Note

Quasifields of symplectic translation planes

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We show that a translation plane is symplectic if and only if at least one of its associated quasifields admits a non-degenerate invariant symmetric bilinear form. As an application we prove that a proper desarguesian, Moufang or nearfield plane can never be symplectic. Moreover, we give a purely algebraic characterization of the quasifields which coordinatize symplectic translation planes.

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1. Introduction

A spread $\Sigma$ of a vector space $V$ over a field $F$ is a collection of mutually complementary subspaces of $V$ such that each element of $V \setminus \{0\}$ is contained in precisely one element of $\Sigma$. With each spread one can associate an affine translation plane $A(\Sigma)$ whose points are the elements of $V$ and whose lines are the cosets of the elements of $\Sigma$. The spread $\Sigma$ and its associated translation plane are called symplectic if there exists a (non-degenerate) symplectic bilinear form $S : V \times V \rightarrow F$ such that all elements of $\Sigma$ are totally isotropic with respect to $S$. If $V$ is of finite dimension over $F$, it was recently shown by Kantor [3] that the kernel $K$ of a non-desarguesian symplectic spread is commutative and that the form is unique up to a constant if $F = K$. The same holds true for pappian spreads.

We show that the translation plane over a quasifield $Q$ is symplectic if and only if $Q$ admits a non-degenerate invariant symmetric bilinear form, cf. Theorem 2.2. We use this to prove that the kernel of such a plane is commutative. This implies in particular that every desarguesian symplectic spread is actually pappian. Moreover, we show that the right and the middle nuclei of a quasifield $Q$ which coordinatizes a symplectic translation plane coincide and that these nuclei are contained in the center of $Q$. 

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We also give a purely algebraic characterization of the quasifields associated with symplectic translation planes by showing that the translation plane over a quasifield $Q$ is symplectic if and only if the subgroup of the additive group of $Q$ generated by all elements of the form $(xy)z - x(zy)$, $x, y, z \in Q$ is different from $Q$, cf. Corollary 4.3.

2. Symplectic spreads and invariant forms on quasifields

Basic information on spreads, translation planes and quasifields can be found in [4] or [5]. A quasifield is an algebraic structure $(Q, +, \cdot)$ which satisfies all axioms for a skew field with the possible exception of the associative law for the multiplication and the distributive law $x(y + z) = xy + xz$. Every quasifield $Q$ is a left vector space over its kernel $K = K(Q) = \{c \in Q \mid c(xy) = (cx)y$ and $c(x + y) = cx + cy \text{ for all } x, y \in Q\}$, which is a skew field. With each quasifield $Q$ one can associate the spread $\Sigma = \Sigma(Q)$ in the $K$-vector space $Q \times Q$, consisting of $\{0\} \times Q$ and of the sets $\{(x, xm) \mid x \in Q\} \subset Q \times Q$ with $m \in Q$. Every spread can be obtained from a, usually not unique, quasifield. The right and the middle nucleus of $Q$ are defined by $N_r(Q) = \{c \in Q \mid (xy)c = x(yc)$ for all $x, y \in Q\}$ and $N_m(Q) = \{c \in Q \mid x(cy) = (xc)y$ for all $x, y \in Q\}$. The center of $Q$ is the set $C(Q) = \{c \in K \mid cx = xc$ for all $x \in Q\}$. Note that the center is contained in the kernel of $Q$.

Let $Q$ be a quasifield with kernel $K$ and let $F$ be a subfield of $K$. A symmetric bilinear form $B : Q \times Q \to F$ is called invariant if we have

$$B(xa, y) = B(x, ya) \quad \text{for all } a, x, y \in Q.$$ 

An equivalent formulation is that all right multiplication mappings of $Q$ are self adjoint with respect to $B$.

Every $F$-linear combination of invariant symmetric bilinear forms is also an invariant symmetric bilinear form, and so the invariant symmetric bilinear forms constitute a vector space over $F$. The set consisting of the zero form and of all symplectic forms on $Q \times Q$ which turn $\Sigma$ into a symplectic spread also forms a vector space over $F$.

**Lemma 2.1.** Let $Q$ be a quasifield with kernel $K$ and let $F$ be a subfield of $K$. Let $B : Q \times Q \to F$ be an invariant symmetric bilinear form. Then $B$ is either non-degenerate or zero.

