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SOME STABLE HOMOTOPY OF COMPLEX PROJECTIVE SPACE†

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§1. INTRODUCTION

IN ORDER to analyze the suspension homomorphism it is natural to study the stable homotopy of $K(Z, n)$- spaces; recently Mahowald and Williams [10] have made detailed calculations in the metastable range. However, this range is negligible in the important space $K(Z, 2)$, alias the complex projective space CP.

In this paper we use some recent techniques to revitalize an old tool, the spectral sequence obtained from the stable homotopy exact couple. Under the usual multiplication on CP, the stable homotopy $\pi_*{}^sCP$ becomes a graded ring and the spectral sequence is a spectral sequence of rings. The free part of this ring is easy to obtain; the torsion seems quite complicated. Nonetheless, our general results give information about differentials d^r for arbitrarily large r. Considerable calculation of $\pi_*{}^sCP$ is feasible.

Using both these methods and *ad hoc* techniques, including comparison with the Adams spectral sequence, we compute, up to some group extensions, the 2-component of the groups π_k^sCP for $k \leqslant 19$. These calculations extend results of Liulevicius, who, using the Adams sequence as the basic tool, published the values of these groups for $k \leqslant 8$ [7] and obtained them for $9 \leqslant k \leqslant 12$ (unpublished).

By a result of Toda, our methods easily apply to calculation of metastable homotopy groups of unitary groups. In particular we here evaluate the 2-component of the homotopy group $\pi_{2n+7}U(n)$ for $n \equiv 5$ (8) and $n \equiv 1$ (16), $n \neq 1$. While the groups $\pi_{2n+7}U(n)$ have been computed elsewhere [11], our results do not coincide in these two congruence classes.

In the case of $CP = K(Z, 2) = BU(1)$, the spectral sequence may be considered as the spectral sequence of a classifying space [13] for the generalized homology theory called stable homotopy, which suggests one possible method of attack on $\pi_*{}^sK(Z, 3)$.

The paper is organized as follows. In §2 we describe the free part of $\pi_*{}^sCP$ and list the results of our calculations of the 2-torsion. Section 3 describes the spectral sequence. General results on differentials are to be found in §4. These results are based on the J, K, and S-theory of stunted complex projective spaces.

We begin the calculation for the 2-torsion in §5 with explicit general results on d^k for $k \leqslant 4$, while in §6 we display E^∞ and list the remaining differentials in the range of the calculation. The Adams spectral sequence for the 2-component appears in §7, where, both for

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completeness and for reference, we display, without proof, its E_∞ through degree 19. The displays of §6 and §7 complete the calculation summarized in (2.2).

Sections 8 through 11 contain deferred proofs. In particular, the proof of (2.4) is completed in §9.

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§2. STATEMENT OF RESULTS ON HOMOTOPY GROUPS

The reduced integral homology $H_* CP$ is well-known to be the polynomial ring with divided powers on one generator x of degree 2. Let $h_* : \pi_*{}^s CP \to H_* CP$ be the stable Hurewicz homomorphism; h_* is a homomorphism of Pontrjagin rings. By the Hurewicz theorem, h_2 is an isomorphism. Let $\alpha \in \pi_*{}^s CP$ correspond to x.

We now describe the "free part" of $\pi_*{}^s CP$; let T be the ideal of torsion elements.

THEOREM 2.1. *The ring $(\pi_*{}^s CP)/T$ is the polynomial ring generated by the class of α; h_* maps $(\pi_*{}^s CP)/T$ isomorphically onto the subring generated by x.*

(2.1) is in essence due to Toda [16], although not in this form. We include a proof in §3.

We next summarize our results on the 2-primary torsion.

THEOREM 2.2. *The 2-primary torsion of the groups $\pi_k{}^s CP$ for $9 \leqslant k \leqslant 19$ is given, up to group extensions, by Table* (2.3).

TABLE 2.3

9	10	11	12	13	14	15	16	17	18	19
Z_8	0	Z_4	0	$Z_4 ? 2Z_2$	Z_2	$Z_2 ? Z_4 ? Z_2$	$2Z_2 ? Z_2$	$Z_{16} ? 2Z_2 ? Z_4$	Z_4	$Z_4 + Z_{64}$

In (2.3) the symbol $A ? B$ denotes a group satisfying an exact sequence $0 \to A \to A ? B \to B \to 0$.

The explicit data in the calculations will enable the reader to obtain homotopy-theoretic descriptions of the elements of these groups and to compute many of the Pontrjagin products. We do not do this systematically here, but remark that (6.1) and (7.1) contain additional information.

We turn now to the new results on $\pi_{2n+7}\, U(n)$. The group $\pi_{2n}\, U(n)$ is well-known to be cyclic of order $n!$; let u_n be a generator. Let σ generate the stable group $\pi_{2n+7}\, S^{2n}$.

THEOREM 2.4. *The 2-component of $\pi_{2n+7}\, U(n)$ for $n \equiv 1$ (16), $n \neq 1$, is Z_8; for $n \equiv 5$ (8) this group is Z_4. In each case a generator is $u_n \sigma$.*

The proof of (2.4) is completed in §9.

The results of (2.2) and (2.4) are not intended to represent the limit of applicability of the methods. The calculation in the 2-component can certainly be pressed further; odd-primary components can easily be obtained in comparable ranges.

§3. DESCRIPTION OF THE SPECTRAL SEQUENCE

The space CP has a natural filtration, namely by the subcomplexes CP^p, and a natural basepoint CP^0. The usual multiplication on CP carries $CP^p \times CP^q$ into CP^{p+q} and thus is a map of filtered spaces [13], where $CP \times CP$ is given the product filtration.

Let $\pi_*{}^s$ denote reduced stable homotopy; $\pi_*{}^s$ is a multiplicative generalized homology theory [17]. Let $\{E^r; d^r\}$, $r \geqslant 1$, be the spectral functors, associated with the theory $\pi_*{}^s$ for a filtered space. For a filtered space X, the identification of $X \wedge S^0$ with X gives E^rX the structure of right differential E^rS^0-module. E^rS^0 is concentrated in first gradation 0, and the graded algebra $E^r_{0,*}S^0$ is isomorphic to G_*, the graded algebra of stable homotopy groups of spheres [15].

