}(N)'$ and $\mathrm{Res}(N)''$ are in fact $\mathscr{A}$-sub-comodules. Without restricting the generality, we can assume that N is a sub-comodule of $\mathrm{Ind}(M)$ for some $M \in \mathfrak{a}$-comod. However, as we have just seen, the block decomposition of $\mathrm{Res} \circ \mathrm{Ind}(M)$ coincides with the block decomposition of M.

Therefore, the block decomposition of $\mathfrak{a}$-comod is "coarser" than that of $\mathscr{A}$-comod.

However, by our assumption on $\mathscr{A}$, its block decomposition is "coarser" than the block decomposition of $\mathfrak{C}(\mathscr{A}, \mathcal{O})$. This implies the assertion of the proposition in view of Theorem 2.8. $\square$

5.3. The category $\mathfrak{C}(\mathscr{A}, \mathcal{O})^0$. We define the category $\mathfrak{C}(\mathscr{A}, \mathcal{O})^0$ as the preimage of $\mathscr{A}$-comod$_0$ under the tautological forgetful functor $\mathfrak{C}(\mathscr{A}, \mathcal{O}) \to \mathscr{A}$-comod. This definition makes sense due to condition ($*$) above. In the course of the proof of Proposition 5.2 we have established the following assertion:

Corollary 5.4. *Under the equivalence of categories* $\mathfrak{C}(\mathscr{A}, \mathcal{O}) \simeq \mathfrak{a}$-comod, *the subcategory* $\mathfrak{C}(\mathscr{A}, \mathcal{O})^0$ *goes over to the regular block* $\mathfrak{a}$-comod$_0$.

5.5. The case of $\mathfrak{u}_\ell$**.** For a regular dominant $\lambda \in X$, let $\mathbf{W}(\lambda) \in \mathbf{U}_\ell$-mod denote the corresponding *Weyl module*. It is well-known that $\mathbf{W}(\lambda)$ has a unique simple quotient, denoted $\mathbf{L}(\lambda)$ and that each simple object in $\mathbf{U}_\ell$-mod is isomorphic to $\mathbf{L}(\lambda)$ for some λ.

The following facts about the block decomposition of the category $\mathbf{U}_\ell$-mod were established in [3]:

Let W_{aff} be the affine Weyl group $Y \rtimes W$. It acts on the lattice X as follows: the translations by Y act via the homomorphism $\phi : Y \to X$ and the action of the finite Weyl group W is centered at $-\rho := -\sum_{i \in I} \omega_i$, where ω_i's are the fundamental weights.

Theorem 5.6. *Two simple modules* $\mathbf{L}(\lambda_1)$ *and* $\mathbf{L}(\lambda_2)$ *are in the same block if and only if* λ_1 *and* λ_2 *belong to the same* W_{aff}*-orbit.*

Moreover, we have the following statement (cf., [15, Theorem 7.4 and Proposition 7.5]):

Proposition 5.7. *Let* $\lambda = \lambda_1 + \lambda_2$ *be the unique decomposition, with* $\lambda_2 = \phi_{\mathrm{sc}}(\check{\mu})$*, where* $\check{\mu} \in X_{\mathrm{sc}}^*$ *is a dominant integral weight of the group* $\check{G}_{\mathrm{sc}}$ *and* λ_1 *is such that* $0 \leqslant \langle \lambda_1, \check{\alpha}_i \rangle < \ell_i$ *for all* $i \in I$*. Then:*

(i) $\mathbf{L}(\lambda_2) \simeq \mathrm{Fr}_{\mathrm{sc}}^*(V^{\check{\mu}})$*, where* $V^{\check{\mu}}$ *is the corresponding irreducible representation of* $\check{G}_{\mathrm{sc}}$*.*
(ii) *The restriction of* $\mathbf{L}(\lambda_1)$ *to* $\mathfrak{u}_\ell$ *remains irreducible.*
(iii) $\mathbf{L}(\lambda) \simeq \mathbf{L}(\lambda_1) \otimes \mathbf{L}(\lambda_2)$*.*

This proposition combined with Theorem 5.6 implies that the category $\mathbf{U}_\ell$-mod satisfies condition $(*)$.

