An Extension Problem for *H*-Unitary Matrices with Applications to Hermitian Toeplitz Matrices

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ABSTRACT

Given a Hermitian matrix H, a matrix U is said to be H-unitary if $U^HHU = H$. We consider the following extension problem: If U_0 is a rectangular matrix such that $U_0^HHU_0 = A$, where A is a leading principal submatrix of H, can U_0 be extended to an H-unitary matrix? After presenting necessary conditions for a more general situation, we state a necessary and sufficient criterion for this problem and give a description of all its solutions. Finally, these results are used to derive some properties of factorizations of Hermitian Toeplitz matrices.

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1. INTRODUCTION

Let H be a Hermitian $n \times n$ matrix. An $n \times n$ matrix U is called H-unitary if

$$U^HHU = H$$

(cf. [10, p. 21]). We consider the following extension problem:

(P) Given an $n \times m$ matrix U_0 $(1 \leqslant m < n)$ such that

$$U_0^HHU_0 = A,$$

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where

$$H=\underbrace{\begin{pmatrix} A & B \\ B^H & C \end{pmatrix}}_{m}\}m,$$

find an $n\times(n-m)$ matrix U_1 such that $U=(U_0\ U_1)$ is an H-unitary matrix.

If H is nonsingular, (P) has a solution iff U_0 has full column rank m. Obviously, this condition is necessary, since H-unitary matrices are nonsingular in this case. The fact that $\operatorname{rank} U_0 = m$ guarantees the solvability of (P) follows from more general results [13, p. 67] on unitary operators in nondegenerate indefinite inner product spaces which were extensively studied by Krein, Iohvidov, and others (see [13] and the references quoted therein).

We are especially interested in the case of singular H. Here the condition $\operatorname{rank} U_0 = m$ is no longer sufficient, as the example

$$H=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\quad U_0=\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\qquad n=3,\quad m=2$$

shows.

In this note, a criterion for the solvability of (P) for arbitrary Hermitian matrices H is presented. In Section 2, we first consider a more general situation and formulate necessary conditions. In Section 3, we return to the matrix case and state a criterion which is necessary and also sufficient. Moreover, a description of all solutions of (P) is given. Finally, in the last section, we apply these results to extension problems of the form (P) which arise in connection with factorizations of Hermitian Toeplitz matrices.

2. A NECESSARY CONDITION

Let $\mathscr{E}$ be a linear space over $\mathbb{C}$ with positive definite inner product $(\cdot,\cdot)$. Given a Hermitian linear operator $H:\mathscr{E}\to\mathscr{E}$, we define by

$$[x,y]=(x,Hy),\qquad x,y\in\mathscr{E},$$

an additional inner product which is in general indefinite and which may be degenerate (see Bognár [2] for notations). Let $U_0:\mathscr{D}(U_0)\to\mathscr{E}$, where $\mathscr{D}(U_0)$

$\subset \mathscr{E}$, be a linear operator which is an isometry, i.e.

$$[U_0 x, U_0 y] = [x, y], \qquad x, y \in \mathscr{D}(U_0).$$

We wish to extend U_0 to an operator $U: \mathscr{E} \to \mathscr{E}$ which is an isometry on the whole space $\mathscr{D}(U) = \mathscr{E}$. For finite dimensional spaces $\mathscr{E}$ this problem reduces to (P).

Denote by

$$\mathscr{E}^0 = \{x \in \mathscr{E} | [x, y] = 0 \text{ for all } y \in \mathscr{E}\} \tag{1}$$

the isotropic part of $\mathscr{E}$, and set

$$\mathscr{E}_0^0 = \mathscr{E}^0 \cap \mathscr{D}(U_0). \tag{2}$$

Note that $\mathscr{E}^0$ is just the kernel of H, and thus $\mathscr{E}^0 = \{0\}$, i.e. $[\cdot,\cdot]$ is nondegenerate, iff H is injective. One can find subspaces $\mathscr{E}_1^0$ and $\mathscr{E}_0^1$ complementary to $\mathscr{E}_0^0$ such that

