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Source algebras of blocks, sources of simple modules, and a conjecture of Feit

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ABSTRACT

We verify a finiteness conjecture of Feit on sources of simple modules over group algebras for various classes of finite groups related to the symmetric groups.

1. Introduction and results

1.1. Amongst the long-standing conjectures in modular representation theory of finite groups is a finiteness conjecture concerning the sources of simple modules over group algebras, due to Feit [12] and first announced at the Santa Cruz Conference on Finite Groups in 1979:

By Green's Theorem [15], given a finite group G and an algebraically closed field F of some prime characteristic p, one can assign to each indecomposable FG-module M a G-conjugacy class of p-subgroups of G, the *vertices of* M. Given a vertex Q of M, there is, moreover, an indecomposable FQ-module L such that M is isomorphic to a direct summand of the induced module $\mathrm{Ind}_Q^G(L)$; such

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a module L is called a (Q-)*source* of M. Any Q-source of M has vertex Q as well, and is determined up to isomorphism and conjugation with elements in $N_G(Q)$. Vertices of simple FG-modules have a number of special features not shared by vertices of arbitrary indecomposable FG-modules; see, for instance, [11] and [22]. Feit's Conjecture in turn predicts also a very restrictive structure of sources of simple modules, and can be formulated as follows:

Conjecture *(Feit). Given a finite p-group Q, there are only finitely many isomorphism classes of indecomposable FQ-modules occurring as sources of simple FG-modules with vertex Q; here G varies over all finite groups containing Q.*

While the conjecture remains open in this generality, weaker versions of it are known to be true: by work of Dade [5], Feit's Conjecture holds when demanding the sources in question have dimension at most d, for a given integer d. Furthermore, Puig [29,31] has shown that Feit's Conjecture holds when allowing the group G to vary over p-soluble groups only, and Puig [30] has also shown that Feit's Conjecture holds for the symmetric groups. The aim of this paper now is to pursue the idea of restricting to suitable classes of groups further, and to prove the following:

Theorem. *Feit's Conjecture holds when letting the group G vary over the following groups only*:

$$\{\mathfrak{S}_n\}_{n\geqslant 1},\quad \{\mathfrak{A}_n\}_{n\geqslant 1},\quad \{\widetilde{\mathfrak{S}}_n\}_{n\geqslant 1},\quad \{\widehat{\mathfrak{S}}_n\}_{n\geqslant 1},\quad \{\widetilde{\mathfrak{A}}_n\}_{n\geqslant 1},\quad \{\mathfrak{B}_n\}_{n\geqslant 2},\quad \{\mathfrak{D}_n\}_{n\geqslant 4}.$$

Here, $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric and alternating groups on n letters, respectively. Moreover, $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ denote the double covers of the symmetric groups, and $\widetilde{\mathfrak{A}}_n$ those of the alternating groups; these groups are described in more detail in 5.1. Finally, $\mathfrak{B}_n$ and $\mathfrak{D}_n$ denote the Weyl groups of type B_n and D_n, respectively; these groups are described in more detail in 6.1.

1.2. We will prove the above-mentioned main result by exploiting the connection between Feit's Conjecture and two other ingredients: Puig's Conjecture regarding source algebras of blocks of group algebras, and a question raised by Puig (see [34]) relating vertices to defect groups:

Conjecture *(Puig). Given a finite p-group P, there are only finitely many isomorphism classes of interior P-algebras that are source algebras of a block of FG; here G varies over all finite groups containing P.*

Question. Suppose that $p > 2$, that G is a finite group, and that D is a simple FG-module with vertex Q. Is the order of the defect groups of the block containing D bounded in terms of the group order $|Q|$?

The corresponding question for $p = 2$ is known to have a negative answer, and we will elaborate on this in Remark 3.3 in more detail. For $p > 2$, to the authors' knowledge, there seem to be no examples known where Puig's Question admits a negative answer, and Zhang [34] has proved a reduction to quasi-simple groups. The Reduction Theorem 3.8 now shows that Puig's Conjecture together with a positive answer to the previous question imply Feit's Conjecture for $p > 2$. A proof of Puig's Conjecture alone, however, might not suffice to prove Feit's Conjecture; see also the remarks following [33, Thm. 38.6].