PROPOSITION 3.1. *$\{E^rCP; d^rCP\}$, $r \geqslant 1$, is a convergent spectral sequence of commutative bigraded differential right E^rS^0-algebras converging to the bigraded $E^\infty S^0$-algebra associated with the graded right G-algebra $\pi_*{}^sCP$.*

(3.1) is known; see for example [13].

Let H denote reduced integral homology and ${}_HE$ the corresponding spectral functors. Then ${}_HE^1CP$ is the polynomial ring with divided powers on one generator $v \in {}_HE^1_{1,1}CP$. The spectral sequence is trivial. v corresponds to $x \in H_2\,CP$.

By inspection of E^1CP, it is apparent that the Hurewicz homomorphism h induces an isomorphism $E^1_{p,p}(h)\colon E^1_{p,p}CP \to {}_HE^1_{p,p}CP$ for each $p \geqslant 1$. Let $\iota_p \in E^1_{E,p}CP$ correspond to v_p. We see also that $E^1_{p,p+q}CP$ is isomorphic to G_q under right action of $E^1_{0,q}S^0$ on ι_p. For $\beta \in G_q = E^1_{0,q}S^0$, let β_p denote the corresponding element of $E^1_{p,p+q}CP$.

PROPOSITION 3.2. *The correspondence carrying $v_p \otimes \beta$ into β_p is an isomorphism ${}_HE^1CP \otimes E^1S^0 \to E^1CP$ of right E^1S^0-algebras.*

Proof. The diagram (3.3) shows that the map $E^1CP \otimes E^1CP \to E^1CP$ is right $E^1S^0 \otimes E^1S^0$-linear.

DIAGRAM 3.3

$$\begin{tikzcd}
CP \wedge CP \wedge S^0 \wedge S^0 \arrow[r] \arrow[d] & CP \wedge S^0 \wedge S^0 \arrow[r] & CP \arrow[d] \\
CP \wedge S^0 \wedge CP \wedge S^0 \arrow[r] & CP \wedge CP \arrow[r] & CP
\end{tikzcd}$$

It is thus sufficient to check the asserted multiplication formula on a set of generators of E^1CP as an E^1S^0-module, namely the ι_p. But $E^1(h)$ is a ring homomorphism, whence the result.

In the sequel we usually write E^r for E^rCP. E^1 thus has the following description. For each $p \geqslant 1$, column p contains a copy of G beginning with $\iota_p \in E^1_{p,p}$. We omit column 0. The multiplication is given by $\beta_p\gamma_q = \binom{p+q}{p}(\beta\gamma)_{p+q}$, where $\beta\gamma$ is computed in G.

We now turn to the proof of (2.1). In this proof and throughout the paper, in stable arguments we omit to write the suspensions, even though the mappings may not exist until all the spaces are each suspended the same suitable number of times. Also we sometimes write number for elements of groups with chosen generators.

Proof of (2.1). Recall that we have chosen $\alpha \in \pi_2{}^s CP$ as the pre-image of x under the Hurewicz map. In the spectral sequence α corresponds to ι_1. Thus α^n corresponds to $n!\,\iota_n$ in the spectral sequence, maps under h to $x^n = n!\,x_n$, and is therefore of infinite order. It remains to show that α^n is not divisible, and thus suffices to show that if tx_n is in the image of h, then $n!$ divides t. Since CP^n is the $(2n+1)$-skeleton of CP, it suffices to verify the assertion in CP^n.

To that end, let $y \in H^2 CP$ have value 1 on x and let $\mu \in \tilde{K}_C(CP^n)$ have ch $\mu = e^y - 1$. Suppose $tx_n \in \operatorname{Im} h$. Then we have the stable diagram $f: S^{2n} \to CP^n$, where $H^{2n}(f)$ is multiplication by t. Thus $\mathrm{ch}_n f^*\mu = f^*\mathrm{ch}_n \mu = t/n!$; hence $t/n!$ is an integer. This completes the proof of (2.1).

We observe that we have proved the following.

COROLLARY 3.4. *$t\iota_n$ is a permanent cycle in the spectral sequence if and only if $n!$ divides t.*

§4. DIFFERENTIALS BY J, K, AND S-THEORY

In this section we state some general formulas for the differentials on certain elements. The main results are (4.7), (4.9), and (4.11).

We first recall the definition of the differential d^k in the stable homotopy exact couple.

Let $f_n: S^{2n-1} \to CP^{n-1}$ be the canonical fibration, so that $CP^n = CP^{n-1} \cup_{f_n} e^{2n}$. For $\beta \in G_q$, if $d^k\beta_n$ is defined, then $d^k\beta_n = \gamma_{n-k}$, where γ is defined by the stable Diagram (4.1).

DIAGRAM 4.1

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_n"] & CP^{n-1} \\ S^{2n-1+q} \arrow[u, "\beta"] \arrow[r] \arrow[d, "id"] & CP^{n-k} \arrow[u] \arrow[d] \\ S^{2n-1+q} \arrow[r, "\gamma"] & S^{2n-2k} \end{tikzcd}$$

By naturality, (4.1) may be replaced by (4.2).

DIAGRAM 4.2

$$\begin{tikzcd} S^{2n-1} \arrow[r, "g_n"] & CP^{n-1}/CP^{n-k-1} \\ S^{2n-1+q} \arrow[u, "\beta"] \arrow[r, "\gamma"] & S^{2n-2k} \arrow[u] \end{tikzcd}$$

where g_n is induced by f_n. Then $CP^n/CP^{n-k-1} = CP^{n-1}/CP^{n-k-1} \cup_{g_n} e^{2n}$. Thus data on the S-type of the complexes CP^n/CP^{n-k-1} will be helpful in determining d^k.