$$\mathscr{E}^0 = \mathscr{E}_0^0 \oplus \mathscr{E}_1^0, \qquad \mathscr{D}(U_0) = \mathscr{E}_0^0 \oplus \mathscr{E}_0^1, \tag{3}$$

and finally a subspace $\mathscr{E}_1^1$ complementary to $\mathscr{E}^0 \oplus \mathscr{E}_0^1$ leading to the decomposition

$$\mathscr{E} = \mathscr{E}^0 \oplus \mathscr{E}^1 \qquad \text{with} \quad \mathscr{E}^1 = \mathscr{E}_0^1 \oplus \mathscr{E}_1^1. \tag{4}$$

Then U_0 can be represented as the operator matrix

$$U_0 = \begin{array}{cl} \begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} & \begin{array}{l} \mathscr{E}^0 \\ \mathscr{E}^1 \end{array}, \\ \begin{array}{cc} \mathscr{E}_0^0 & \mathscr{E}_0^1 \end{array} & \end{array} \tag{5}$$

and we define

$$\tilde{U}_0 = V_{11}: \mathscr{E}_0^1 \to \mathscr{E}^1. \tag{6}$$

Roughly speaking, $\tilde{U}_0$ is just U_0 considered modulo $\mathscr{E}^0$.

Now suppose that U_0 admits an extension to an isometric operator

$$U=\begin{array}{cc} \begin{pmatrix} * & * \\ * & \tilde{U} \end{pmatrix} & \begin{array}{l} \mathscr{E}^0 \\ \mathscr{E}^1 \end{array} \\ \begin{array}{cc} \mathscr{E}^0 & \mathscr{E}^1 \end{array} & \end{array}$$

on the whole space $\mathscr{E}$. Obviously,

$$\tilde{U}: \mathscr{E}^1 \to \mathscr{E}^1 \tag{7}$$

is an isometric extension of $\tilde{U}_0$ on $\mathscr{E}^1$. Since $\mathscr{E}^1$ is a complementary subspace to the isotropic part $\mathscr{E}^0$, the restriction of the inner product $[\cdot,\cdot]$ to $\mathscr{E}^1 \times \mathscr{E}^1$ is nondegenerate (see [2, p. 11]), and therefore any isometry on $\mathscr{E}^1$ is injective [2, p. 31]. In particular, $\tilde{U}_0$ has to be injective. If in addition, the operator (7) is surjective, then $V_{10}=0$ in (5). This follows from

$$0=[x,y]=[Ux,Uy]=[\tilde{U}x,V_{10}y], \qquad x\in\mathscr{E}^1, \quad y\in\mathscr{E}_0^0\subset\mathscr{E}^0.$$

Thus, we have proved the following

THEOREM 1. *Let $U_0: \mathscr{D}(U_0)\to\mathscr{E}$ be an isometry with respect to the inner product $[\cdot,\cdot]$, and $\mathscr{E}_0^1, \mathscr{E}^1, \tilde{U}_0$ be defined by* (1)–(6). *Then U_0 can be extended to an isometric* (*with respect to* $[\cdot,\cdot]$) *operator on $\mathscr{E}$ only if $\tilde{U}_0: \mathscr{E}_0^1 \to \mathscr{E}^1$ is an injective operator.*

Moreover, if U_0 admits an isometric extension on $\mathscr{E}$ such that the operator (7) *is surjective, then $V_{10}=0$ in* (5).

REMARK 1. In general, the necessary condition given in Theorem 1 is not sufficient. The injectiveness of $\tilde{U}_0$ does not even guarantee that $\tilde{U}_0$ can be extended to $\mathscr{E}^1$. Sufficient conditions for this nondegenerate extension problem on $\mathscr{E}^1$ for the special case that $\mathscr{E}^1$ is a Krein space [2, p. 100] can be found in Azizov [1], Iohvidov, Krein, and Langer [13], Bognár [2], and Young [17].