Let $\mathfrak{C}(\mathscr{A}_G, \mathscr{O}_G)^0$ denote the corresponding sub-category of $\mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}})$. By applying Proposition 5.2 and Corollary 5.4, we obtain the following theorem:

Theorem 5.8. *We have a bijection between the sets* $\mathsf{Bl}(\mathbf{U}_\ell\text{-mod}) \simeq \mathsf{Bl}(\mathfrak{u}_\ell\text{-mod})$ *and an equivalence of categories:*

$$\mathfrak{a}_G\text{-comod}_0 \simeq \mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}})^0.$$

Recall (cf. [14] or [4]) that to every element $\lambda \in X$ we attached an irreducible object of $\mathfrak{u}_\ell$-mod, denoted $L(\lambda)$, which depends only on the image of λ in the quotient $X/\phi(Y)$, and Proposition 5.7(ii) implies that if λ satisfies $\langle \lambda, \check{\alpha}_i \rangle < \ell_i$, then $L(\lambda) \simeq \mathrm{Res}(\mathbf{L}(\lambda))$.

The following corollary repeats in fact Section 2.9 of [4]:

Corollary 5.9. *For two elements λ_1 and λ_2 of X, the modules $L(\lambda_1)$ and $L(\lambda_2)$ belong to the same block of $\mathfrak{u}_\ell$-mod if and only if λ_1 and $\lambda_2 \in X$ are W_{aff}-conjugate.*

5.10. The graded case. For completeness, let us analyze the block decomposition of the category $\dot{\mathfrak{u}}_\ell$-mod in light of Theorem 5.2. However, all that we are going to obtain is already contained in [4].

Recall that for $\lambda \in X$, $\dot{L}(\lambda)$ denotes the corresponding irreducible object of $\dot{\mathfrak{u}}_\ell$-mod, and $L(\lambda) = \mathrm{Res}^{\dot{\mathfrak{u}}_\ell}_{\mathfrak{u}_\ell}(\dot{L}(\lambda))$.

The following is known, due to [4]:

Proposition 5.11. *The translation functors $P_{\check{\lambda}}$, $\check{\lambda} \in Y$ preserve the block decomposition of $\dot{\mathfrak{u}}_\ell$-mod.*

Using this proposition, we can apply Proposition 5.2 to the category $\dot{\mathfrak{u}}_\ell$-mod and the group $\check{T}$. Thus, we obtain the following result of [4]:

Corollary 5.12. *There is a natural bijection $\mathrm{Bl}(\dot{\mathfrak{u}}_\ell\text{-mod}) \simeq \mathrm{Bl}(\mathfrak{u}_\ell\text{-mod})$. The modules $\dot{L}(\lambda_1)$ and $\dot{L}(\lambda_2)$ belong to the block in $\dot{\mathfrak{u}}_\ell$-mod if and only if λ_1 and λ_2 are W_{aff}-conjugate.*

Let $\dot{\mathfrak{C}}(\mathscr{A}_G, \mathscr{O}_{\check{G}})^0$ denote the preimage of $\mathbf{U}_\ell\text{-mod}_0$ under the obvious forgetful functor. From Proposition 5.2 and Theorem 4.7, we obtain the following theorem (cf. [1] for the first assertion):

Theorem 5.13. *There is an isomorphism of sets $\mathrm{Bl}(\mathbf{U}_\ell\text{-mod}) \simeq \mathrm{Bl}(\dot{\mathfrak{u}}_\ell\text{-mod})$ and an equivalence of categories:*

$$\dot{\mathfrak{a}}_G\text{-comod}_0 \simeq \dot{\mathfrak{C}}(\mathscr{A}_G, \mathscr{O}_{\check{G}})^0.$$

5.14. The case of $\mathfrak{u}_{\ell,\mathrm{sc}}$. Observe that if we consider the triple $\mathscr{A} = \mathscr{A}_G$, $\mathscr{O} = \mathscr{O}_{\check{G}_{\mathrm{sc}}}$, $\mathfrak{a} = \mathfrak{a}_{G_{\mathrm{sc}}}$, then condition (*) above will not be satisfied. Instead, we have the following assertion:

