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Numerical solution of Lotka Volterra prey predator model by using Runge–Kutta–Fehlberg method and Laplace Adomian decomposition method

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Abstract This paper reflects some research outcome denoting as to how Lotka–Volterra prey predator model has been solved by using the Runge–Kutta–Fehlberg method (RKF). A comparison between Runge–Kutta–Fehlberg method (RKF) and the Laplace Adomian Decomposition method (LADM) is carried out and exact solution is found out to verify the applicability, efficiency and accuracy of the method. The obtained approximate solution shows that the Runge–Kutta–Fehlberg method (RKF) is a more powerful numerical technique for solving a system of nonlinear differential equations.

1. Introduction

The Lotka–Volterra model describes an arbitrary number of ecological competitors (or predator–prey) model which is dynamic by nature [1]. This model, based on the ecological system was framed and gradually gained its popularity in the technological arena. The simple prey–predator model is among the most popular models, being frequently used to demonstrate a simple non-linear control system.

In the concerned field of science and technology, numerous significant physical phenomenons are frequently modeled by nonlinear differential equations. Such equations are often stiff or impractical to solve analytically. Yet, analytical approximate methods to obtain fairly accurate solutions have gained much significance in recent years [18]. There are numerous methods, undertaken to find out approximate solutions to nonlinear problems: Homotopy Perturbation method (HPM), Homotopy Analysis method (HAM) [21], Differential Transform method (DTM) [15–17], Variational Iteration method (VIM) [22], Adomian Decomposition method (ADM), Laplace Adomian Decomposition method (LADM) and Runge–Kutta–Fehlberg method (RKF) and Chebyshev Spectral methods [19,20] are some proven instances. The purpose of this paper was to bring out the analytical expressions of Lotka–Volterra prey predator model and the solution of nonlinear differential equations by using the new approach

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to Runge–Kutta–Fehlberg method (RKF) in an elegant way. Thus all these methods entail to multidimensional aspects.

In the recent period, much interest is focused on [6,7,9] the application of Laplace Adomian Decomposition method (LADM) in order to solve an extensive variety of linear and nonlinear problems. Unlike in numerical methods, Laplace Adomian decomposition method is free from rounding off errors. So we emphasize on this method. The Laplace Adomian Decomposition method (LADM) [4,5] was first introduced by Khuri [8] and has been successfully used to find the solution of linear and nonlinear system of differential equations. This method has yielded dependable results in the cases of nonlinear models and its wide range application is found in deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions, etc. So this method is magnificent and easily affordable.

One of the most popular methods with a constant step size is the fourth order Runge–Kutta method (RK4). Reasonably the Runge–Kutta method can obtain [10,11,14] the accuracy of a Taylor Series approximation without the need of higher derivative calculations. This method can be considered as the basic form of other methods. However, in terms of error estimation, the one-step method with an adaptive step size like the Runge–Kutta Fehlberg method (RKF) [12,13] gives better error estimation than that of one-step method with a constant step size like the Runge–Kutta method. The Fehlberg Runge–Kutta method is a method derived out of the calculation of two Runge–Kutta methods of different order. Where subtracting the results from each other an estimate of the error is obtained. The one-step Algorithm method with an adaptive step size automatically organizes the step size as a recompose to the calculation truncation errors. This method has shown dependable results in the case of nonlinear models and hence, its application is found in wide range of deterministic and stochastic problems, linear and nonlinear, in physics, biology and chemical reactions, etc.

The main aim of this paper was to carry out systematic analysis of the comparisons among exact solution, Laplace Adomian Decomposition method (LADM) and Runge–Kutta–Fehlberg method (RKF) on the Lotka–Volterra prey predator model.

2. Laplace Adomian Decomposition method (LADM)

To consider the following system of nonlinear differential equation

$$y_1' = f_1(t, y_1, \ldots, y_n),$$
$$y_2' = f_2(t, y_1, \ldots, y_n),$$
$$\vdots$$
$$y_n' = f_n(t, y_1, \ldots, y_n),$$

where each equation represents the first derivative of each unknown function as a mapping depending on the independent variable t and n unknown functions $f_1, f_2, \ldots, f_n$ and the initial conditions $y_1(0), y_2(0), \ldots, y_n(0)$ are prescribed.