REMARK 2. If $\mathscr{E}^1$ is of finite dimension, then $\tilde{U}$ is always surjective and thus necessarily $V_{10}=0$ in this case.

3. THE COMPLETE SOLUTION OF (P)

Now we return to problem (P). Throughout this section let H be a given Hermitian $n \times n$ matrix and U_0 an $n \times m$ matrix $(1 \leq m < n)$ with

$$U_o^H H U_0 = A, \qquad \text{where} \quad H = \begin{array}{c} \left(\begin{array}{cc} A & B \\ B^H & C \end{array} \right) \begin{array}{l} \}\, m \\ \end{array} \\ \begin{array}{cc} \underbrace{}_{m} & \end{array} \end{array} . \tag{8}$$

The ranks of H and A are denoted by n_0 and m_0, respectively. We will solve problem (P) completely by first reducing it to a normal form which then admits a simple description of all solutions. We start with the following

Lemma 1. *There is a nonsingular $n \times n$ matrix*

$$T = \begin{array}{c} \left(\begin{array}{cc} T_{11} & T_{12} \\ 0 & T_{22} \end{array} \right) \begin{array}{l} \}\, m \\ \end{array} \\ \begin{array}{cc} \underbrace{\phantom{T_{11}}}_{m} & \phantom{T_{22}} \end{array} \end{array} \tag{9}$$

such that

$$T^H H T = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ \hline 0 & 0 & I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \}\, d_0 \\ \}\, m_0 \\ \}\, k \\ \\ \}\, n_0 - m_0 - 2k \\ \}\, d_1 \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} m \tag{10}$$

where Λ_1 and Λ_2 are real signature matrices, i.e. diagonal matrices with diagonal elements ±1. *In particular,*

$$T_{11}^H A T_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{11}$$

The normal form (10) *is uniquely determined up to permutations of the diagonal elements of* Λ_1 *and* Λ_2, *respectively.*

Proof. Using the eigenvalue decomposition of A, one can find a nonsingular $m \times m$ matrix W_1 such that

$$\begin{pmatrix} W_1^H & 0 \\ 0 & I_{n-m} \end{pmatrix} H \begin{pmatrix} W_1 & 0 \\ 0 & I_{n-m} \end{pmatrix} = \left(\begin{array}{cc|c} \Lambda_1 & 0 & \\ & & W_1^H B \\ 0 & 0 & \\ \hline B^H & W_1 & C \end{array}\right) \quad (=: H_1),$$

where Λ_1 is an $m_0 \times m_0$ signature matrix. Let

$$W_1^H B = \begin{pmatrix} D \\ E \end{pmatrix} \begin{matrix} \}\, m_0 \\ \end{matrix},$$

$k = \operatorname{rank} E$, and set $d_0 = m - m_0 - k$. There exist a nonsingular matrix R and a permutation matrix P such that

$$REP = \begin{pmatrix} 0 & 0 \\ I_k & F \end{pmatrix} \begin{matrix} \}\, d_0 \\ \end{matrix}.$$

By multiplying H_1 from the right by

$$\left(\begin{array}{cc|c} I_{m_0} & 0 & -\Lambda_1^{-1} DP \\ 0 & R^H & 0 \\ \hline 0 & 0 & P \end{array}\right)$$

and by its corresponding Hermitian matrix from the left, we transform H_1 into

$$H_2 = \left(\begin{array}{ccc|cc} \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & F \\ \hline 0 & 0 & I_k & G_1 & G_2 \\ 0 & 0 & F^H & G_2^H & G_3 \end{array}\right) \begin{matrix} \\ \}\, d_0 \\ \\ \\ \\ \end{matrix}$$