Let H_k be the complex line bundle over CP^{k-1} determined by $\tau \oplus 1 = kH_k$. Then CP^{n+k-1}/CP^{n-1} is the Thom space $T(nH_k)$. Let M_k be the Atiyah–Todd number, given explicitly in [5, p. 343]. Let J and the realification r be as in [3]. As usual, J may also denote the stable Hopf–Whitehead homomorphism. The following assembles for reference results of James [6], Atiyah and Todd [5], and Adams and Walker [3].

THEOREM 4.3. (a) *The S-type of* $T(nH_k) = CP^{n+k-1}/CP^{n-1}$ *depends only on* $JrnH_k$.
(b) $JrnH_k = 0$ *if and only if* $n \equiv 0\ (M_k)$.
(c) *If* $p + q + k \equiv 0\ (M_k)$, *the complexes* $T(pH_k)$ *and* $T(qH_k)$ *are S-dual.*
(d) $T(nH_k)$ *is S-coreducible if and only if* $n \equiv 0\ (M_k)$.
(e) $T(nH_k)$ *is S-reducible if and only if* $n + k \equiv 0\ (M_k)$.

By (4.2) we have that γ in the formula $d^k\beta_n = \gamma_{n-k}$ depends on both the S-type of CP^{n-1}/CP^{n-k-1} and the stable attaching map g_n; thus γ is determined by the S-type of $CP^n/CP^{n-k-1} = T((n-k)H_{k+1})$. This last, by (4.3a), depends only on $Jr(n-k)H_{k+1}$. Applying (4.3b), it follows that γ is a function of the congruence class of n mod M_{k+1}, rather than on n, except for $k > n$, when the differential goes off the page and is zero. Thus the columns of E^k are periodic with period M_k. Stated formally, we have the following.

PROPOSITION 4.4. *Suppose* $p \geqslant k - 1$. *Suppose* $s \equiv 0\ (M_k)$. Then $E^k_{p+s,q+s}$ and $E_{p,q}$ *are isomorphic, with* β_{p+s} *corresponding to* β_p *for* $\beta \in G$. *If* $s \equiv 0\ (M_{k+1})$, *the isomorphism commutes with* d^k.

We next investigate the first non-trivial differential on each column.

PROPOSITION 4.5. $d^k\iota_s$ *is defined if and only if* $s + 1 \equiv 0\ (M_k)$, *in which case* $d^k\iota_s = 0$ *if and only if* $s + 1 \equiv 0\ (M_{k+1})$.

Proof. From (4.2) we see that the combined assertion $d^k\iota_s$ is defined and zero is equivalent to the statement $g_s: S^{2s} \to CP^{s-1}/CP^{s-k-1}$ is S-null-homotopic, which in turn is equivalent to the S-reducibility of $CP^s/CP^{s-k-1} = T((s-k)H_{k+1})$. This last condition is by (4.3e) equivalent to $(s-k) + (k+1) \equiv 0\ (M_{k+1})$. The result follows.

We wish to specifically evaluate $d^k\iota_s$ for $s + 1 \equiv 0\ (M_{k+1})$. In the statement of the result, e_C is the invariant defined in [1]; we also require the following notation.

DEFINITION 4.6. Let e_k be the coefficient of z^k in the power series expansion of

$$\left(\frac{\log(1+z)}{z}\right)^{M_k}.$$

THEOREM 4.7. *Suppose* $s + 1 \equiv tM_k\ (M_{k+1})$. *Then* $d^k\iota_s = \beta_{s-k}$, where
(a) $\beta \in \operatorname{Im} J$
(b) $e_C\beta \equiv te_k \bmod 1$
(c) $\beta_{s-k} = 0 \in E^k_{s-k,s+k+1}$ *if and only if* $te_k \equiv 0 \bmod 1$ *if and only if* $s + 1 \equiv 0\ (M_{k+1})$.

(4.7) in effect locates and computes the first non-trivial differential on each column; it is the complex analogue of a result of Toda [14, Th. 1]. The proof of (4.7) is in §8.

We next show how the S-dual of the information of (4.7) may sometimes be used to evaluate higher differentials. Suppose for $\gamma \in G_q$ and $\alpha \in G_r$ it happens that $d^t\gamma_{n+k+t}$ is defined, with value δ_{n+k}, and that $\delta\alpha = 0$. Then surely $d^{t+1}(\gamma\alpha)_{n+k+t}$ is defined. We consider a stronger condition, namely that the stable diagram (4.8) exists.

Whenever (4.8) exists, $d^{t+k}(\gamma\alpha)_{n+k+t}$ is defined (and in fact is represented by the bottom row of (4.8)). Our result, stated in terms of secondary composition [15], gives a formula for this differential in certain congruence classes.

DIAGRAM 4.8

$$\begin{tikzcd}
S^{2n+2k+2t-1} \arrow[r, "g_{n+k+t}"] & CP^{n+k+t-1}/CP^{n-1} \\
S^{2n+2k+2t+q-1} \arrow[r] \arrow[u, "\gamma"] & CP^{n+k}/CP^{n-1} \arrow[u] \\
S^{2n+2k+2t+q-1+r} \arrow[r] \arrow[u, "\alpha"] & S^{2n} \arrow[u]
\end{tikzcd}$$

PROPOSITION 4.9. *Under the above notation, suppose that* $n \equiv 0$ (M_k), $s \equiv -n-1$ (M_{k+1}), $d^k \iota_s = \beta_{s-k}$, *and that* (4.8) *exists. Then* $d^{t+k}(\gamma\alpha)_{n+k+t}$ *may be written as* ψ_n, *where* $-\psi \in \langle \beta, \delta, \alpha \rangle$.

The proof of (4.9) appears in §8.

Note that under the hypotheses of (4.9), $d^k \iota_s$ is given by (4.7). Note also that the existence of (4.8) is a stronger condition than that $d^{t+k}(\gamma\alpha)_{n+k+t}$ be defined; as k increases, the verification of the existence of (4.8) becomes more difficult. This verification invariably involves an analysis of the cell structure of CP^{n+k}/CP^{n-1}.

Finally we state a partial result on higher differentials for elements on the diagonal, namely we compute e_C of the differential.

Definition 4.10. Let $a_{n+k,k}$ be the coefficient of y^{n+k} the power series expansion of $(e^y - 1)^n$.