Now we can present the above system of differential equation by using the ith equation term as

$$y_i' = F_i(t, y_1, \ldots, y_n) + N_i(t, y_1, \ldots, y_n) + g_i(t, y_1, \ldots, y_n),\quad i = 1, 2, \ldots, n, \tag{1}$$

where F_i is a linear operator of the 1st-order derivative which is assumed to be invertible easily, g_i is a source term and N_i is a nonlinear operator of $f_i(t, y_1, \ldots, y_n)$.

Taking Laplace transform on both sides of Eq. (1), we get

$$\mathcal{L}[y_i'(t)] = \mathcal{L}[F_i(t, y_1, \ldots, y_n)] + \mathcal{L}[N_i(t, y_1, \ldots, y_n)] + \mathcal{L}[g_i(t, y_1, \ldots, y_n)], \quad i = 1, 2, \ldots, n. \tag{2}$$

Using the differential property of Laplace transform and using the initial condition, we get

$$s\mathcal{L}[y_i(t)] - y_i(0) = \mathcal{L}[F_i(t, y_1, \ldots, y_n)] + \mathcal{L}[N_i(t, y_1, \ldots, y_n)] + \mathcal{L}[g_i(t, y_1, \ldots, y_n)], \quad i = 1, 2, \ldots, n,$$

or,

$$\mathcal{L}[y_i(t)] = \frac{1}{s}y_i(0) + \frac{1}{s}\mathcal{L}[F_i(t, y_1, \ldots, y_n)] + \frac{1}{s}\mathcal{L}[N_i(t, y_1, \ldots, y_n)] + \frac{1}{s}\mathcal{L}[g_i(t, y_1, \ldots, y_n)], \quad i = 1, 2, \ldots, n. \tag{3}$$

Now we represent the unknown functions $y_i(t)$ by an infinite series of the form

$$y_i(t) = \sum_{n=0}^{\infty} y_{in}(t), \quad i = 1, 2, \ldots, n. \tag{4}$$

Here the components $y_{in}(t)$ are usually determined recurrently and the nonlinear operator $N_i(t, y_1, \ldots, y_n)$ can be decomposed into an infinite series of polynomials given by

$$N_i(t, y_1, \ldots, y_n) = \sum_{n=0}^{\infty} A_{in}(t), \quad i = 1, 2, \ldots, n,$$

where $A_{in}, i = 1, 2, \ldots, n$ are Adomian polynomials of $y_0, y_1, \ldots\ldots\ldots, y_n$ defined by

$$A_{in} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(t, \sum_{k=0}^{n}\lambda^k y_{1k}, \sum_{k=0}^{n}\lambda^k y_{2k}, \ldots, \sum_{i=0}^{n}\lambda^k y_{nk},\right)\right]_{\lambda=0},$$
$$n = 0, 1, 2, \ldots; \quad i = 1, 2, \ldots, n.$$

Therefore,

$$\mathcal{L}\left[\sum_{n=0}^{\infty} y_{in}(t)\right] = \frac{y_i(0)}{s} + \frac{1}{s}\mathcal{L}[g_i(t, y_1, \ldots, y_n)] + \frac{1}{s}\mathcal{L}\left[F_i\left(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n}, \ldots, \sum_{n=0}^{\infty} y_{nn}\right)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} A_{in}\right], \quad i = 1, 2, \ldots, n.$$

In general, the recursive relation is given by

$$\mathcal{L}[y_{i0}(t)] = \frac{y_i(0)}{s} + \frac{1}{s}\mathcal{L}[g_i(t, y_1, \ldots, y_n)], \quad i = 1, 2, \ldots, n, \tag{5}$$

and

$$\mathcal{L}[y_{in+1}(t)] = \frac{1}{s}\mathcal{L}\left[F_i\left(\sum_{n=0}^{\infty} y_{1n}, \sum_{n=0}^{\infty} y_{2n}, \ldots, \sum_{n=0}^{\infty} y_{nn}\right)\right] + \frac{1}{s}\mathcal{L}\left[\sum_{n=0}^{\infty} A_{in}\right], \quad i = 1, 2, \ldots, n. \tag{6}$$

Applying the inverse Laplace transform to both sides of (5) and (6), we obtain $y_{in}, (n \geqslant 0)$, $i = 1, 2, \ldots, n$, which is then substituted into (4).