Setting

$$Z = \left(\begin{array}{ccc|cc} 0 & I_{m_0} & 0 & 0 & 0 \\ I_{d_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_k & -\frac{1}{2}G_1 & -G_2+G_1F \\ \hline 0 & 0 & 0 & I_k & -F \\ 0 & 0 & 0 & 0 & I_{n-m-k} \end{array}\right),$$

one further obtains

$$H_3 := Z^H H_2 Z = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 \\ \hline 0 & 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & G \end{array}\right),$$

where

$$G = G^H = G_3 + F^H G_1 F - G_2^H F - F^H G_2.$$

Finally, the matrix

$$\left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & I_k & 0 \\ 0 & 0 & W_2^H \end{array}\right) H_3 \left(\begin{array}{c|cc} I_m & 0 & 0 \\ \hline 0 & I_k & 0 \\ 0 & 0 & W_2 \end{array}\right),$$

where the nonsingular matrix W_2 satisfies

$$W_2^H G W_2 = \begin{pmatrix} \Lambda_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_2 \text{ a signature matrix,}$$

is of the desired form (10), and the product T of all used transformation matrices is of the type (9).

From (10)

$$T_{11}^H(A \quad B)T = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \end{array}\right),$$

and thus

$$\operatorname{rank}(A \quad B) = m_0 + k = \operatorname{rank} A + k.$$

Hence k (and therefore d_0, d_1) in (10) are independent of the transformation matrix. This concludes the proof of the lemma. ∎

In view of Lemma 1, (8) is equivalent to

$$U_0'^H H' U_0' = A', \tag{12}$$

where $U_0' = T^{-1}U_0T_{11}$, and H' and A' are given by (10) and (11), respectively. We wish to extend U_0' to an H'-unitary $n \times n$ matrix. According to the previous section, such an extension can only exist if a corresponding nondegenerate extension problem has a solution. Obviously, one obtains such a nondegenerate problem by simply deleting the first d_0 and the last d_1 rows and columns of H' and the first d_0 rows and columns of A'. Thus we arrive at

$$\tilde{U}_0^H \tilde{H} \tilde{U}_0 = \tilde{A}, \tag{13}$$

where

$$\tilde{H} = \left(\begin{array}{cc|cc} \Lambda_1 & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ \hline 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & \Lambda_2 \end{array}\right), \qquad \tilde{A} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and $\tilde{U}_0$ is the $n_0 \times (m_0 + k)$ matrix defined by

$$U_0' = \left(\begin{array}{cc} * & * \\ V_{10} & \tilde{U}_0 \\ \underbrace{*}_{d_0} & * \end{array}\right) \begin{array}{l} \}\, d_0 \\ \}\, n_0 \\ \}\, d_1 \end{array}.$$

Note that (12) implies

$$\tilde{U}_0^H \tilde{H} V_{10} = 0, \qquad V_{10}^H \tilde{H} V_{10} = 0. \tag{14}$$

In view of Theorem 1 and Remark 2, (P) can only have a solution if

$$V_{10} = 0. \tag{15}$$

A simple calculation shows that if (15) holds, the set of all $n \times (n-m)$ matrices U_1' which extend U_0' to an H'-unitary matrix $U' = (U_0' \;\; U_1')$ is given by

$$U_1' = \begin{array}{c} \\ \left(\begin{array}{cc} Z_1 & Z_2 \\ \tilde{U}_1 & 0 \\ Z_3 & Z_4 \end{array}\right) \begin{array}{l} \}\, d_0 \\ \}\, n_0 \\ \}\, d_1 \end{array} \\ \hphantom{Z_1 \quad} \underbrace{\hphantom{Z_4}}_{d_1} \end{array}, \tag{16}$$

where Z_1, Z_2, Z_3, Z_4, and $\tilde{U}_1$ are arbitrary complex matrices with $\tilde{U} = (\tilde{U}_0 \;\; \tilde{U}_1)$ $\tilde{H}$-unitary. In particular, the extension problem (12) has a solution iff (13) has one.

From now on, we assume that

$$\operatorname{rank} \tilde{U}_0 = m_0 + k. \tag{17}$$

In view of Theorem 1, this is a necessary condition for the solvability of (P).

Remark 3. (13) implies the linear independence of the first m_0 columns of $\tilde{U}_0$. Therefore (17) is always fulfilled if $k = 0$.