Proposition 5.15. *The natural restriction functor $\mathfrak{u}_\ell\text{-mod} \to \mathfrak{u}_{\ell,\mathrm{sc}}\text{-mod}$ induces an equivalence $\mathfrak{u}_\ell\text{-mod}_0 \to \mathfrak{u}_{\ell,\mathrm{sc}}\text{-mod}_0$.*

Proof. For $\lambda \in X$, let us denote by $L(\lambda)_{\mathrm{sc}}$ the restriction of $L(\lambda)$ to $\mathfrak{u}_{\ell,\mathrm{sc}}$. By construction, it depends only on the class of λ in $X/\phi(X^*_{\mathrm{sc}})$.

Let us consider the forgetful functor $\mathrm{Res}^{\mathfrak{u}_\ell}_{\mathfrak{u}_{\ell,\mathrm{sc}}} : \mathfrak{u}_\ell\text{-mod} \to \mathfrak{u}_{\ell,\mathrm{sc}}\text{-mod}$. Note that in terms of $\mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}})$ and $\mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}_{\mathrm{sc}}})$, it acts as follows:

$$M \in \mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}}) \mapsto \mathscr{O}_{\check{G}_{\mathrm{sc}}} \underset{\mathscr{O}_{\check{G}}}{\otimes} M \in \mathfrak{C}(\mathscr{A}_G, \mathscr{O}_{\check{G}_{\mathrm{sc}}}).$$

This functor has a right adjoint, which we will denote by $\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}$. On the level of $\mathfrak{C}(\mathscr{A}_G, \mathcal{O}_{\check{G}_{\mathrm{sc}}})$, $\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}$ is the natural forgetful functor.

Note that for $M \in \dot{\mathfrak{u}}_{\ell}$-mod we have:

$$\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}} \circ \mathrm{Res}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}(M) \simeq \bigoplus_{\check{\lambda} \in X_{\mathrm{sc}}^*/X^*} M \otimes \mathbb{C}^{\check{\lambda}}.$$

(Recall that $\mathbb{C}^{\check{\lambda}}$ is a one-dimensional $\dot{\mathfrak{u}}_{\ell}$-module, and, hence, we are allowed to tensor any $\mathfrak{u}_{\ell}$-module by it on the right.) In particular, $\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}(L(\mu)_{\mathrm{sc}}) = \oplus_{\check{\lambda} \in X_{\mathrm{sc}}^*/X^*} L(\mu + \phi_{\mathrm{sc}}(\check{\lambda}))$.

Therefore, two irreducible objects $L(\mu^1)_{\mathrm{sc}}$ and $L(\mu^2)_{\mathrm{sc}}$ of $\mathfrak{u}_{\ell,\mathrm{sc}}$-mod belong to the same block if and only if there exists $\check{\lambda} \in X_{\mathrm{sc}}^*$, such that $L(\mu^1 + \phi_{\mathrm{sc}}(\check{\lambda}))$ and $L(\mu^2)$ belong to the same block of $\mathfrak{u}_{\ell}$-mod, i.e. μ^1 and μ^2 belong to the same orbit of the extended affine Weyl group $W_{\mathrm{aff}}^{\mathrm{ext}} \simeq X_{\mathrm{sc}}^* \ltimes W$.

Thus, we obtain that the functor $\mathrm{Res}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}$ maps $\mathfrak{u}_{\ell}$-mod$_0$ to $\mathfrak{u}_{\ell,\mathrm{sc}}$-mod$_0$. We claim now that this functor has a left quasi-inverse. Namely, it is given by

$$N \mapsto pr_0(\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}(N)),$$

where pr_0 denotes the functor of projection onto the regular block in $\mathfrak{u}_{\ell}$-mod. Indeed, for $M \in \mathfrak{u}_{\ell}$-mod$_0$ we have

$$pr_0(\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}} \circ \mathrm{Res}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}(M)) \simeq M,$$

because for $\check{\lambda} \in X_{\mathrm{sc}}^*$, the object $pr_0(M \otimes \mathbb{C}^{\check{\lambda}})$ is non-zero only if $\check{\lambda} \in X^*$, which follows from the description of blocks of $\mathfrak{u}_{\ell}$-mod in terms of W_{aff}.