**Proof.** Assume that $B$ is degenerate. Then there exists $x \in Q \setminus \{0\}$ such that $B(x, y) = 0$ for all $y \in Q$. Let $y, z \in Q$, then we get $B(xy, z) = B(x, zy) = 0$. Since $x \neq 0$ this implies $B = 0$. □

**Theorem 2.2.** Let $Q$ be a quasifield with kernel $K$ and let $F$ be a subfield of $K$. Let $B : Q \times Q \to F$ be an invariant symmetric bilinear form. Then $S_B : \{(x, y) \mid x \in Q\} \to F$ with $S_B((x_1, x_2), (y_1, y_2)) = B(x_1 y_2) - B(x_2 y_1)$ is a symplectic form and the spread $\Sigma$ associated with $Q$ is symplectic in the $F$-vector space $Q \times Q$ with respect to $S_B$. The form $S_B$ is non-degenerate if and only if $B$ is. Any symplectic form on $Q \times Q$ which turns $\Sigma$ into a symplectic spread is obtained in this way.

The mapping $B \mapsto S_B$ is an $F$-linear bijection between the vector space of all invariant symmetric bilinear forms on $Q$ and the vector space of all symplectic forms on $Q \times Q$ for which $\Sigma$ is a symplectic spread.

**Proof.** Let $S : (Q \times Q) \times (Q \times Q) \to F$ be a symplectic form such that all elements of $\Sigma$ are totally isotropic with respect to $S$. There exist bilinear forms $B_1, \ldots, B_4 : Q \times Q \to F$ such that $S((x_1, x_2), (y_1, y_2)) = B_1(x_1 y_1) + B_2(x_1 y_2) + B_3(x_2 y_1) + B_4(x_2 y_2)$ for all $x_1, \ldots, y_2 \in Q$. The vertical subspace $\{0\} \times Q$ is an element of $\Sigma$ and hence totally isotropic. This yields $S((0, x_2), (0, y_2)) = B_4(x_2 y_2) = 0$ for all $x_2, y_2 \in Q$ and hence $B_4 = 0$. Similarly, the horizontal subspace $Q \times \{0\}$ is contained in $\Sigma$, which implies $B_3 = 0$. The diagonal subspace $\{(x, x) \mid x \in Q\}$ is also contained in $\Sigma$. This yields $S((x, x), (y, y)) = B_2(x, y) + B_3(x, y) = 0$ for all $x, y \in Q$ and hence $B_3 = -B_2$. Now $S$ is symplectic and so $S((x, y), (x, y)) = B_2(x, y) - B_2(y, x) = 0$, i.e. $B = B_2$ is symmetric. Obviously, $S = S_B$.

For each $a \in Q$ the spread element $L(a) = \{(x, xa) \mid x \in Q\}$ is totally isotropic and hence $S((x, xa), (y, ya)) = B(x, ya) - B(xa, y) = 0$ for all $a, x, y \in Q$. It follows that $B$ is an invariant bilinear form.
Assume that $B$ is degenerate, then $B = 0$ by 2.1 and hence $S = 0$. If $B$ is non-degenerate then $S = S_B$ also is.

Assume now that $B : Q \times Q \to F$ is an invariant symmetric bilinear form. Reversing the above arguments, one sees that $S_B$ is a symplectic form on $Q \times Q$ and that all elements of $\Sigma$ are totally isotropic with respect to $S_B$.

Obviously, the mapping $B \mapsto S_B$ is $F$-linear, and we just have proved that it is bijective. \qed

**Corollary 2.3.** Let $Q$ be a quasifield, then the translation plane associated with $Q$ is symplectic if and only if $Q$ admits a non-degenerate invariant symmetric bilinear form.

**Remark 2.4.**

(1) Assume that $Q$ is only a prequasifield, i.e. $Q$ satisfies all axioms for a quasifield except the existence of a multiplicative identity element. Then one can still construct the spread $\Sigma(Q)$, but it will not contain the diagonal subspace. Inspection of the proof of 2.2 shows that there is still a non-degenerate bilinear form $B : Q \times Q \to F$, but it will not necessarily be symmetric and the invariance property becomes $B(x, ya) = B(y, xa)$ for all $x, y, a \in Q$.

(2) The question if a spread is symplectic is independent of the quasifield used for its coordinatization. This means that if a quasifield $Q$ admits a non-degenerate invariant symmetric bilinear form then any other quasifield which coordinatizes the same translation plane as $Q$ also does.

## 3. Kernel and nuclei

In this section we investigate the kernel and the nuclei of the quasifields associated with a symplectic translation plane. In particular, we prove that they are all commutative.