PROPOSITION 4.11. Suppose $d^k t\iota_{n+k}$ is defined and suppose $\gamma_n \in d^k t\iota_{n+k}$. Then $e_C \gamma \equiv t a_{n+k,k} \bmod 1$.

Proof. Suppose $\gamma_n \in d^k t\iota_{n+k}$. Forming (4.2) and extending each row to the right in a cofibre sequence, we obtain the S-diagram (4.12).

DIAGRAM 4.12

$$\begin{tikzcd}
CP^{n+k-1}/CP^{n-1} \arrow[r] & CP^{n+k}/CP^{n-1} \arrow[r] & S^{2n+2k} \\
S^{2n} \arrow[r] \arrow[u] & S^{2n} \cup_{\gamma} e^{2n+2k} \arrow[r] \arrow[u, "f"] & S^{2n+2k} \arrow[u, "t"]
\end{tikzcd}$$

In (4.12) $H^{2n}(f)$ is an isomorphism and $H^{2n+2k}(f)$ is multiplication by t. Consider $\mu^n \in \tilde{K}_C(CP^{n+k}/CP^{n-1})$. $\operatorname{ch} \mu^n = (e^y - 1)^n$ and $\operatorname{ch}_n \mu^n = y^n$. Since $H^{2n}(f)$ is an isomorphism, $e_C \gamma$ is represented by $\operatorname{ch}_{n+k} f^* \mu^n = t \operatorname{ch}_{n+k} \mu^n = t a_{n+k,k}$. This completes the proof.

§5. GENERAL CALCULATIONS IN THE 2-COMPONENT

Let p be a prime. By the p-component of a group we mean its quotient by the subgroup of torsion elements of order prime to p; by the p-component of an integer, the highest power of p dividing it. We assert without proof the validity of the following simplifications in the calculation of the p-compnent of $\pi_*{}^s CP$.

First, we replace the E^1-term by its p-component. Second, in the infinite cyclic terms $E^r_{n,n}$ we replace q in $q\iota_n$ with its p-component. Third, in a formula $d^k \beta_{n+k} = q\gamma_n$, we replace q

by its p-component. Fourth, we replace the period M_k for the columns of E^k by the p-component of M_k.

For example, computing in the 2-component, the statement $d^4 4\iota_9 = 4\sigma_5$ is adequate, but, fully interpreted, asserts: there exist odd numbers s and t such that $d^4 4s\iota_9$ is defined and has value $4t\sigma_5$. Further, the statement provides the value in the 2-component of $d^4 4\iota_n$ for $n \equiv 9 \bmod 2^6$, although $M_5 = 2^6 3^2 5$.

We now turn to the 2-component exclusively and describe d^1, d^2, and d^3 and the formula for d^4 on the diagonal. The nomenclature of elements of G is as in [15].

Let $\beta \in G$.

PROPOSITION 5.1. $d^1\beta_{n+1} = n\eta\beta_n$.

PROPOSITION 5.2. $d^2\beta_{n+2} = \lambda\nu\beta_n$, *where* λ *is given by Table* (5.3).

TABLE 5.3

n mod 8	0	1	2	3	4	5	6	7
λ	0	2	1	1	2	0	1	1

PROPOSITION 5.4. $d^3\beta_{n+3} = 0$ *if* $n \equiv 0$ (2). *If* $n \equiv 1$ (2), *then* $d^3\beta_{n+3} = \gamma_n$, *where* γ *is an element of the secondary composition given by Table* (5.5).

TABLE 5.5

n mod 8	1	3	5	7
$\gamma \in$	$\langle 2\nu, \eta, \beta\rangle + \langle \eta, \nu, \beta\rangle$	$\langle \nu, \eta, \beta\rangle + \langle \eta, 2\nu, \beta\rangle$	$\langle \eta, \nu, \beta\rangle$	$\langle \nu, \eta, \beta\rangle$

PROPOSITION 5.6. *The value of* d^4 *on the diagonal is given by Table* (5.7), *a row of which asserts that* $d^4 t\iota_{n+4} = \lambda\sigma_n$ *and that* $t\iota_{n+4}$ *generates* $E^4_{n+4,n+4}$.

TABLE 5.7

Congruence class of n	t	λ
0 (8)	8	0
1, 2 (16)	4	8
9, 10 (16)	4	0
3 (16)	1	2
11 (32)	1	4

Congruence class of n	t	λ
27 (64)	1	8
59 (64)	1	0
4 (8)	8	8
5 (8)	4	4
6, 7 (8)	2	2

The proofs of (5.1), (5.2), (5.4), and (5.6) are in §9.

§6. SPECIFIC CALCULATIONS IN THE 2-COMPONENT

Here we work entirely in the 2-component and compute E^∞ in total degree $\leqslant 19$ by computing all differentials on elements of total degree $\leqslant 20$.

THEOREM 6.1. *Table* (6.2) *lists* E^∞ *for the* 2-*component of* $\pi_*{}^s CP$ *in total degree* $\leqslant 19$.

In (6.2) an entry at (p, q) represents a group in $E^\infty_{p,q}$, together with a generator. (6.2) follows by direct calculation from the following evaluation of the differentials.

TABLE 6.2

18	$Z_2(\nu\kappa)$								
17	$Z_2(\eta^*)$	$Z_8(2\rho)$ + $Z_2(\eta\kappa)$							
16	$Z_{16}(\rho)$	$Z_2(\sigma^2)$							
15	$Z_2(\kappa)$ + $Z_2(\sigma^2)$			$Z_4(\zeta)$					
14									
13		$Z_2(\zeta)$		$Z_2(\nu^3)$ + $Z_2(\eta\varepsilon)$					
12	$Z_2(\zeta)$				$Z_4(\sigma)$				
11		$Z_2(\eta\varepsilon)$	$Z_2(\varepsilon)$	$Z_4(2\sigma)$	$Z_2(\nu^2)$			$Z_2(4\nu)$	
10			$Z_2(\sigma)$						
9		$Z_4(2\sigma)$				$Z_2(\nu)$			$Z(128\iota)$
8	$Z_8(\sigma)$				$Z_2(2\nu)$			$Z(128\iota)$	
7	$Z_2(\nu^2)$						$Z(16\iota)$		
6						$Z(16\iota)$			
5		$Z_2(\nu)$			$Z(8\iota)$				
4	$Z_2(\nu)$			$Z(8\iota)$					
3			$Z(2\iota)$						
2		$Z(2\iota)$							
1	$Z(\iota)$								
	1	2	3	4	5	6	7	8	9

In the remainder of this section, we consider differentials only on elements of total degree $\leqslant 20$. Recall that d^1 and d^2 are given by (5.1) and (5.2) respectively.