Next, we show that $\tilde{U}_0$ can be extended to an $\tilde{H}$-unitary matrix.

Lemma 2. *There is a nonsingular $n_0 \times n_0$ matrix S such that*

$$S^H \tilde{H} S = \tilde{H} \quad \textit{and} \quad \tilde{U}_0 = S \begin{pmatrix} I_{m_0+k} \\ 0 \end{pmatrix}.$$

Proof. In a first step, we construct an $\tilde{H}$-unitary matrix S_1 such that

$$\tilde{U}_0 = S_1 \begin{pmatrix} R & 0 \\ 0 & V \end{pmatrix}, \qquad R \text{ an upper triangular } m_0 \times m_0 \text{ matrix}. \tag{18}$$

Diagonalizing $\tilde{H}$ with

$$\Omega = \begin{pmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}}I_k & -\sqrt{\frac{1}{2}}I_k & 0 \\ 0 & \sqrt{\frac{1}{2}}I_k & \sqrt{\frac{1}{2}}I_k & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

we get

$$\Omega^H \tilde{H} \Omega = \operatorname{diag}(\Lambda_1, I_k, -I_k, \Lambda_2) =: \Lambda$$

and

$$Y^H \tilde{H} Y = \Lambda_1. \tag{19}$$

Here $Y = (y_1 \ \cdots \ y_{m_0})$, and y_j denotes the jth column of $\Omega^H \tilde{U}_0$. From (19), $y_1^H \tilde{H} y_1 = (\Lambda_1)_{11} \neq 0$, and therefore there exists a Λ-unitary hyperbolic Householder matrix J_1 (see [3] and [4]) such that

$$J_1 y_1 = \begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad r_{11} \neq 0.$$

Continuing this process, one can find further Λ-unitary Householder matrices $J_2, \dots, J_{m_0}$ with

$$J_{m_0} \cdots J_2 J_1 Y = \begin{pmatrix} R \\ 0 \end{pmatrix}, \qquad R \text{ upper triangular and nonsingular.}$$

The matrix

$$S_1 := \Omega J_1^{-1} J_2^{-1} \cdots J_{m_0}^{-1} \Omega^H$$

is $\tilde{H}$-unitary, and

$$S_1^{-1} \tilde{U}_0 = \begin{pmatrix} R & W \\ 0 & V \end{pmatrix}.$$

Furthermore, (13) implies $W = 0$, and therefore S_1 fulfills (18).

Equation (17) ensures $\operatorname{rank} V = k$, and there exists a nonsingular $(n_0 - m_0)\times(n_0 - m_0)$ matrix Z with

$$V = Z\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

Using (18) and setting $S_2 = \operatorname{diag}(R, Z)$, we obtain

$$\tilde{U}_0 = S_1 S_2 \begin{pmatrix} I_{m_0+k} \\ 0 \end{pmatrix}. \tag{20}$$

The $\tilde{H}$-unitarity of S_1, (13), and (20) imply that $\tilde{H}_2 := S_2^H \tilde{H} S_2$ has the form

$$\tilde{H}_2 = \left(\begin{array}{cc|c} \Lambda_1 & 0 & 0 \\ 0 & 0 & F \\ \hline 0 & F^H & M \end{array}\right).$$

Finally, one can find a nonsingular matrix

$$S_3 = \begin{pmatrix} I_{m_0+k} & * \\ 0 & * \end{pmatrix}$$

with $S_3^H \tilde{H}_2 S_3 = \tilde{H}$. Then, $S := S_1 S_2 S_3$ is $\tilde{H}$-unitary and, in view of (20), the first $m_0 + k$ columns of S are just those of $\tilde{U}_0$. ■

REMARK 4. For the special case that $\tilde{H}$ and $\tilde{A}$ are certain diagonal matrices, the result of Lemma 2 was also given by Glover [9].