**Proposition 3.1.** Let $Q$ be a quasifield which admits a non-degenerate invariant bilinear form $B : Q \times Q \to F$ with respect to some subfield $F$ of the kernel $K$ of $Q$, then $K$ is commutative.

**Proof.** Let $x, y \in Q$, $c \in K$, then we get

$$B(cx, y) = B((cx)y, 1) = B(c(xy), 1) = B(c, xy) = B(cy, x).$$

If also $x \in K$, then we get

$$B(cx, y) = B(cy, x) = B(1, xc) = B(1, xc) = B(y, xc)$$

and since $B$ is non-degenerate this implies $xc = cx$. \qed

**Remark 3.2.** Proposition 3.1 was already proved by Kantor [3] under the assumptions that $Q$ is a finite-dimensional $F$-vector space and that the translation plane associated with $Q$ is non-desarguesian.

**Lemma 3.3.** Let $Q$ be a quasifield with kernel $K$ and let $B : Q \times Q \to F$ be a non-degenerate invariant symmetric bilinear form, where $F$ is a subfield of $K$. Let $c \in Q \setminus \{0\}$. Then $B_c : Q \times Q \to F$, $(x, y) \mapsto B(xc, y)$ is also a non-degenerate symmetric bilinear form. $B_c$ is invariant if and only if $c$ is contained in center of $Q$.

**Proof.** For all $x, y \in Q$ we have

$$B_c(x, y) = B(xc, y) = B(x, yc) = B(yc, x) = B_c(y, x)$$

and hence $B_c$ is symmetric. $B_c$ is certainly linear in the second argument and by symmetry also in the first, so it is a symmetric bilinear form. It is non-degenerate since $B$ is.

Assume that $B_c$ is invariant. Let $a, x, y \in Q$, then we get

$$B((xa)c, y) = B(xc, ya) = B_c(x, ya) = B_c(xa, y) = B((xa)c, y)$$
and since $B$ is non-degenerate this implies $(xc)a = (xa)c$. Setting $x = 1$ we see that $c$ commutes with every element of $Q$. But then we also get $(cx)a = (xa)c = c(xa)$ and $c(a + x) = (a + x)c = ac + xc = ca + cx$, which implies $c \in C(K)$.

Assume now that $c$ belongs to the center of $Q$, then the reverse argument shows that $B_c$ is invariant. □

**Proposition 3.4.** Let $Q$ be a quasifield with kernel $K$ and assume that there exists a non-degenerate invariant symmetric bilinear form $B : Q \times Q \to F$ for some subfield $F$ of $K$. Then we have $N_r(Q) = N_m(Q) \subseteq C(Q)$, in particular $N_r(Q) = N_m(Q)$ is commutative.

**Proof.** Let $c \in N_r(Q)$ and let $a, x, y \in Q$, then we have

$$B_c(xa, y) = B((xa)c, y) = B(x(ac), y) = B(x, y(ac)) = B(x, (ya)c) = B_c(x, ya).$$

So $B_c$ is invariant and 3.3 implies that $N_r(Q) \subseteq C(Q)$.

Let $c \in N_m(Q)$, then we also have $c^{-1} \in N_m(Q)$, cf. [4], proof of 1.22. Let $a, x, y \in Q$, then we have

$$B_c(x, ya) = B(xc, (yc)(c^{-1}a)) = B((xc)(c^{-1}a), yc) = B(xa, yc) = B_c(xa, y).$$

Thus we also have $N_m(Q) \subseteq C(Q)$.

Let $c \in N_r(Q)$ and let $x, y \in Q$, then we get

$$(xc)y = (cx)y = c(xy) = (xy)c = x(y)c = x(cy).$$

and hence $c \in N_m(Q)$.

Analogously, for $c \in N_m(Q)$ and $x, y \in Q$ we get

$$(xy)c = c(xy) = (cx)y = (xc)y = x(y)c = x(cy),$$

and hence $c \in N_r(Q)$.

It follows that $N_r(Q) = N_m(Q) \subseteq C(Q)$, which proves the proposition. □

**Corollary 3.5.** Let $A$ be a symplectic translation plane, let $p_1$, $p_2$ be distinct points on the translation axis and let $L_1$, $L_2$ be lines distinct from the translation axis with $p_i \in L_i$, $i = 1, 2$. Then the groups of $(p_1, L_1)$- and $(p_2, L_1)$-homologies are isomorphic and abelian.