LEMMA 6.3. *The non-zero values of* d^3 *are* $d^3 2v_4 = \varepsilon_1$ *and* $d^3 2v_8 = \varepsilon_5$.

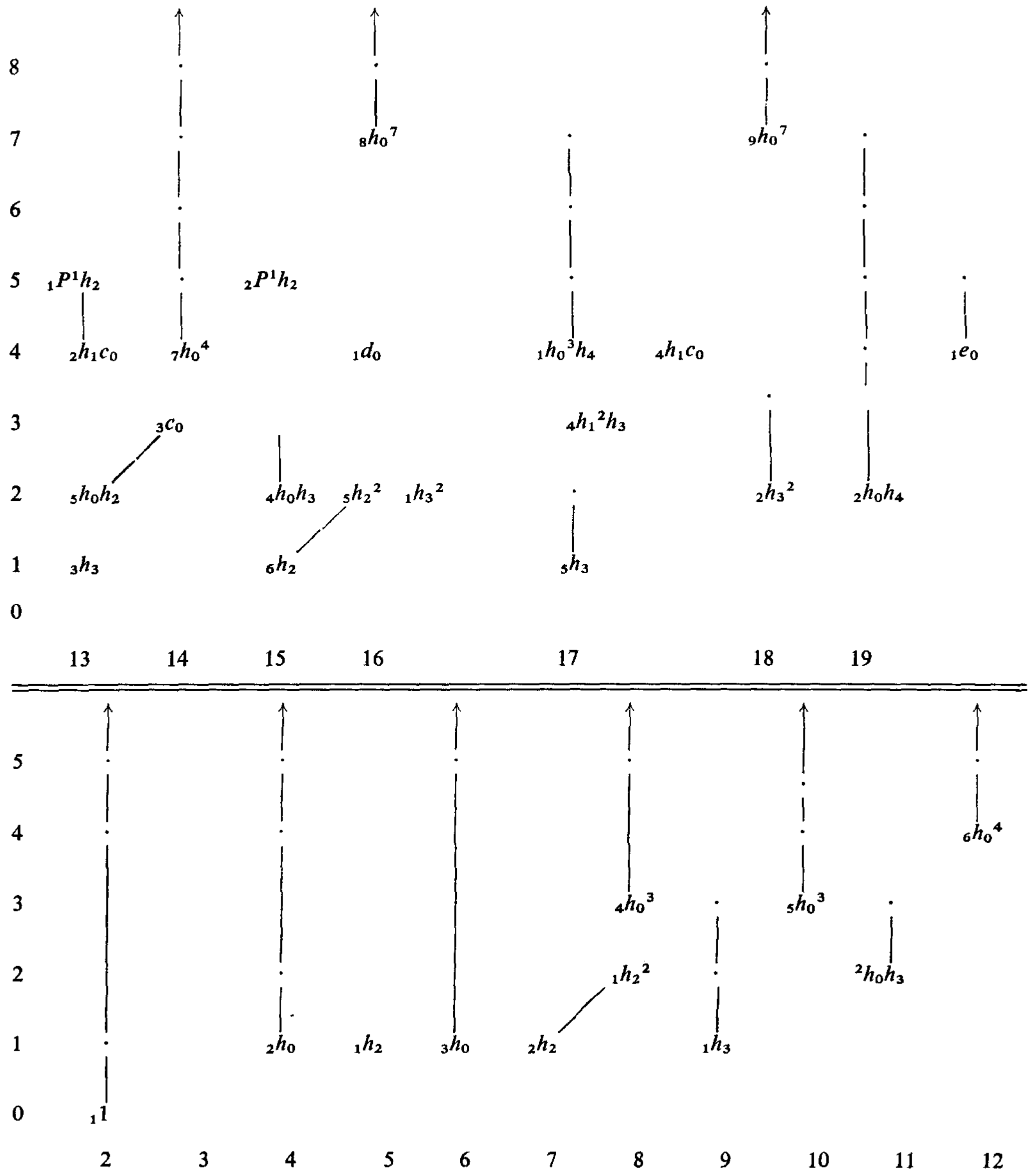
Recall also that d^4 on the diagonal is given by (5.6).

LEMMA 6.4. d^4 *vanishes on the off-diagonal elements.*

LEMMA 6.5. *The non-zero values of* d^5 *are* $d^5 8\iota_6 = \mu_1$, $d^5 16\iota_8 = \mu_3$, *and* $d^5 16\iota_{10} = \mu_5$.

LEMMA 6.6. *The non-zero values of* d^6 *are* $d^6 8\iota_7 = 2\zeta_1$, $d^6 32\iota_8 = 2\zeta_2$, $d^6 16\iota_9 = \zeta_3$, *and* $d^6 32\iota_{10} = 4\zeta_4$.

TABLE 7.2



LEMMA 6.7. $d^7 = 0$.

LEMMA 6.8. *The non-zero values of* d^8 *are* $d^8 64\iota_9 = 16\rho_1$ *and* $d^8 64\iota_{10} = 16\rho_2$.

LEMMA 6.9. *The non-zero value of* d^9 *is* $d^9 128\iota_{10} = \bar{\mu}_1$.

This completes the evaluation of the differentials on elements of total degree $\leqslant 20$. The proofs of (6.3) through (6.9) are in §10.