Despite the fact that these conjectures have been around for quite a while, and belong to folklore in modular representation theory of finite groups, we have not been able to find a reference where they have been stated formally. Thus they are restated here as Conjectures 3.5 and 3.7, respectively, in a category-theoretic language we are going to develop, and which will also be used to formulate our Reduction Theorem 3.8.

Our strategy for proving our main Theorem 3.10 is as follows: by work of Kessar [19–21], Puig [30], and Scopes [32], Puig's Conjecture is known to be true for the groups considered here. We will, therefore, show that Puig's Question has an affirmative answer when allowing the groups to vary over the groups in the theorem only, by determining in Theorem 3.9 explicit upper bounds on the respective defect group orders, and regardless of whether p is even or odd. It should be pointed out that such bounds can also be derived from work of Zhang [34], which are, however, much weaker than the ones we get, and are thus hardly useful when actually trying to compute vertices of simple modules in practice.

The strategy to prove the bounds on defect group orders in terms of vertices, in turn, is for part of the cases based on Knörr's Theorem, see Remark 3.12, ensuring the existence of a self-centralizing Brauer pair for any subgroup of a group G being a vertex of a simple FG-module. This reduces Puig's Question to asking, more strongly, whether defect group orders can even be bounded in terms self-centralizing Brauer pairs. This idea turns out to be successful for the alternating groups and the double covers of the symmetric and alternating groups. In particular, in Theorem 4.5 and the subsequent Remark 4.6 we derive a detailed picture of the self-centralizing Brauer pairs for the alternating groups in characteristic $p = 2$, which might be of independent interest.

Moreover, we would like to point out that, in particular, for the case of the alternating groups, we have been examining various examples explicitly, where the computer algebra systems GAP [14] and MAGMA [2] have been of great help; we will specify later on where precisely these have been invoked.

1.3. The paper is organized as follows: in Section 2 we introduce our notational set-up, define our notions of the category of interior algebras and vertex-source pairs of indecomposable modules over group algebras, and recall the notion of source algebras. Then, in Section 3, we formulate Feit's and Puig's Conjectures in our category-theoretic language, prove the Reduction Theorem 3.8, and state Theorem 3.9 in order to prove the main Theorem 3.10. Sections 4–6 are then devoted to proving Theorem 3.9 for the alternating groups, the double covers of the symmetric and alternating groups, and the Weyl groups appearing in our main theorem, where in the former two cases we pursue the idea of using self-centralizing Brauer pairs, while for the Weyl groups appearing in our main theorem we are content with looking at vertices directly. Finally in Section 7 we briefly deal with semidirect products with abelian kernel in general.

Throughout this article, let p be a prime number, and let F be a fixed algebraically closed field of characteristic p. All groups appearing will be finite and, whenever G is a group, any FG-module is understood to be a finitely generated left module. Hence we may assume that the groups considered here form a small category, that is, its object class is just a set, and similarly module categories may be assumed to be small as well. This will, for instance, allow us to speak of the *set* of all finite groups. We assume the reader to be familiar with modular representation theory of finite groups in general, and the standard notation commonly used, as exposed for example in [25] and [33]. For background concerning the representation theory of the symmetric groups and their covering groups, we refer the reader to [18] and [17], respectively.

2. Interior algebras, source algebras, and vertex-source pairs

In this section we introduce a category-theoretic language, which will be used to restate Feit's and Puig's Conjectures later in this article. The language we use is that of interior algebras, see for example [33, Ch. 10], where we additionally have to allow the group acting to vary.

2.1. *A category of interior algebras.* We define a category $\mathcal{A}$ whose objects are the triples (G, α, A), where G is a finite group, A is a finite-dimensional, associative, unitary F-algebra, and $\alpha : G \longrightarrow A^\times$ is a group homomorphism into the group of multiplicative units $A^\times$ in A. Given objects (G, α, A) and (H, β, B) in $\mathcal{A}$, the morphisms $(G, \alpha, A) \longrightarrow (H, \beta, B)$ are the pairs (φ, Φ) where $\varphi : G \longrightarrow H$ is a group homomorphism and $\Phi : A \longrightarrow B$ is a homomorphism of F-algebras satisfying

$$\Phi\bigl(\alpha(g)a\bigr) = \beta\bigl(\varphi(g)\bigr)\Phi(a) \quad \text{and} \quad \Phi\bigl(a\alpha(g)\bigr) = \Phi(a)\beta\bigl(\varphi(g)\bigr), \tag{1}$$

for all $g \in G$ and all $a \in A$. We emphasize that the algebra homomorphism Φ need not be unitary, in general. If it additionally is, that is, if we have $\Phi(1_A) = 1_B$ then the above compatibility condition (1) simplifies to

$$\Phi\bigl(\alpha(g)\bigr) = \beta\bigl(\varphi(g)\bigr), \quad \text{for all } g \in G.$$