By Lemmas 1 and 2, (P) is reduced to the problem of finding all $\tilde{H}$-unitary extensions of

$$S^{-1}\tilde{U}_0 = \left(\begin{array}{cc} I_{m_0} & 0 \\ 0 & I_k \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right).$$

A simple calculation shows that these are given by

$$\begin{pmatrix} I_{m_0} & 0 & 0 & 0 \\ 0 & I_k & N - \frac{1}{2}Y^H\Lambda_2 Y & -Y^H\Lambda_2 X \\ 0 & 0 & I_k & 0 \\ 0 & 0 & Y & X \end{pmatrix},$$

where N, X, Y are arbitrary complex matrices with $N = -N^H$ and X Λ_2-unitary. Thus, in view of (16), we have proved the following

THEOREM 2. *The extension problem* (P) *has a solution iff* (15) *and* (17) *are satisfied. The set of all matrices U_1 which extend U_0 to an H-unitary $U = (U_0\ U_1)$ is given by*

$$U_1 = \left[T \begin{pmatrix} Z_1 & \overset{d_1}{Z_2} \\ SM & 0 \\ Z_3 & Z_4 \end{pmatrix} \begin{matrix} \}\, d_0 \\ \\ \}\, d_1 \end{matrix} \qquad -U_0T_{12} \right] T_{22}^{-1},$$

where

$$M = \begin{pmatrix} 0 & 0 \\ N - \frac{1}{2}Y^H\Lambda_2 Y & -Y^H\Lambda_2 X \\ I & 0 \\ \underbrace{Y}_{k} & \underbrace{X}_{l} \end{pmatrix} \begin{matrix} \}\, m_0 \\ \}\, k \\ \}\, k \\ \}\, l := n_0 - m_0 - 2k \\ \\ \end{matrix}$$

and N, X, Y, Z_j, $j = 1,\dots,4$, *are arbitrary complex matrices of appropriate dimension with* $N = -N^H$ *skew-Hermitian and* X Λ_2*-unitary. The transformation matrices* T *and* S *are as in Lemma* 1 *and* 2.

COROLLARY 1. (P) *has a solution, if one of the conditions*

(a) *rank*$A =$ *rank*H,
(b) *rank*$A =$ *rank*$H - 1$,
(c) H *and* A *have the same number of negative* (*positive*) *eigenvalues*,
(d) H *is positive* (*negative*) *semidefinite*,
(e) A *is nonsingular*

is satisfied.

Proof of the corollary. Since (a), (b), and (d) each imply (c), we only have to consider the conditions (c) and (e). From (10) we deduce that in both cases $k = 0$, and thus (17) is satisfied in view of Remark 3. Moreover, $d_0 = 0$ in the case (e) and (15) is trivially fulfilled. Now assume that (c) holds. It follows from (10) that either $n_0 = m_0$ or $\Lambda_2 = \pm I$. Using the matrix S from Lemma 2, it follows from

$$\tilde{U}_0 = S\begin{pmatrix} I_{m_0} \\ 0 \end{pmatrix}$$

and the first relation in (14) that

$$S^{-1}V_{10} = \begin{pmatrix} 0 \\ W_0 \end{pmatrix} \begin{matrix} \}\, m_0 \\ \}\, n_0 - m_0 \end{matrix}.$$

Therefore $V_{10} = 0$ if $n_0 = m_0$. Otherwise the second equation in (14) implies $\pm W_0^H W_0 = 0$ and thus again $V_{10} = 0$. This concludes the proof of the corollary. ∎