§7. THE ADAMS SPECTRAL SEQUENCE FOR $\pi_*{}^s CP$

Using the methods of [8], it is not hard to compute the E_2-term of the Adams spectral sequence [2] for the 2-component of $\pi_*{}^s CP$ in stems $\leqslant 20$. The differentials may then be evaluated by various means, including comparison with the homotopy exact couple. We do not reproduce the computation here, but state the result.

THEOREM 7.1. E_∞ *of the Adams spectral sequence for the* 2-*component of* $\pi_*{}^s CP$ *in degree* $\leqslant 19$ *is given by Table* (7.2).

In (7.2) we have used the nomenclature and notations of [8]. In particular, the subprefix denotes cell filtration. The vertical axis is s, Adams filtration degree, and the horizontal axis is $t - s$, homotopy dimension. For example, the entry ${}_2P^1h_2$ at (5, 15) asserts: there is an element of $E^{5,20}$ having as representative in $E_2^{5,20}$ a class in the image of $\mathrm{Ext}_A^{5,20}(\tilde{H}^*(CP^2), Z_2)$. Under the projection this class maps to $P^1h_2 \in \mathrm{Ext}_A^{5,20}(\tilde{H}^*(S^4), Z_2) = \mathrm{Ext}_A^{5,16}(Z_2, Z_2)$.

The vertical and diagonal lines indicate some non-zero multiplications by h_0 and h_1 respectively. They do not exclude the possibility of other such, although we know of none

While the calculations of section 6 are used in part of the deduction of (7.1), it is our intention to use (7.1) in the proof of (6.6) and (6.7). The portion of (7.1) so used is indeed obtained independently of (6.6) and (6.7).

We remark here that interaction between the homotopy exact couple and the Adams spectral sequence seems a useful tool in computation; for another example, the group extension for $\pi_{19}^s CP$ is determined by comparing the composition series given by the two spectral sequences. This type of analysis has been extensively used by Mahowald and Tangora [9].

We remark also that with (6.1) and (7.1), the proof of (2.2) is by inspection.

§8. PROOF OF (4.7) AND (4.9)

Proof of (4.7). By (4.3e) there is nothing to prove if $M_k = M_{k+1}$, which is the case for k odd, $k \neq 1$, according to [5, p. 344]. (4.7) is well-known for $k = 1$ (see also (5.1)). Thus we assume k is even.

To study $d^k\iota_s$, the Thom complex of interest is $T((s-k)H_{k+1}) = CP^s/CP^{s-k-1}$. In this complex, since $d^k\iota_s$ is defined, the attaching map for the top cell is simply $i_*\beta$, where $i: S^{2s-2k} \to CP^{s-1}/CP^{s-k-1}$ is the inclusion.

The S-dual to $T((s-k)H_{k+1})$ is $T(nH_{k+1}) = CP^{n+k}/CP^{n-1}$, where $n \equiv -tM_k\ (M_{k+1})$. In this complex the only cell attached to the bottom cell is the top cell. We thus have an obvious map $f: CP^{n+k}/CP^{n-1} \to S^{2n} \cup_\beta e^{2n+2k} = X$ inducing isomorphisms on H^{2n} and

H^{2n+2k}. By S-duality the attaching map in X is indeed β. We wish (a) to prove that X is a Thom complex, (b) to evaluate $e_C\beta$, and (c) to show that $e_C\beta$ determines $d^k\iota_s$.

For (a) and (c), consider Diagram (8.1), with notations as in [3].

DIAGRAM 8.1

$$\begin{tikzcd}
 & \tilde{K}_C(CP^k) \arrow[d, "r"] & \\
 & \tilde{K}_R(CP^k) \arrow[d, "J"] & \tilde{K}_R(S^{2k}) \arrow[l] \arrow[d, "J"] \\
J(CP^{k-1}) & J(CP^k) \arrow[l] \arrow[d, "\simeq"] & J(S^{2k}) \arrow[l, "p"'] \arrow[d, "\theta'_C"] \\
 & J'_C(CP^k) & J'_C(S^{2k}) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

In (8.1) the homomorphisms J and θ'_C are epimorphisms by definition. The upper row is exact by (6.3) of [3]; the lower, by (4.9) of [3]. The isomorphism is by (4.3) and (6.1) of [3], recalling k even.

Consider $nH_{k+1} - n$, the projection of nH_{k+1} into $\tilde{K}_C(CP^k)$. Applying Jr and restricting to $J(CP^{k-1})$, we get zero, since $n \equiv 0\ (M_k)$. Thus there is an element $\gamma \in \tilde{K}_R(S^{2k})$ such that $Jr(nH_{k+1} - n) = Jp\gamma$. Let ξ be a real vector bundle over S^{2k} such that $\xi - \dim \xi = \gamma \in \tilde{K}_R(S^{2k})$. Then $S^{2n} \cup_\beta e^{2n+2k}$ has the S-type of $T(\xi)$. Thus $\beta = J\xi$, and (4.7a) is proved.

While γ need not be unique in $\tilde{K}_R(S^{2k})$, the lower part of (8.1) shows that $\theta'_C J\gamma$ is a unique element of $J'_C(S^{2k})$. But by [1] elements of $J'_C(S^{2k})$ are measured by the invariant e_C. Since $d^k\iota_s = 0$ if and only if $Jr(nH_{k+1} - n) = 0$, we have by (8.1) that $d^k\iota_s$ is determined as an element of $E^k_{s-k,\,s+k-1}$ by $e_C\beta$. This proves (4.7c).

We turn now to (4.7b). To evaluate $e_C\beta$, consider the mapping $f\colon CP^{n+k}/CP^{n-1} \to X$, and as usual let y generate H^2CP. Let $w \in \tilde{K}_C(X)$ be such that $\mathrm{ch}\, w = S^{2n} + \lambda e^{2n+2k}$, where $\lambda \equiv e_C\beta \bmod 1$. Then $\mathrm{ch} f^*w = y^n + \lambda y^{n+k}$.