Anyway, whenever $(\varphi, \Phi) : (G, \alpha, A) \longrightarrow (H, \beta, B)$ and $(\psi, \Psi) : (H, \beta, B) \longrightarrow (K, \gamma, C)$ are morphisms in $\mathcal{A}$, their composition is defined to be $(\psi, \Psi) \circ (\varphi, \Phi) := (\psi \circ \varphi, \Psi \circ \Phi)$, where the compositions of the respective components are the usual compositions of group homomorphisms and algebra homomorphisms, respectively. Hence $\mathcal{A}$ is indeed a category, from now on called the *category of interior algebras*; an object (G, α, A) in $\mathcal{A}$ is called an *interior G-algebra*, and $(\varphi, \Phi) : (G, \alpha, A) \longrightarrow (H, \beta, B)$ is called a *morphism of interior algebras*.

We just remark that for the conjugation automorphisms $\kappa_a \in \mathrm{Aut}(A)$ and $\lambda_b \in \mathrm{Aut}(B)$ induced by some $a \in (A^{\alpha(G)})^\times$ and $b \in (B^{\beta(H)})^\times$, respectively, we also have the morphism $(\varphi, \lambda_b \circ \Phi \circ \kappa_a) : (G, \alpha, A) \longrightarrow (H, \beta, B)$. This defines an equivalence relation on the morphisms of interior algebras $(G, \alpha, A) \longrightarrow (H, \beta, B)$, and the equivalence class

$$(\varphi, \widehat{\Phi}) := \bigl\{(\varphi, \lambda_b \circ \Phi \circ \kappa_a) \bigm| a \in \bigl(A^{\alpha(G)}\bigr)^\times, b \in \bigl(B^{\beta(H)}\bigr)^\times\bigr\}$$

is called the associated *exomorphism of interior algebras*.

By the above definition, (G, α, A) and (H, β, B) are isomorphic in $\mathcal{A}$ if and only if there exists a morphism $(\varphi, \Phi) : (G, \alpha, A) \longrightarrow (H, \beta, B)$ such that φ is an isomorphism of groups and Φ is an (automatically unitary) isomorphism of algebras. So, in particular, if (G, α, A) is an interior algebra and $\varphi : H \longrightarrow G$ is an isomorphism of groups then also $(H, \alpha \circ \varphi, A)$ is an interior algebra, and $(H, \alpha \circ \varphi, A)$ and (G, α, A) are isomorphic via (φ, id_A). Analogously, if (G, α, A) is an interior algebra and if $\Phi : A \longrightarrow B$ is an isomorphism of algebras then $(G, \Phi \circ \alpha, B)$ is also an interior algebra, and (G, α, A) and $(G, \Phi \circ \alpha, B)$ are isomorphic via (id_G, Φ).

2.2. *An equivalence relation.* Let G and H be groups, let M be an FG-module, and let N be an FH-module. Let further $\alpha : G \longrightarrow E_M^\times$ and $\beta : H \longrightarrow E_N^\times$ be the corresponding representations, where $E_M := \mathrm{End}_F(M)$ and $E_N := \mathrm{End}_F(N)$. Then (G, α, E_M) and (H, β, E_N) are interior algebras.

(a) We say that the pairs (G, M) and (H, N) are *equivalent* if there are a group isomorphism $\varphi : G \longrightarrow H$ and a vector space isomorphism $\psi : M \longrightarrow N$ such that, for all $g \in G$ and all $m \in M$, we have

$$\psi\bigl(\alpha(g) \cdot m\bigr) = \beta\bigl(\varphi(g)\bigr) \cdot \psi(m).$$

This clearly is an equivalence relation on the set of all such pairs.