To finish the proof of the proposition it remains to show that if N is a non-zero object in $\mathfrak{u}_{\ell,\mathrm{sc}}$-mod$_0$, then $pr_0(\mathrm{Ind}_{\mathfrak{u}_{\ell,\mathrm{sc}}}^{\mathfrak{u}_{\ell}}(N))$ is non-zero either. For that, it is enough to suppose that N is irreducible, i.e. of the form $L(\lambda)_{\mathrm{sc}}$, and our assertion follows from the explicit description of $\mathfrak{u}_{\ell,\mathrm{sc}}$-mod$_0$ given above. $\square$

6. Geometric interpretation

6.1. Affine flag variety. Our goal in this section is to give a geometric description of the category $\mathfrak{u}_{\ell}$-mod$_0$. Namely, we will show that it can be described as the category of certain perverse sheaves on the affine flag variety corresponding to the group G, which have the Hecke eigen-property. It is via this description that one can link $\mathfrak{u}_{\ell}$-mod$_0$ to certain categories which appear in the geometric Langlands correspondence and to other interesting categories arising in representation theory.

First, we will briefly recall several definitions concerning the affine Grassmannian and the affine flag variety. We refer the reader to [5,17] or [9] for a more detailed discussion.

Consider the ring $\mathbb{C}[[t]]$ of Taylor series and the field $\mathbb{C}((t))$ of Laurent series. The loop group $G((t))$ (resp., the group of positive loops $G[[t]]$) has a structure of an group-indscheme (resp., of a group-scheme). The quotient $G((t))/G[[t]]$ is an ind-scheme of ind-finite type, called *the affine Grassmannian* of the group G, denoted Gr.

By definition, the group-scheme $G[[t]]$ acts (on the left) on Gr. The orbits of this action are finite-dimensional quasi-projective varieties and they are in a natural bijection with the dominant elements $Y^+ \subset Y$. For $\check{\lambda} \in Y^+$, we will denote by $\overline{\mathrm{Gr}}^{\check{\lambda}}$ the closure of the corresponding orbit. Thus, it makes sense to talk about the category of $G[[t]]$-equivariant perverse sheaves on Gr. By definition, every such perverse sheaf is supported on $\overline{\mathrm{Gr}}^{\check{\lambda}}$ for $\check{\lambda}$ sufficiently large. We will denote this category by $\mathrm{P}_{G[[t]]}(\mathrm{Gr})$. This is an abelian category and it possesses an additional structure of the *convolution product* $\mathrm{P}_{G[[t]]}(\mathrm{Gr}) \star \mathrm{P}_{G[[t]]}(\mathrm{Gr}) \to \mathrm{P}_{G[[t]]}(\mathrm{Gr})$, which makes $\mathrm{P}_{G[[t]]}(\mathrm{Gr})$ into a tensor category.

We have the following fundamental theorem [10,17]:

Theorem 6.2. *There is an equivalence of tensor categories $\check{G}$-mod $\simeq \mathrm{P}_{G[[t]]}(\mathrm{Gr})$. Under this equivalence, the intersection cohomology sheaf $\mathrm{IC}_{\overline{\mathrm{Gr}}^{\check{\lambda}}}$ goes over to the highest weight module $V^{\check{\lambda}}$.*

Now we pass to the definition of the affine flag variety.

Let $\mathrm{Iw} \subset G[[t]]$ be the Iwahori subgroup. By definition, Iw is the preimage of the Borel subgroup $B \subset G$ under the natural evaluation map $G[[t]] \to G$. The quotient $G((t))/\mathrm{Iw}$ is also an ind-scheme of ind-finite type, called *the affine flag variety* of G, denoted Fl. By definition, we have a projection $\mathrm{Fl} \twoheadrightarrow \mathrm{Gr}$, whose fibers are (non-canonically) isomorphic to the usual flag manifold G/B.