**Proof.** If one coordinatizes $A$ by a quasifield $Q$ such that $L_1 \setminus \{p_1\} = Q \times \{0\}$ and $L_2 \setminus \{p_2\} = \{0\} \times Q$, then the homology groups in question are isomorphic to $N_m(Q) \setminus \{0\}$ and $N_r(Q) \setminus \{0\}$, respectively, cf. e.g. [4], 1.22. So the result follows from 3.4. □

**Remark 3.6.** For finite quasifields, 3.4 has already been proved by Johnson and Vega [1], Theorem 2, and for finite semifields also by Lunardon [6], Theorem 3.

**Proposition 3.7.** Let $Q$ be a nearfield or a skew field which admits a non-degenerate invariant symmetric bilinear form with respect to some subfield $F$ of $K$. Then $Q$ is commutative and hence a field.

**Proof.** If the multiplication of $Q$ is associative, then $N_r(Q) = N_m(Q) = Q$, and so the result follows from 3.4. If $Q$ is a skew field one can also use 3.1. □

**Corollary 3.8.** If the translation plane associated with a symplectic spread is desarguesian, then it is pappian.

Since every proper Moufang plane contains proper desarguesian subplanes we also have the following

**Proposition 3.9.** Every symplectic Moufang plane is pappian.
Proof. Let $Q$ be a proper alternative field with kernel $K$ and assume that there exists a non-degenerate invariant symmetric bilinear form $B : Q \times Q \to K$. If there exists a quaternion subalgebra $H$ of $Q$ such that the restriction of $B$ to $H$ is not the zero form then the result follows from 3.7.

So we may assume that the restriction of $B$ to any quaternion subalgebra of $Q$ is zero. Each element $x \in Q$ is contained in a quaternion subalgebra $H$ of $Q$ with $K \subset H$, where $K$ denotes the kernel of $Q$, cf. e.g. [7], 6.15. This implies that $K$ is contained in the radical of $Q$, contradicting the non-degeneracy of $B$. □

4. Algebraic characterization of symplectic spreads

An invariant bilinear form $B : Q \times Q \to F$ on a quasifield $Q$ is completely determined by the linear form $\varphi : Q \to F, x \mapsto B(1, x)$ since $B(x, y) = B(1, xy)$ holds for all $x, y \in Q$. In this section we investigate the question which linear forms on the $F$-vector space $Q$ yield invariant bilinear forms, where $F$ is a subfield of $Q$.

Lemma 4.1. Let $Q$ be a quasifield with kernel $K$ and let $F$ be a subfield of $K$. Then the $F$-subspace and the subgroup of the additive group of $Q$ generated by the set $A = \{(xy)z - x(zy) \mid x, y, z \in Q\}$ coincide.

Proof. It is sufficient to show that for all $x, y, z \in Q$ and for all $c \in F \subseteq K$ there holds $c((xy)z - x(zy)) \in A$. This follows from

$$c((xy)z - x(zy)) = c((xy)z) - c(x(zy)) = ((cx)y)z - (cx)(zy).$$ □

Theorem 4.2. Let $Q$ be a quasifield with kernel $K$ and let $F$ be a subfield of $K$. Put $A = \{(xy)z - x(zy) \mid x, y, z \in Q\}$ and $W = \langle A \rangle$. Let $\varphi : Q \to F$ be linear with $W \subseteq \ker \varphi$, and define $B_\varphi : Q \times Q \to F$ by $B_\varphi(x, y) = \varphi(xy)$. Then $B_\varphi$ is an invariant symmetric bilinear form on $Q$, and every invariant symmetric bilinear form on $Q$ is obtained in this way.

The form $B_\varphi$ is degenerate if and only if $\varphi = 0$.

The mapping $\varphi \mapsto B_\varphi$ is an $F$-linear bijection between the vector space of all linear forms on $Q$ which contain $W$ in their kernel and the vector space of all invariant symmetric bilinear forms on $Q$. In particular, there exists a non-degenerate invariant symmetric bilinear form on $Q$ if and only if $W \neq Q$.

Proof. Let $B : Q \times Q \to F$ be an invariant symmetric bilinear form and define the linear form $\varphi : Q \to F$ by $\varphi(x) = B(1, x)$. Let $x, y, z \in Q$, then we get

$$B(x, y) = B(xy, 1) = B(1, xy) = \varphi(xy)$$

and

$$\varphi((xy)z - x(zy)) = B(1, (xy)z) - B(1, x(zy)) = B(z, xy) - B(zy, x) = 0.$$ It follows that $B = B_\varphi$ and that $W$ is contained in the kernel of $\varphi$.