4. APPLICATIONS TO HERMITIAN TOEPLITZ MATRICES

There is a close connection between H-unitary matrices and block Toeplitz matrices with block size $q \geqslant 1$. Let H and U be given $nq \times nq$ matrices where H is Hermitian and U is H-unitary. For an arbitrary $nq \times q$ matrix R_1, we define

$$R_j = UR_{j-1}, \qquad j = 2, \dots, n, \tag{21}$$

and set $R = (R_1 \;\, R_2 \;\, \cdots \, R_n)$. It follows from the H-unitarity of U that

$$T_n = R^H H R \tag{22}$$

has the form

$$T_n = \begin{pmatrix} C_1 & C_2^H & \cdots & C_n^H \\ C_2 & C_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_2^H \\ C_n & \cdots & C_2 & C_1 \end{pmatrix} \tag{23}$$

with $q \times q$ matrices C_j and $C_1 = C_1^H$, i.e. T is an Hermitian block Toeplitz matrix. Conversely, given any matrix (23) with a factorization (22), one would like to find an associated H-unitary matrix U. Equation (21) defines U only on the subspace spanned by the columns of $R_1, \dots, R_{n-1}$, and the wish to obtain an H-unitary matrix on the whole space $\mathbb{C}^{nq}$ leads to an extension problem of the form (P). Note that this problem does not occur when studying similar problems for infinite dimensional spaces, where equations of the type (21) (but now with $n = \infty$) are sufficient to define an isometric operator on the whole space (see Krein and Langer [14] and Sz.-Nagy and Korányi [15]). This is typical for many questions concerning Toeplitz matrices, where very often the infinite dimensional case is the more natural one (cf. Widom [16]).

Theorem 3. *Let T_n be a Hermitian block Toeplitz matrix of the form* (23) *with a factorization*

$$T_n = R^H H R$$

where $R = (R_1 \ R_2 \ \cdots \ R_n)$ (R_j $nq \times q$ matrices) is nonsingular. Let Q be an unitary matrix such that $Q^H R$ is an $n \times n$ block upper triangular matrix of block size q, and set

$$Q^H H Q = \begin{pmatrix} A & B \\ B^H & C \end{pmatrix} \begin{matrix} \}(n-1)q \\ \}q \end{matrix}.$$

Then, there exists an H-unitary matrix U such that

$$R_j = U^{j-1} R_1, \qquad j = 1, \dots, n, \tag{24}$$

if H is nonsingular or if H is singular and one of the following conditions is satisfied:

(a) *$\operatorname{rank} A = \operatorname{rank} H$,*
(b) *$\operatorname{rank} A = \operatorname{rank} H - 1$,*
(c) *H and A have the same number of negative (positive) eigenvalues,*
(d) *H is positive (negative) semidefinite,*
(e) *A is nonsingular.*

Proof. A simple transformation shows that we only need to consider the case $Q = I$. So let

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ 0 & R_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{n-1n} \\ 0 & \cdots & 0 & R_{nn} \end{pmatrix} =: \left(\begin{array}{c|c} R_0 & \begin{matrix} R_{1n} \\ \vdots \\ R_{n-1n} \end{matrix} \\ \hline 0 & r_{nn} \end{array}\right)$$

(R_{jk} $q \times q$ matrices). With the partition $U = (U_1\ U_2\ \cdots\ U_n)$ (U_j $nq \times q$ matrices), (24) is equivalent to

$$U_j = \left(R_{j+1} - \sum_{i=1}^{j-1} U_i R_{ij} \right) R_{jj}^{-1}, \qquad j = 1, \dots, n-1, \tag{25}$$

and this already defines $U_0 := (U_1\ \cdots\ U_{n-1})$. Rewriting (25) as

$$U_0 R_0 = (R_2\ \cdots\ R_n), \tag{26}$$

we obtain

$$R_0^H U_0^H H U_0 R_0 = \begin{pmatrix} R_2^H \\ \vdots \\ R_n^H \end{pmatrix} H(R_2, \dots, R_n)$$

$$= \begin{pmatrix} C_1 & C_2^H & \cdots & C_{n-1}^H \\ C_2 & C_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_2^H \\ C_{n-1} & \cdots & C_2 & C_1 \end{pmatrix} = T_{n-1}.$$

On the other hand,

$$T_{n-1} = \begin{pmatrix} R_1^H \\ \vdots \\ R_{n-1}^H \end{pmatrix} H(R_1, \dots, R_{n-1}) = R_0^H A R_0,$$

and therefore $U_0^H H U_0 = A$. If H is nonsingular, Theorem 2 [(26) implies that U_0 has full column rank] and, if H is singular, Corollary 1 ensure that we can extend U_0 to an H-unitary matrix $U = (U_0\ U_n)$. ∎

For the special case of positive semidefinite Toeplitz matrices ($q = 1$), the result of Theorem 3 is well known (e.g. Cybenko [6]). In this case, there is also a close connection with the finite trigonometric moment theorem of Carathéodory [5] (cf. Delsarte and Genin [7]). Finally, we note that the more general result of Theorem 3 can be used to obtain certain generalizations of Carathéodory's theorem [8].