Let N_{Iw} be the unipotent radical of Iw. By definition, N_{Iw} is the preimage of $N \subset B$ under $\mathrm{Iw} \to B$. Since N_{Iw} is normal in Iw and $\mathrm{Iw}/N_{\mathrm{Iw}} \simeq T$, the quotient $\widetilde{\mathrm{Fl}} := G((t))/N_{\mathrm{Iw}}$ is a principal T-bundle over $\mathrm{Fl} := G((t))/\mathrm{Iw}$. We will call $\widetilde{\mathrm{Fl}}$ the enhanced affine flag variety. The group-scheme $G[[t]]$ acts on the left on both Fl and $\widetilde{\mathrm{Fl}}$.

We define the category $\mathrm{P}_{G[[t]]}(\mathrm{Fl})$ to be the abelian category of $G[[t]]$-equivariant perverse sheaves on Fl. We define the category $\widetilde{\mathrm{P}}_{G[[t]]}(\mathrm{Fl})$ to be the subcategory of the category of $G[[t]]$-equivariant perverse sheaves on $\widetilde{\mathrm{Fl}}$, which consists of T-monodromic objects.[3] Note that the pull-back functor identifies $\mathrm{P}_{G[[t]]}(\mathrm{Fl})$ with the sub-category of $\widetilde{\mathrm{P}}_{G[[t]]}(\mathrm{Fl})$ consisting of T-equivariant objects.

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[3] Recall that a perverse sheaf is called T-monodromic when it has a filtration, whose subquotients are T-equivariant perverse sheaves.

It is known (cf. [9] for details) that the convolution tensor structure on the category $P_{G[[t]]}(\mathrm{Gr})$ extends to an action of $P_{G[[t]]}(\mathrm{Gr})$ on $P_{G[[t]]}(\mathrm{Fl})$.

Similarly, one can define an action

$$P_{G[[t]]}(\mathrm{Gr}) \star \widetilde{P}_{G[[t]]}(\mathrm{Fl}) \to \widetilde{P}_{G[[t]]}(\mathrm{Fl}).$$

6.3. Hecke eigen-sheaves. Let $\underline{\widetilde{P}}_{G[[t]]}(\mathrm{Fl})$ denote the ind-completion of the category $\widetilde{P}_{G[[t]]}(\mathrm{Fl})$. We define the category $\mathfrak{U}$ as follows: its objects are pairs

$$(\mathscr{F} \in \underline{\widetilde{P}}_{G[[t]]}(\mathrm{Fl}), \{\alpha_V \ \forall V \in \check{G}\text{-mod}\}),$$

where each α_V is a map

$$\mathscr{F}_V \star \mathscr{F} \to \underline{V} \otimes \mathscr{F},$$

where $\mathscr{F}_V \in P_{G[[t]]}(\mathrm{Gr})$ is the perverse sheaf corresponding to $V \in \check{G}$-mod via the equivalence of categories of Theorem 6.2. The maps α_V must satisfy the three conditions of Section 2.2. Morphisms in $\mathfrak{U}$ between $(\mathscr{F}, \alpha_V)$ and $(\mathscr{F}', \alpha'_V)$ are maps $\mathscr{F} \to \mathscr{F}'$, which intertwine between α_V and α'_V. As in Proposition 2.3, one shows that the maps α_V as above are automatically isomorphisms.

The rest of this section (and of the paper) is devoted to the proof of the following theorem.

Theorem 6.4. *For ℓ sufficiently large, there is an equivalence of categories between $\mathfrak{U}$ and $\mathfrak{u}_\ell$-mod$_0$.*

Unfortunately, the proof relies on two results, whose proofs are unavailable in the published literature. Therefore, the reader may regard Theorem 6.4 as a conjecture, which can be deduced from Theorems 6.7 and Theorem 6.12 stated below.

6.5. Twisted D-modules on $\widetilde{\mathrm{Fl}}$. The first step in the passage $\mathfrak{U} \to \mathfrak{u}_\ell$-mod$_0$ is the functor from perverse sheaves on Fl to modules over the Kac–Moody algebra due to [12].

Recall that to an invariant symmetric from $c : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, which is *integral* (i.e. induces an integral-valued form on the cocharacter lattice Y), we can associate a line bundle $\mathscr{L}_c$ on Gr (cf. [12]). By pulling it back to Fl and $\widetilde{\mathrm{Fl}}$ we obtain the corresponding line bundles on the latter ind-schemes.