For numerical computation, we get the expression as

$$\phi_n(t) = \sum_{k=0}^{n} y_{ik}(t), \quad i = 1, 2, \ldots, n,$$

which is the nth term approximation of $y_i(t)$.

3. Runge–Kutta–Fehlberg method (RKF)

To consider the following system of ith equation with initial value problem

$$y_i' = f_i(t, y_1, \ldots, y_n); y_i(t_0) = y_0, i = 1, 2, \ldots, n, \tag{7}$$

where each equation is first order differential equation.

The RKF is one way to try to resolve this problem.

The problem is to solve the initial value problem in above equation by means of ([14], [15]) Runge–Kutta methods of order 4 and order 5.

First we need some definitions:

$$k_1 = hf_i(t, y_1, \ldots, y_n),$$

$$k_2 = hf_i\left(t + \frac{1}{4}h, y_1 + \frac{1}{4}k_1, y_2 + \frac{1}{4}k_1, \ldots, y_n + \frac{1}{4}k_1\right),$$

$$k_3 = hf_i\left(t + \frac{3}{8}h, y_1 + \frac{3}{32}k_1 + \frac{9}{32}k_2, y_2 + \frac{3}{32}k_1 + \frac{9}{32}k_2, \ldots, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right),$$

$$k_4 = hf_i\left(t + \frac{12}{13}h, y_1 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3, y_2 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3, \ldots, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right),$$

$$k_5 = hf_i\left(t + h, y_1 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4, y_2 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4, \ldots, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right),$$

$$k_6 = hf_i\left(t + h, y_1 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5, y_2 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5, \ldots, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right),$$

where $i = 1, 2, \ldots, n$.

Then an approximation to the solution of initial value problem is made using Runge–Kutta method of order 4:

$$y_{ik+1} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5, i = 1, 2, \ldots, n. \tag{8}$$

Here the local error $\approx O(h^5)$.

A better value for the solution is determined using a Runge–Kutta method of order 5:

$$y_{ik+1} = y_k + \frac{16}{135}k_1 + \frac{6656}{12,825}k_3 + \frac{28,561}{56,430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6, \quad i = 1, 2, \ldots, n. \tag{9}$$

with local error $\approx O(h^6)$ and global $\approx O(h^5)$.

A formula for the estimation of error of the Runge–Kutta–Fehlberg method is

$$E = \frac{1}{360}k_1 + \frac{128}{4275}k_3 + \frac{2197}{7524}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6.$$

Since the $k_1; k_2; \ldots; k_6$ are known in every step we can always test the accuracy of the method.

The optimal step size sh can be determined by multiplying the scalar s times the step size h. The scalar s is

$$s \leqslant \left(\frac{\epsilon h}{2|\overline{y_{ik+1}} - y_{ik+1}|}\right)^{1/4} = 0.0840896\left(\frac{\epsilon h}{|\overline{y_{ik+1}} - y_{ik+1}|}\right)^{1/4},$$

where ϵ is the specified error control tolerance.

Note: RK4 method requires four function evaluations and RK5 method requires six evaluations, i.e., total ten for RK4 and RK5 methods. Fehlberg devised a method to get RK4 and RK5 methods results using only six function evaluations by using some of k values in both methods where $k = \frac{\partial f}{\partial y}$.