(b) The case $G = H$ deserves particular attention: pairs (G, M) and (G, N) are equivalent, via (φ, ψ) say, if and only if we have

$$\beta\bigl(\varphi(g)\bigr) \cdot n = \psi\bigl(\alpha(g) \cdot \psi^{-1}(n)\bigr)$$

for all $g \in G$ and $n \in N$, that is, if and only if M and N are in the same $\mathrm{Aut}(G)$-orbit on the set of isomorphism classes of FG-modules. Moreover, M and N are isomorphic as FG-modules if and only if φ can be chosen to be the identity id_G. In particular, there are at most $|\mathrm{Aut}(G)|$ isomorphism classes of FG-modules in the equivalence class of (G, M).

Lemma 2.3. *We keep the notation of* 2.2. *Then the pairs* (G, M) *and* (H, N) *are equivalent if and only if the associated interior algebras* (G, α, E_M) *and* (H, β, E_N) *are isomorphic in* $\mathcal{A}$.

Moreover, if $G = H$ then M and N are isomorphic as FG-modules if and only if (G, α, E_M) *and* (G, β, E_N) *are isomorphic in $\mathcal{A}$ via an isomorphism of the form* $(\mathrm{id}_G, ?)$.

Proof. If (G, M) and (H, N) are equivalent via $\varphi : G \longrightarrow H$ and $\psi : M \longrightarrow N$ then

$$\Psi : E_M \longrightarrow E_N, \qquad \gamma \longmapsto \psi \circ \gamma \circ \psi^{-1}$$

is an isomorphism of algebras, and we have $\Psi(\alpha(g)) = \psi \circ \alpha(g) \circ \psi^{-1} = \beta(\varphi(g))$, for all $g \in G$, thus the interior algebras (G, α, E_M) and (H, β, E_N) are isomorphic in $\mathcal{A}$ via (φ, Ψ).

Let, conversely, (G, α, E_M) and (H, β, E_N) be isomorphic via (φ, Ψ), where $\varphi : G \longrightarrow H$ is a group isomorphism and $\Psi : E_M \longrightarrow E_N$ is an isomorphism of algebras. Then, letting $i \in E_M$ be a primitive idempotent, we may assume that $M = E_M i$ and, letting $j := \Psi(i) \in E_N$, we may similarly assume that $N = E_N j$. Moreover, $\Psi(E_M i) = \Psi(E_M)\Psi(i) = E_N j$ shows that $\psi := \Psi|_{E_M i} : E_M i \longrightarrow E_N j$ is a vector space isomorphism, where for all $g \in G$ and $\gamma \in E_M$ we have

$$\psi\bigl(\alpha(g) \cdot \gamma i\bigr) = \Psi\bigl(\alpha(g) \cdot \gamma i\bigr) = \beta\bigl(\varphi(g)\bigr) \cdot \Psi(\gamma) j = \beta\bigl(\varphi(g)\bigr) \cdot \psi(\gamma i),$$

implying that (G, M) and (H, N) are equivalent via (φ, ψ). This proves the first statement.

The second statement can be found in [33, La. 10.7]. It also follows from the above observations, by recalling that M and N are isomorphic FG-modules if and only if the group isomorphism $\varphi : G \longrightarrow G$ inducing an equivalence of pairs can be chosen to be the identity id_G. $\square$

Lemma 2.4. *We keep the notation of* 2.2*, and let* (G, M) *and* (H, N) *be equivalent. Then the equivalence classes of pairs* (G, M') *where* M' *is an indecomposable direct summand of the* FG*-module* M *coincide with the equivalence classes of pairs* (H, N') *where* N' *is an indecomposable direct summand of the* FH*-module* N*. In particular, the* FG*-module* M *is indecomposable if and only if the* FH*-module* N *is.*

Proof. Let (G, α, E_M) and (H, β, E_N) be isomorphic via (φ, Ψ); such an isomorphism exists, by Lemma 2.3. Given an indecomposable direct summand M' of M, let $i \in (E_M)^{\alpha(G)}$ be the associated primitive idempotent, so that $M' = iM$, with associated representation

$$\alpha' : G \longrightarrow E_{iM} = iE_M i, \qquad g \longmapsto i\alpha(g)i.$$