Thus, we can consider the category of $\mathscr{L}_c$-twisted right D-modules on $\widetilde{\mathrm{Fl}}$, cf. [5,12]. As $\mathscr{L}_c$ is $G[[t]]$-equivariant, it makes sense to consider the category $\mathrm{D}_c^{G[[t]]}(\widetilde{\mathrm{Fl}})$ of $G[[t]]$-equivariant T-monodromic $\mathscr{L}_c$-twisted right D-modules on $\widetilde{\mathrm{Fl}}$.

Let us now consider the Kac–Moody algebra $\hat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$:

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0,$$

defined with respect to the pairing $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ given by c. Let us denote by $\hat{\mathfrak{g}}_c$-mod the category of continuous representations of $\hat{\mathfrak{g}}$ on which $1 \in \mathbb{C} \subset \hat{\mathfrak{g}}$ acts as identity.

Let us denote by $\hat{\mathfrak{g}}_c^{G[[t]]}$-mod the subcategory of $\hat{\mathfrak{g}}_c$-mod consisting of finite length representations, on which the action of $\mathfrak{g}[[t]] \subset \hat{\mathfrak{g}}$ integrates to the action of the group-scheme $G[[t]]$. According to [11], this is an Artinian category and we will denote by $\hat{\mathfrak{g}}_c^{G[[t]]}$-mod$_0$ its regular block.

From now on let us suppose that c is such that $c_{\mathrm{crit}} - c$ is positive definite on Y, where c_{crit} corresponds to $-\frac{1}{2}$ (the Killing form). When $\mathfrak{g}$ is simple, c_{crit} is $-\check{h}$ times the *canonical* integral from on Y, where $\check{h}$ is the dual Coxeter number.

The following theorem has been established in [5,12].

Theorem 6.6. *The functor of global sections of a twisted D-module defines an exact and faithful functor*:

$$\mathrm{D}_c^{G[[t]]}(\widetilde{\mathrm{Fl}}) \to \hat{\mathfrak{g}}_c^{G[[t]]}\text{-mod}_0.$$

However, a stronger statement is true:[4]

Theorem 6.7. *The above functor* $\mathrm{D}_c^{G[[t]]}(\widetilde{\mathrm{Fl}}) \to \hat{\mathfrak{g}}_c^{G[[t]]}$-mod$_0$ *is in fact an equivalence of categories.*

The Riemann–Hilbert correspondence yields an equivalence of categories between $\mathrm{D}_c^{G[[t]]}(\widetilde{\mathrm{Fl}})$ and $\widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{Fl})$, cf. [12]. Therefore, we obtain the following corollary:

Corollary 6.8. *There is an equivalence of categories* $\widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{Fl}) \simeq \hat{\mathfrak{g}}_c^{G[[t]]}$-mod$_0$.

6.9. The Kazhdan–Lusztig equivalence of categories. Now let ℓ be as in Section 1.2 and set $c = c_{\mathrm{crit}} - (\cdot, \cdot)_\ell$, where $(\cdot, \cdot)_\ell$ has been introduced in Section 1.3. Again, when $\mathfrak{g}$ is simple, c is $-\check{h} - \frac{\ell}{d}$ times the canonical form, where $d = \max(d_i)$.

The following theorem has been established in [11]:

Theorem 6.10. *When ℓ is sufficiently large, there is an equivalence of categories* $\mathbf{U}_\ell$-mod $\simeq \hat{\mathfrak{g}}_c^{G[[t]]}$-mod.

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[4] This result is probably well-known to many experts. The proof that we have in mind has been explained to us by M. Finkelberg and R. Bezrukavnikov.

By combining this theorem with Corollary 6.8, we obtain the following corollary:

Corollary 6.11. *There is an equivalence of categories*: $\widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{F1}) \simeq \mathbf{U}_\ell\text{-mod}_0$.