In the rest of this section, only ordinary ($q = 1$) Toeplitz matrices (23) are considered.

An infinite matrix T is Toeplitz iff $E^H T E = T$ with the shift matrix $E = (\delta_{i,j+1})$. Analogous characterizations of finite Toeplitz matrices naturally lead to Frobenius matrices

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & & 0 & a_2 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{n-1} \\ 0 & \cdots & 0 & 1 & a_n \end{pmatrix}. \tag{27}$$

Gragg [11] showed that for each positive definite Toeplitz matrix T_n there exist a Frobenius matrix U and a $\delta \in \mathbb{R}$ with

$$T_n = U^H T_n U + \delta e_n e_n^H, \qquad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{28}$$

Here, we are interested in representations (28) with $\delta = 0$.

THEOREM 4. *Let T_n be an $n \times n$ Hermitian Toeplitz matrix and T_{n-1} its $(n-1)\times(n-1)$ leading principal submatrix. The following conditions are equivalent*:

(a) *There exists a matrix U of the form* (27) *such that*

$$T_n = U^H T_n U. \tag{29}$$

(b) (i) *T_n is nonsingular or*
  (ii) *T_n is singular and $\operatorname{rank} T_n = \operatorname{rank} T_{n-1}$.*

Proof. Application of Theorem 3 to the trivial factorization $T_n = R^H HR$ with $H = T_n$, $R = I$ shows that (b) guarantees the existence of a matrix U satisfying (29). Moreover, because of (24), U is a Frobenius matrix. Thus, (b) implies (a). Conversely, we now assume that (a) holds, but that (b) is not satisfied, i.e., T_n is singular with $n_0 := \operatorname{rank} T_n \neq \operatorname{rank} T_{n-1}$. It follows from Iohvidov's results [12] on the rank of Toeplitz matrices that $\operatorname{rank} T_{n-1} = n_0 - 2$, $n > 2$, and we set $s = n - n_0$. Then, the matrix

$$W = \begin{pmatrix} \bar{c}_2 & \cdots & \bar{c}_n \\ & T_{n-1} & \end{pmatrix}$$

with the $n-1$ last columns of T_n has rank $n_0 - 1$, and there are linearly independent vectors $v_j \in \mathbb{C}^{n-1}$, $j = 1, \ldots, s$, with $Wv_j = 0$. Thus the vectors

$$\begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ v_s \end{pmatrix}$$

span the null space $\mathscr{E}^0$ of T_n. We may assume that none of the vectors $v_2, \ldots, v_s$ has more leading zeros than v_1, and this ensures

$$\begin{pmatrix} v_1 \\ 0 \end{pmatrix} \notin \mathscr{E}^0. \tag{30}$$

Setting $U_0 = (e_2 \ \cdots \ e_n)$, where e_j denotes the jth unit vector in $\mathbb{C}^n$, we have $U_0^H T_n U_0 = T_{n-1}$, and (a) just states that the problem of extending U_0 by a vector $u_n \in \mathbb{C}^n$ to a T_n-unitary matrix $U = (U_0 \ u_n)$ has a solution. However, the necessary condition given in Theorem 1 is violated, as easily follows from (30) and

$$U_0 v_1 = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \in \mathscr{E}^0.$$

This contradiction concludes the proof. ■

REMARK 5. Obviously, any Hermitian matrix satisfying (29) with a Frobenius matrix (27) is Toeplitz.

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