4. Analysis of multispecies Lotka–Volterra model

Mathematical models of population growth have been formed to provide an inconceivable significant angle of true ecological situation. The meaning of each parameter in the models has been defined biologically. For n species, we consider the following [2,3] general Lotka–Volterra model:

$$\frac{dN_i}{dt} = N_i\left(\alpha_i - \sum_{j=1}^{n}\beta_{ij}N_j\right), \quad i = 1, 2, \ldots, n; \; i \neq j. \tag{10}$$

These above cited equations may represent either predator–prey or competition cases.

Lotka–Volterra model (two species):

The Lotka–Volterra model in case of two species is a prey predator equation which is defined as follows:

$$\frac{dN_1}{dt} = N_1(\alpha - \beta N_2),$$
$$\frac{dN_2}{dt} = N_2(\delta N_1 - \gamma),$$

where the parameters α, β, γ, δ are all positive and $N(0) > 0$ and N_1 is a population size of prey species and N_2 is a population size of predator species. α is the per capita reduction in prey per predator and γ denotes the per capita increase in predator per prey. β and δ are mortality rate of prey and predator species respectively. $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the growth rates of the prey and predator species over time t.

5. Numerical results and discussion

A comparison of the numerical solutions, so obtained from RKF and LADM is made with exact solutions (for multiple

Table 1 Numerical comparison when initially we have $N_1(0) = 4$, $N_2(0) = 9$, $\alpha = 0.1$, $\beta = 0.0014$, $\gamma = 0.0012$, $\delta = 0.08$, $h = 1$.

t	Exact		LADM (4th iteration)		RKF	
	N_1	N_2	N_1	N_2	N_1	N_2
0	4.00000000	9.00000000	4.00000000	9.00000000	4.00000000	9.00000000
1	4.36734648	8.34983166	4.36423471	8.34470435	4.36733225	8.34981749
2	4.77260007	7.75022106	4.76588351	7.74074456	4.77236772	7.74999782
3	5.21966481	7.19734593	5.20875110	7.18416528	5.21846454	7.19623279
4	5.71285281	6.68767985	5.69703097	6.67132682	5.70898082	6.68421422
5	6.25692620	6.21797028	6.23534323	6.19887998	6.24727466	6.20963389
6	6.85714350	5.78521821	6.82877602	5.76374307	6.83670418	5.76818362
7	7.51931075	5.38665958	7.48293069	5.36308097	7.48062749	5.35555521
8	8.24983784	5.01974814	8.20397089	4.99428600	8.18240270	4.96744045
9	9.05580068	4.68213978	8.99867598	4.65496056	8.94538793	4.59953116
10	9.94500969	4.37167823	9.87449896	4.34290125	9.77294127	4.24751914
11	10.92608546	4.08638201	10.83962917	4.05608454	10.66842086	3.90709618
12	12.00854112	3.82443260	11.90305989	3.79265371	11.63518479	3.57395409
13	13.20287427	3.58416389	13.07466132	3.55090713	12.67659119	3.24378468
14	14.52066904	3.36405263	14.36525868	3.32928773	13.79599815	2.91227974
15	15.97470369	3.16271017	15.78671596	3.12637363	14.99676380	2.57513108

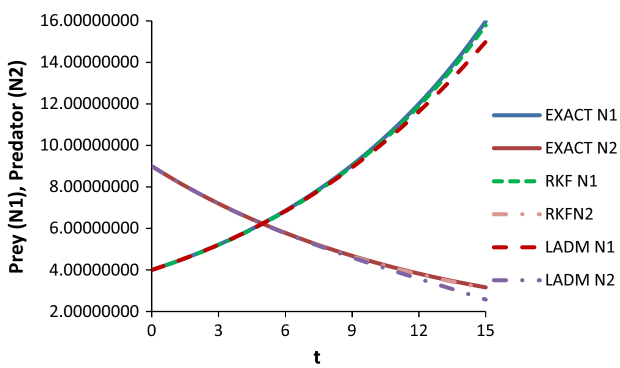

Figure 1 Evaluation between the exact solution and the solutions obtained by using LADM and RKF methods in 2D.