To prove Theorem 6.4, we will need the following property of the equivalence stated in Corollary 6.11:

Theorem 6.12. *Under the above equivalence of categories* $\widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{F1}) \simeq \mathbf{U}_\ell\text{-mod}_0$ *the functors* $\check{G}\text{-mod} \times \widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{F1}) \to \widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{F1})$ *given by*

$$V, \mathscr{F} \mapsto \mathscr{F}_V \star \mathscr{F} \quad and \quad V, M \mapsto \mathrm{Fr}^*(V) \otimes M$$

are naturally isomorphic.

This result has not been stated explicitly in [11]. We will supply the proof in a later publication.

Now, by passing to the ind-completions of the categories $\widetilde{\mathbf{P}}_{G[[t]]}(\mathrm{F1})$ and $\mathbf{U}_\ell\text{-mod}_0$, we obtain that Theorem 6.4 is a consequence of Theorem 2.4 and Theorem 6.12.

Acknowledgments

References

[1] H.H. Andersen, J.C. Jantzen, W. Soergel, Representations of quantum groups at pth roots of unity and of semisimple groups in characteristic p: independence of p, Asterisque 220 (1994).

[2] H.H. Andersen, J. Paradowski, Fusion categories arising from semisimple Lie algebras, Comm. Math. Phys. 169 (1995) 563–588.

[3] H.H. Andersen, P. Polo, K. Wen, Representations of quantum algebras, Invent. Math. 104 (1991) 1–59.

[4] H.H. Andersen, P. Polo, K. Wen, Injective modules over quantum algebras, Amer. J. Math. 114 (1992) 571–604.

[5] A. Beilinson, V. Drinfeld, Quantization of the Hitchin's integrable system and Hecke eigensheaves, preprint, 2000, available electronically at www.math.uchicago.edu/˜benzvi.

[6] R. Bezrukavnikov, M. Finkelberg, V. Schechtman, Factorizable sheaves and quantum groups, Lecture Notes in Mathematics, Vol. 1691, Springer, Berlin, 1998.

[7] M. Finkelberg, I. Mirkovic, Semi-infinite flags-I. Case of global curve $\mathbb{P}^1$, in: Differential Topology, Infinite-Dimensional Lie Algebras and Applications, American Mathematical Society Translation Series 2, Vol. 194, American Mathematical Society, Providence, RI, 1999, pp. 81–112; math.AG/9707010, 1997.

[8] B. Feigin, M. Finkelberg, A. Kuznetsov, I. Mirkovic, Semiinfinite flags II. Local and global intersection cohomology of quasimaps' spaces, in: Differential Topology, Infinite-Dimensional Lie Algebras and Applications, American Mathematical Society Translational Series 2, Vol. 194, American Mathematical Society, Providence, RI, 1999, pp. 113–148.

[9] D. Gaitsgory, Construction of central elements of the affine Hecke algebra via nearby cycles, Invent. Math. 144 (2001) 253–280.

[10] V. Ginzburg, Perverse sheaves on a loop group and Langlands' duality, math.AG/9511007, 1995.

[11] D. Kazhdan, G. Lusztig, Tensor structures arising from affine Lie algebras, I, J. Amer. Math. Soc. 6 (1993) 905–947; II, J. Amer. Math. Soc. 6 (1993) 949–1011; III, J. Amer. Math. Soc. 7 (1994) 335–381; IV, J. Amer. Math. Soc. 7 (1994) 383–453.

[12] M. Kashiwara, T. Tanisaki, The Kazhdan–Lusztig conjecture for affine Lie algebras with negative level, Duke Math. J. 77 (1995) 21–62.

[13] G. Lusztig, Introduction to Quantum Groups, Progress in Mathematics, Vol. 110, Birkhäuser, Boston, 1993.

[14] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. 3 (1990) 257–296.

[15] G. Lusztig, Modular representations and quantum groups, Contemp. Math. 82 (1989) 59–77.

[16] S. Montgomery, Hopf algebras and their actions on rings, Regional Conference Series in Mathematics, Vol. 82, American Mathematical Society, Providence, RI, 1993.

[17] I. Mirkovic, K. Vilonen, Perverse sheaves on loop Grassmannians and Langlands duality, preprint math.AG/9703010, (1997).