Let now $\varphi : Q \to F$ be a linear form with $W \subseteq \ker \varphi$, and let $B = B_\varphi : Q \times Q \to F$ be defined by $B(x, y) = \varphi(xy)$ for $x, y \in Q$. Setting $x = 1$ we get $(xy)z - x(zy) = yz - zy \in W \subseteq \ker \varphi$ for all $y, z \in Q$. This implies

$$B(y, z) = \varphi(yz) = \varphi(zy) = B(z, y),$$

and so $B$ is symmetric.

Let $x, y, z \in Q$ and $c \in F$, then we get

$$B(x + y, z) = \varphi((x + y)z) = \varphi(xz + yz) = B(x, z) + B(y, z)$$

and

$$B(cx, y) = \varphi(cx) = \varphi(cxy) = c\varphi(xy) = cB(x, y).$$

It follows that $B$ is linear in the first argument and then by symmetry also in the second.
Finally, we compute
\[ B(xy, z) = \varphi((xy)z) = \varphi(x(zy)) = B(x, zy), \]
and so \( B \) is an invariant symmetric bilinear form.

Obviously, \( \varphi = 0 \) if and only if \( B_\varphi = 0 \).
The mapping \( \varphi \mapsto B_\varphi \) is linear, and we just have proved that it is bijective. \( \square \)

**Corollary 4.3.** The translation plane associated with \( Q \) is symplectic if and only if \( W \neq Q \).

**Remark 4.4.**

1. Since \( W \) does not depend on \( F \) the question of whether or not a spread is symplectic in the \( F \)-vector space \( Q \times Q \) does also not depend on \( F \), a fact already noted by Kantor [3], Theorem 3 and Lunardon [6], Theorem 1.
2. The question if a spread is symplectic is independent of the quasifield used for its coordinatization. So if the condition \( W \neq Q \) is satisfied for one quasifield \( Q \) it also holds for any quasifield which coordinatizes the same translation plane as \( Q \).
3. For a proper skew field \( Q \) an alternative proof for 3.7 runs as follows. \( W \) contains all elements of the form \( (xy)z - x(zy) = x(yz) - x(zy) = x(yz - zy) \) for \( x, y, z \in Q \), which comprise all of \( Q \) since \( Q \) is not commutative. So \( W = Q \) and by 4.2 the corresponding spread is not symplectic.

**Proposition 4.5.** Let \( Q \) be a quasifield with kernel \( K \) and let \( F \) be a subfield of \( K \). Put \( A = \{(xy)z - x(zy) \mid x, y, z \in Q\} \) and \( W = (A) \). Let \( B : Q \times Q \to F \) be a non-degenerate invariant symmetric bilinear form. Then we have \( W^\perp = K \), where \( \perp \) denotes the orthogonality relation induced by \( B \).

If the dimension of \( Q \) as an \( F \)-vector space is finite, then \( Q/W \) and \( K \) are isomorphic as \( F \)-vector spaces.

**Proof.** Let \( x, y, z \in Q \) and let \( c \in K \), then we get
\[
B(c, (xy)z - x(zy)) = B(c, (xy)z) - B(c, x(zy)) = B(cz, xy) - B((cz)y, x) = 0
\]
and hence \( K \subseteq W^\perp \).

Let now \( c \in W^\perp \) and let \( x, y, z \in Q \), then we get
\[
0 = B(c, (xy)z - x(zy)) = B(c, (xy)z) - B(c, x(zy)) = B(cz, xy) - B(c(zy), x) = B((cz)y, x) - B(c(zy), x).
\]
Since \( B \) is non-degenerate this implies \( c \in K \) and hence \( W^\perp \subseteq K \).

If the dimension of \( Q \) as an \( F \)-vector space is finite, then \( Q/W \) is isomorphic to \( W^\perp \) and hence to \( K \). \( \square \)

**Proposition 4.6.** Let \( Q \) be a quasifield and let \( F \) be a subfield of the kernel \( K \) of \( Q \) such that the dimension of \( Q \) as an \( F \)-vector space is finite. Assume that there exists a non-degenerate invariant symmetric bilinear form \( B : Q \times Q \to F \). Then the space of all invariant symmetric \( F \)-valued bilinear forms on \( Q \) is isomorphic to the dual space of \( K \) viewed as a vector space over \( F \). By 2.2 the same holds true for the space of all \( F \)-valued symplectic forms on \( Q \times Q \) for which \( \Sigma(Q) \) is symplectic.