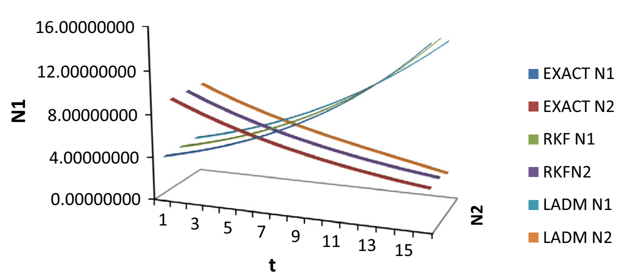

Figure 2 Evaluation between the exact solution and the solutions obtained by using LADM and RKF methods in 3D.

species). Table 1 shows comparison among the RKF, 3-term LADM and the exact solution for the single species in the case $N_1(0) = 4$, $N_2(0) = 9$, $\alpha = 0.1$, $\beta = 0.0014$, $\gamma = 0.0012$, $\delta = 0.08$, $h = 1$ (see Figs. 1 and 2 and Table 2).

The graphical representations of this model reveal that the exact solution and RKF are overlapping with each other whereas there is a least difference between exact solution and that of LADM. So, from the above evaluation we can reach to the decision that RKF is a trustworthy numerical technique and the above figure shows that the growth rate of prey species increases and the growth rate of predator species decreases whereas initially we have $N_1 = 4$ and $N_2 = 9$.

Table 2 Error term of Laplace Adomian Decomposition method (ELADM) and Error term of Runge–Kutta–Fehlberg method (ERKF).

t	ELADM		ERKF	
	N_1	N_2	N_1	N_2
0	0.00E+00	0.00E+00	0.00E+00	0.00E+00
1	3.11E−03	5.13E−03	1.42E−05	1.42E−05
2	6.72E−03	9.48E−03	2.32E−04	2.23E−04
3	1.09E−02	1.32E−02	1.20E−03	1.11E−03
4	1.58E−02	1.64E−02	3.87E−03	3.47E−03
5	2.16E−02	1.91E−02	9.65E−03	8.34E−03
6	2.84E−02	2.15E−02	2.04E−02	1.70E−02
7	3.64E−02	2.36E−02	3.87E−02	3.11E−02
8	4.59E−02	2.55E−02	6.74E−02	5.23E−02
9	5.71E−02	2.72E−02	1.10E−01	8.26E−02
10	7.05E−02	2.88E−02	1.72E−01	1.24E−01
11	8.65E−02	3.03E−02	2.58E−01	1.79E−01
12	1.05E−01	3.18E−02	3.73E−01	2.50E−01
13	1.28E−01	3.33E−02	5.26E−01	3.40E−01
14	1.55E−01	3.48E−02	7.25E−01	4.52E−01
15	1.88E−01	3.63E−02	9.78E−01	5.88E−01

Note: The above table shows that the results are free from any error in respect of calculations between exact solution and RKF but, some errors do exist between calculations of LADM and exact solution. That is why RKF is a high accurate numerical technique to adopt.

6. Conclusions and future research scope

This article highlights the numerical solutions of Lotka–Volterra prey predator model where a well established method, called multistep RKF method, is introduced. This numerical technique is having high accuracy rate compared to LADM. So we can conclude that the RKF is more accurate and reliable numerical technique for solutions of linear and nonlinear system of differential equations in population models. The graphical representation makes it clear that RKF gives quite good results after a considerable time interval. This method is magnificently very useful and will undoubtedly be applicable in broad arena. The advantage of the RKF over the LADM is that there is no need for the evaluations of the Adomian polynomials and it provides an efficient numerical solution. In our present activities the RKF method has successfully been applied to system of nonlinear differential equations application in prey predator model. Hence, introduction of this method in population dynamics may pave the way of a new horizon in the days to come. In future, we shall be able to solve nonlinear differential equations application as different population dynamics model by using some methods such as Homotopy Analysis Method (HAM), Chebyshev Spectral Method (CSM) and Differential Transform method (DTM).

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