Following ideas and notations of [3], we may write $f^*w = \sum_{0 \leqslant i \leqslant k} w_i \mu^{n+i}$, where $\mathrm{ch}\, \mu = e^y - 1$ and $w_i \in Z$. Writing z for $\mathrm{ch}\, \mu$, we have

$$y^n + \lambda y^{n+k} = z^n \sum w_i z^i. \tag{8.2}$$

Define a_i by

$$\sum a_i z^i = \left(\frac{\log(1+z)}{z}\right)^n \tag{8.3}$$

Now $a_0 = 1$. For $n = M_k$, $a_i \in Z$, $0 \leqslant i < k$, and $a_k = e_k$. For $n = M_{k+1}$, $a_k \in Z$. It follows that for $n \equiv -tM_k\ (M_{k+1})$, $a_k \equiv -te_k \bmod 1$.

On the other hand, $y \equiv z$ mod higher powers, and $y/z = \sum a_i z^i$. Computing, using (8.2) and (8.3), and recalling $w_k \in Z$, we find that $a_k \equiv -\lambda \bmod 1$. This completes the proof of (4.7).

Proof of (4.9). We retain the notations of the proof of (4.7) and note in particular that n and s satisfy the same congruences as in (4.7) and its proof.

Under the hypotheses of (4.9), we may enlarge Diagram (4.8) to the stable Diagram (8.4).

DIAGRAM 8.4

$$\begin{tikzcd}
S^{2n+2k+2t-1} \arrow[r] & CP^{n+k+t-1}/CP^{n-1} & & \\
S^{2n+2k+2t+q-1} \arrow[r] \arrow[u, "\gamma"] & CP^{n+k}/CP^{n-1} \arrow[r] \arrow[u] & X \arrow[r] & S^{2n+2k} \\
S^{2n+2k+2t+q+r-1} \arrow[r] \arrow[u, "\alpha"] & S^{2n} \arrow[r] \arrow[u] & S^{2n} \arrow[u] &
\end{tikzcd}$$

The composite across the middle row of (8.4) represents δ; the bottom row is ψ. Recall that X has the cell structure $S^{2n} \underset{\beta}{\cup} e^{2n+2k}$.

(4.9) is now a direct consequence of [15, Prop. 1.8].

§9. PROOFS FOR (2.4) AND §5

Proof of (5.1). CP^{n+1}/CP^{n-1} has the cell structure $S^{2n} \cup e^{2n+2}$. The attaching map is non-trivial, and hence η, if and only if $Sq^2 y^n = y^{n+1}$, which is equivalent to $n \equiv 1$ (2).

Proof of (5.2). To see that $d^2\beta_{n+2} = \lambda\nu\beta_n$ for some value of λ, we observe that the stable cell structure of $CP^{n+2}/CP^{n-1} = S^{2n} \cup e^{2n+2} \cup e^{2n+4}$ is as follows. For $n \equiv 0(2)$, CP^{n+2}/CP^{n-1} has the S-type of $(S^{2n} \vee S^{2n+2}) \cup e^{2n+4}$, where the top cell is attached by $\lambda\nu \oplus \eta \in \Pi^s_{2n+3}(S^{2n}) \oplus \Pi^s_{2n+3}(S^{2n+2}) = G_3 \oplus G_1$. For $n \equiv 1(2)$, the top cell is attached to $CP^{n+1}/CP^{n-1} = S^{2n} \underset{\eta}{\cup} e^{2n+2}$, but the projection of the attaching map to $S^{2n+2} = CP^{n+1}/CP^n$ is zero. Therefore this attaching map stably factors through the inclusion $i: S^{2n} \to CP^{n+1}/CP^{n-1}$ and has form $i_*\lambda\nu$. It remains in each case to determine λ.

Following [5], for each prime p we define the function ν_p by $n = \Pi p^{\nu_p(n)}$. Since $\nu_2 M_3 = 3$, λ depends only on the congruence class of n mod 8. That $\lambda \equiv 1 \bmod 2$ for $n \equiv 2, 3$ (4) is obvious, since for these cases $Sq^4 y^n = y^{n+2}$. The assertions for the remaining congruences are the only ones consistent with the structure of E^2 as differential ring. For example, for $n \equiv 3$ (8), we have $d^2\iota_{n+2} = \nu_n$. Multiplying by ι_1, $d^2 2\iota_{n+3} = 4\nu_{n+1}$, yielding the asserted result for $n \equiv 4$ (8).

Proof of (5.4). Using (5.1) and (5.2), one computes the cell-structure of CP^{n+3}/CP^{n-1}. (5.4) now follows by inspection, using [15, Prop. 1.8] to interpret the differentials in terms of secondary composition.

Proof of (5.6). We need consider congruence mod 64, since $\nu^2 M_5 = 6$. That $t\iota_{n+4}$ generates $E^4_{n+4,n+4}$ follows by calculation, using (5.1) and (5.2). Now an element $\gamma \in G_7$ is known to be uniquely determined by $e_C\gamma$. On the other hand, by (4.11) the determination of $e_C d^4 t\iota_{n+4}$ is reduced to arithmetic. This arithmetic leads to (5.6).

Proof of (2.4). By naturality, our general formulas for differentials are valid in the spectral sequence for $\pi_*^s(CP^{n+k}/CP^{n-1})$ obtained from the stable homotopy exact couple. Using only the results of §5, it is a routine task to compute all differentials on elements of total degree $\leqslant 2n + 8$ in the spectral sequence for CP^{n+4}/CP^{n-1}. For $n \equiv 1$ (16), this computation is just a portion of that leading to (6.1). Thus for such n (see column 1 of (6.2)) $\pi^s_{2n+7}(CP^{n+4}/CP^{n-1}) = Z_8$ generated by $\iota_n\sigma$, where ι_n is the basic class in π^s_{2n}.

A similar calculation yields for $n \equiv 5$ (8) that $\pi^s_{2n+7}(CP^{n+4}/CP^{n-1}) = Z_4$ generated by $\iota_n \sigma$. The crucial result is the entry of (5.7) for $n \equiv 5$ (8).

(2.4) now follows by a result of Toda [16, Th. 4.3].

§10. PROOFS FOR §6

Proof of (6.3). By (5.4) and (5.5), (6.3) follows in principle by inspection, provided one can compute the requisite secondary compositions. Most of these appear in or follow readily from results of [15]. However, a key step in the evaluation of $d^3\zeta_4$ is the following lemma, for the proof of which we are indebted to M. E. Mahowald.

