Indag. Mathem., N.S., 4 (4), 479–481 December 20, 1993

Answer to a query concerning the mapping $w = z^{1/m}$

Dedicated to Jaap Korevaar on the occasion of his 70th birthday

by T.L. McCoy[1] and A.B.J. Kuijlaars[2]

[1] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA*
[2] *Faculty of Mathematics and Computer Science, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, the Netherlands*

Communicated by Prof. J.J. Duistermaat at the meeting of June 21, 1993

For $0 < r < 1$ and $m > 0$, let $C_{r,m}$ denote the disk $|w - c_{r,m}| < \rho_{r,m}$ with center

$$c_{r,m} := \frac{(1+r)^{1/m} + (1-r)^{1/m}}{2}$$

and radius

$$\rho_{r,m} := \frac{(1-r)^{1/m} - (1-r)^{1/m}}{2}.$$

The purpose of this note is to consider a question of L. Petkovič who asks (paraphrasing slightly) (a) how does one establish that the image $D_{r,m}$ of $|z-1| < r$ under $w = z^{1/m}$ has diameter $2\rho_{r,m}$ and (b) how does one establish that $D_{r,m}$ lies inside $C_{r,m}$? Part (a) follows quite easily from part (b): if $D_{r,m} \subset C_{r,m}$ then $D_{r,m}$ does not have diameter greater than $2\rho_{r,m}$; on the other hand, the mapping $w = z^{1/m}$ sends the points $z = 1 \pm r$ to the points $w = (1 \pm r)^{1/m} = c_{r,m} \pm \rho_{r,m}$ so that $D_{r,m}$ cannot have diameter less than $2\rho_{r,m}$. Thus we need only concern ourselves with part (b) of Petkovič's question. L. Petkovič's question originally appeared a few years ago in the (now-discontinued) Queries column of the A.M.S. Notices ([3, Query 359]) but was never answered. Petkovič asks the question for $m \in \mathbf{N}$, but the question makes sense for all real $m \geq 1$. We shall prove

Theorem. *For $0 < r < 1$ and every real $m \geq 1$, $D_{r,m} \subset C_{r,m}$. For $m > 1$, the boundary $\partial D_{r,m}$ meets $\partial C_{r,m}$ only at the two points $(1 \pm r)^{1/m}$.*

Proof. For $m = 1$, there is nothing to prove since $D_{r,m} = C_{r,m}$. So we assume $m > 1$. The boundary of $D_{r,m}$ is a simple closed curve with parametrization

$$w(\theta) = (1 + r\,e^{i\theta})^{1/m}, \quad 0 \leq \theta < 2\pi.$$

This curve is tangent to the circle $\partial C_{r,m}$ in the points $c_{r,m} \pm \rho_{r,m}$. To prove that the curve $w(\theta)$ lies inside $\partial C_{r,m}$ (except for the points $c_{r,m} \pm \rho_{r,m}$), we compute its curvature.

In general, the curvature κ of a curve $w(\theta)$ in the complex plane is given by

$$\kappa = \frac{\operatorname{Im} \bar{\dot{w}} \ddot{w}}{|\dot{w}|^3}$$

where dots denote differentiation with respect to θ, see [2]. For $w(\theta) = z(\theta)^{1/m}$ with $z(\theta) = 1 + r\,e^{i\theta}$, we compute

$$\dot{w}(\theta) = \frac{ri\,e^{i\theta}}{m}\, z(\theta)^{(1/m)-1}$$

and by logarithmic differentiation

$$\ddot{w}(\theta) = \dot{w}(\theta) \left[\frac{((1/m) - 1)\, ri\,e^{i\theta}}{z(\theta)} + i \right].$$

Hence

$$\begin{aligned} \operatorname{Im} \overline{\dot{w}(\theta)}\, \ddot{w}(\theta) &= |\dot{w}(\theta)|^2 \operatorname{Re} \left[\frac{((1/m) - 1)\, r\,e^{i\theta}}{z(\theta)} + 1 \right] \\ &= |\dot{w}(\theta)|^2 \left[\frac{((1/m) - 1)\, r(\cos\theta + r)}{|z(\theta)|^2} + 1 \right] \end{aligned}$$

and finally

$$\kappa(\theta) = \frac{((1/m) - 1)\, r(\cos\theta + r) + |z(\theta)|^2}{|\dot{w}(\theta)|\, |z(\theta)|^2} = \frac{m + r^2 + (m+1)\, r\cos\theta}{r\, |z(\theta)|^{1+(1/m)}}.$$

Since $m + r^2 + (m+1)\, r\cos\theta \geq m + r^2 - (m+1)\, r = (m - r)(1 - r) > 0$, the curvature is strictly positive, and so the domain $D_{r,m}$ is strictly convex. A further computation gives

$$\dot{\kappa}(\theta) = \frac{-(m^2 - 1)(r + \cos\theta)\sin\theta}{m\, |z(\theta)|^{3+(1/m)}}.$$

We see that $\dot{\kappa}(\theta)$ has precisely four simple zeros in $[0, 2\pi)$, namely at $\theta = 0, \pi$ and $\pm \arccos(-r)$. By a result given by Blaschke [1, p. 161], see also [2, p. 30], this implies that the curve $w(\theta)$ has at most four points of intersection with any circle. The circle $\partial C_{r,m}$ is tangent to $w(\theta)$ in the points $c_{r,m} \pm \rho_{r,m}$ and so there are no more points of intersection. Hence $w(\theta)$ lies either completely inside or completely outside $\partial C_{r,m}$. (Note that we have interpreted tangential intersections to count as multiple intersections in the Blaschke result. This 'double counting' interpretation is indeed valid, for otherwise we could always produce a circle close to $\partial C_{r,m}$ which meets $w(\theta)$ in at least five points.)

We shall be finished if we can show that in the point $c_{r,m} + \rho_{r,m}$ the curvature of $w(\theta)$ is greater than the curvature of $\partial C_{r,m}$. So we want to show that $\kappa(0) > 1/\rho_{r,m}$, or

$$\frac{r(1+r)^{1/m}}{m+r} < \frac{(1+r)^{1/m} - (1-r)^{1/m}}{2}.$$

Rewriting this, we need to show

$$(1) \qquad (m+r)(1-r)^{1/m} < (m-r)(1+r)^{1/m}, \quad m > 1, \quad 0 < r < 1.$$

For $r = 0$, we have equality in (1). Further it is easy to check that, for $0 < r < 1$, the derivative with respect to r of the left hand side is less than the derivative of the right hand side. Hence the inequality (1) holds and the proof is finished. □

REFERENCES

1. Blaschke, W. – Kreis und Kugel. 2. Auflage, De Gruyter, Berlin, 1956.
2. Klingenberg, W. – A course in differential geometry. Springer-Verlag, New York, 1978.
3. Petković, L. – Query 359. Notices AMS Vol. **33** (4), 629 (1986).