**Proof.** By 4.2 the space in question is isomorphic to the \( F \)-dual of \( Q/W \) and by 4.5 the space \( Q/W \) is isomorphic to \( K \). \( \square \)

**Proposition 4.7.** Assume that the conditions from 4.6 are satisfied. Then for each invariant symmetric bilinear form \( B' : Q \times Q \to F \) there exists \( c \in K \) with \( B'(x, y) = B(cx, y) \) for all \( x, y \in Q \).
Proof. By 3.1 the kernel $K$ is commutative, and hence the mapping $B^c : Q \times Q \to F$, $(x, y) \mapsto B(cx, y)$ is a bilinear form for each $c \in K$. It is also symmetric since

$$B^c(x, y) = B(cx, y) = B(c, xy) = B(c(yx), 1) = B((cy)x, 1) = B(cy, x) = B^c(y, x)$$

for all $x, y \in Q$. The invariance of $B^c$ follows from

$$B^c(xy, z) = B(c(xy), z) = B((cx)y, z) = B(cx, zy) = B^c(x, zy)$$

for all $x, y, z \in Q$. The set $\{B^c | c \in K\}$ forms an $F$-vector space isomorphic to $K$. The result now follows from 4.6 and the fact that a finite-dimensional vector space and its dual space are isomorphic. □

Remark 4.8.

(1) In the special case $F = K$, proposition 4.7 says that two non-degenerate symplectic forms on $Q \times Q$ for which the spread $\Sigma(Q)$ is symplectic differ only by an element of $K$. This was already proved by Kantor [3], Theorem 3.

(2) Kantor [2] has discovered a connection between semifields which coordinatize symplectic translation planes and commutative semifields. In our setting this can be described as follows. Let $Q$ be a semifield with kernel $K$ and let $F$ be a subfield of $N_F(Q)$ such that $Q$ is of finite dimension over $F$. Let $B : Q \times Q \to F$ be a non-degenerate invariant symmetric bilinear form. From 3.4 we infer that for each $a \in Q$ the left multiplication with $a$, given by $\lambda_a : Q \to Q$, $x \mapsto ax$, is linear over $F$ and hence has an adjoint $\lambda^*_a : Q \to Q$, defined by $B(ax, y) = B(x, \lambda^*_a(y))$ for all $x, y \in Q$. Define a new multiplication $\circ : Q \times Q \to Q$ by $x \circ y = \lambda^*_a(y)$ for $x, y \in Q$. This multiplication is bilinear over $F$ since $\lambda^*_a = c\lambda^*_a$ for all $c \in F$, $x \in Q$. From

$$B(x \circ y, z) = B(\lambda^*_a(y), z) = B(y, xz) = B(yz, x) = B(z, \lambda^*_a(x)) = B(z, y \circ x)$$

for all $x, y, z \in Q$ we infer that $\circ$ is commutative. For all $c \in F$, $x, y \in Q$ we have $B(cx, y) = cB(x, y) = B(x, cy)$ and hence $\lambda^*_a = \lambda^*_c$. This implies in particular that $1 \circ x = x \circ 1 = x$ for all $x \in Q$. Thus $Q$ with this new multiplication is a commutative semifield.

Conversely, let $Q$ be a commutative semifield of finite dimension over its kernel $K$, and let $\varphi : Q \to K$ be a non-trivial linear form. Define $B : Q \times Q \to K$ by $B(x, y) = \varphi(xy)$, then $B$ is a non-degenerate symmetric bilinear form. Define a new multiplication $\ast : Q \times Q \to Q$ by $x \ast y = \lambda^*_a(y)$. This new multiplication is bilinear over $K$, and the form $B$ is invariant. This follows from

$$B(x \ast y, z) = B(\lambda^*_a(y), z) = B(y, xz) = B(y, zx) = B(\lambda^*_a(y), x) = B(z \ast y, x)$$

for all $x, y, z \in Q$. Setting $x = 1$ respectively $z = 1$, we see that $1 \ast y = y \ast 1 = y$ for all $y \in Q$. It follows that $Q$ with the new multiplication is a semifield which admits an invariant bilinear form.

If the dimension of $Q$ over $F$ or $K$ is infinite the construction will not work in general since it is not clear if each left multiplication mapping has an adjoint.

References