LEMMA 10.1. $\langle \zeta, \nu, \eta \rangle \equiv 0 \bmod \eta\rho$.

Proof. Let z generate $\pi_{11}SO$, so that $Jz = \zeta$. Then in the diagram $S^{15} \xrightarrow{\eta} S^{14} \xrightarrow{\nu} S^{11} \xrightarrow{z} SO$ we have $\nu\eta = 0$ and $z\nu = 0$. By naturality it follows that $\langle \zeta, \nu, \eta \rangle \equiv 0 \bmod \operatorname{Im} J$. But $\eta\rho$ generates $\operatorname{Im} J$ in G_{16}. This completes the proof.

Proof of (6.4). By dimensional arguments, the only elements on which we need evaluate d^4 are σ_5, $2\sigma_6$, and μ_5. Now $2\sigma_6 = d^4 2\iota_{10}$; thus $d^4 2\sigma_6 = 0$.

We dispense with μ_5 as follows. Inspection of the cell structure of CP^5 shows that $d^4\mu_5$ is represented by $\langle 2\nu, \nu, \mu \rangle_1 \in E^4_{1,17} = Z_2$ generated by $\eta^*{}_1$. But by results of [15], $\langle 2\nu, \nu, \mu \rangle = \langle \zeta, \eta, \nu \rangle$, which is settled by the technique of the proof of (10.1).

We state the value of $d^4\sigma_5$ as a lemma and postpone the proof until §11.

LEMMA 10.2. $d^4\sigma_5 = 0$.

Proof of (6.5). The three non-zero values of d^5 are obtained by direct application of (4.9), under the substitutions $t = 4$, $k = 1$, $\beta = \eta$, $\delta = 8\sigma$, and $\alpha = 2\iota$ in each case. For $n = 1$, we take $\gamma = 4\iota$, while for $n = 3$ or 5, take $\gamma = 8\iota$.

It remains to show that d^5 vanishes on $8\iota_7$ and $16\iota_9$. These results follow from the evaluation of d^5 on the preceding columns by multiplication by ι_1.

Proof of (6.6). Since the 2-component of 7! is 16, by (3.4) it follows that $d^6 8\iota_7$ has order precisely 2 in $E^6_{1,12} = Z_4$ generated by ζ_1; hence $d^6 8\iota_7 = 2\zeta_1$. The values of d^6 on $32\iota_8$, $16\iota_9$, and $32\iota_{10}$ follow by successive multiplication by ι_1.

By dimensional arguments it remains only to find $d^6 4\nu_8$; we state the value as a lemma and defer the proof to §11.

LEMMA 10.3. $d^6 4\nu_8 = 0$.

Proof of (6.7). By dimensional arguments, we need evaluate d^7 only on $4\nu_8$. We prove the following in §11.

LEMMA 10.4. $d^7 4\nu_8 = 0$.

Proof of (6.8). Since the 2-component of 9! is 128, it follows by (3.4) that $d^8 64\iota_9$ is of order precisely 2 and thus is $16\rho_1$. The value of d^8 on $64\iota_{10}$ follows by multiplication by ι_1.

Proof of (6.9). This differential follows from (4.9), taking $n = 1$, $t = 8$, $k = 1$, $\beta = \eta$ $\delta = 16\rho$, $\alpha = 2\iota$, and $\gamma = 64\iota$.

§11. PROOFS FOR §10

Proof of (10.2). Inspection of the cell structure of CP^5 shows that $d^4\sigma_5$ is represented by $\langle 2\nu, \nu, \sigma\rangle_1 \in E^4_{1,15} = Z_2 + Z_2$ generated by $\sigma_1{}^2$ and κ_1. Now $\langle 2\nu, \nu, \sigma\rangle$ contains the two elements 0 and σ^2. To complete the proof of (10.2), we establish the following, which shows that $d^4\sigma_5$ cannot be $\sigma_1{}^2$.

LEMMA 11.1. *Let K be the stable complex $S^2 \underset{\sigma^2}{\cup} e^{17}$. Then there is no S-map $f: K \to CP^5$ such that $H^2(f; Z_2) \neq 0$.*

Proof. (See [12] for these techniques.) Let L be the stable complex $S^9 \underset{\sigma}{\cup} e^{17}$. Let $g: L \to K$ be such that $Sq_g{}^8: H^2(K) \to H^9(L)$ is non-zero. Let ψ be the secondary cohomology operation arising from the Adem relation $Sq^8Sq^8 + Sq^{12}Sq^4 + Sq^{14}Sq^2 = 0$, valid on a class in the image of reduction mod 2. Suppose f exists. Let h be the composition fg. By the Peterson–Stein formula, it follows that $0 = h^*\psi(y) = Sq^8\,Sq_h{}^8(y) + Sq^{12}Sq_h{}^4(y) + Sq^{14}\,Sq_h{}^2(y) = Sq^8Sq_h{}^8(y) \neq 0 \bmod 0$, a contradiction. This completes the proof of (11.1), and hence of (10.2).

Proof of (10.3) *and* (10.4). This proof will be based on comparison with the Adams spectral sequence; see Table (7.2).

We first observe that, regardless of d^6 and d^7 on $4\nu_8$, our homotopy spectral sequence tells the order of $\pi^s_{17}\,CP$. It is this observation which assures us that the entries of column 18 of (7.2) are indeed permanent cycles in the Adams spectral sequence; otherwise column 17 would have too few non-zero elements.

But by dimensional reasons, neither ${}_2h_3{}^2$ or ${}_2h_0\,h_3{}^2$ can be boundaries in the Adams sequence. It follows that $\pi^s_{18}\,CP$ contained an element of order 4.

Now compare this last with the homotopy spectral sequence. The only two non-zero entries representing 2-torsion of total degree 18 are $\eta^*{}_1$ and $\sigma_2{}^2$. It follows that neither is killed in the spectral sequence, but these are the potential non-zero values of d^7 and d^6 on $4\iota_8$. This completes the proof.

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