Hence, for $j := \Psi(i) \in E_N$ we have

$$\beta\bigl(\varphi(g)\bigr) j = \beta\bigl(\varphi(g)\bigr)\Psi(i) = \Psi\bigl(\alpha(g)i\bigr) = \Psi\bigl(i\alpha(g)\bigr) = \Psi(i)\beta\bigl(\varphi(g)\bigr) = j\beta\bigl(\varphi(g)\bigr),$$

for all $g \in G$. Thus $j \in (E_N)^{\beta(H)}$ is a primitive idempotent, giving rise to the indecomposable direct summand jN of N, with associated representation

$$\beta' : H \longrightarrow E_{jN} = jE_N j, \qquad g \longmapsto j\beta(g)j.$$

Moreover, we have an isomorphism of algebras

$$\Psi' := \Psi|_{iE_M i} : iE_M i \longrightarrow jE_N j, \qquad ixi \longmapsto \Psi(ixi) = j\Psi(x)j.$$

Then we have

$$\Psi'\bigl(\alpha'(g)\bigr) = \Psi'\bigl(i\alpha(g)i\bigr) = j\Psi\bigl(\alpha(g)\bigr)j = j\beta\bigl(\varphi(g)\bigr)j = \beta'\bigl(\varphi(g)\bigr),$$

for all $g \in G$. Thus the interior algebras $(G, \alpha', iE_M i)$ and $(H, \beta', jE_N j)$ are isomorphic in $\mathcal{A}$ via (φ, Ψ'), that is, the pairs (G, iM) and (H, jN) are equivalent. $\square$

Remark 2.5. (a) Let (G, α, A) be an interior algebra. For any A-module M with associated representation $\delta : A \longrightarrow E_M := \mathrm{End}_F(M)$ we obtain an FG-module $\mathrm{Res}_\alpha(M)$ by restriction along α, that is, the associated representation is given as $\delta \circ \alpha : G \longrightarrow E_M^\times$. Thus we get a functor

$$\mathrm{Res}_\alpha : A\text{-}\mathbf{mod} \longrightarrow FG\text{-}\mathbf{mod}, \qquad M \longmapsto \mathrm{Res}_\alpha(M).$$

Let (H, β, B) be an interior algebra, and let $(\varphi, \Phi) : (G, \alpha, A) \longrightarrow (H, \beta, B)$ be a morphism in $\mathcal{A}$. Hence, by restriction along β and φ, respectively, we similarly get functors

$$\mathrm{Res}_\beta : B\text{-}\mathbf{mod} \longrightarrow FH\text{-}\mathbf{mod} \quad \text{and} \quad \mathrm{Res}_\varphi : FH\text{-}\mathbf{mod} \longrightarrow FG\text{-}\mathbf{mod}.$$

Moreover, for any B-module N with associated representation $\gamma : B \longrightarrow E_N := \mathrm{End}_F(N)$ we obtain an A-module $\mathrm{Res}_\Phi(N) := \Phi(1_A)N$ whose associated representation is given as

$$A \longrightarrow \mathrm{End}_F\big(\Phi(1_A)N\big) = \Phi(1_A)E_N\Phi(1_A), \qquad x \longmapsto \Phi(1_A)\gamma\big(\Phi(x)\big)\Phi(1_A).$$

This gives rise to a functor $\mathrm{Res}_\Phi : B\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod}$.

(b) If additionally Φ is unitary, that is, $\Phi(1_A) = 1_B$ then the representation associated with $\mathrm{Res}_\Phi(N)$ is obtained by restriction along Φ. Moreover, from $\Phi(\alpha(g)) = \beta(\varphi(g))$, for all $g \in G$, we infer that we have the following equality of functors

$$\mathrm{Res}_\alpha \circ \mathrm{Res}_\Phi = \mathrm{Res}_\varphi \circ \mathrm{Res}_\beta : B\text{-}\mathbf{mod} \longrightarrow FG\text{-}\mathbf{mod}.$$

In other words, for any B-module N with associated representation $\gamma : B \longrightarrow E_N$, we have

$$(\gamma \circ \Phi \circ \alpha)(g) \cdot n = (\gamma \circ \beta \circ \varphi)(g) \cdot n,$$

for all $g \in G$ and all $n \in N$. In particular, if φ is an isomorphism then $(G, \mathrm{Res}_\alpha(\mathrm{Res}_\Phi(N)))$ and $(H, \mathrm{Res}_\beta(N))$ are equivalent via (φ, id_N).

Definition 2.6. Let G be a group, and let M be an indecomposable FG-module. Assume that $V \leqslant G$ is a vertex of M and that S is a V-source of M. Then the elements of the equivalence class of the pair (V, S) are called the *vertex-source pairs* of (G, M).

Proposition 2.7. *If G is a group and M is an indecomposable FG-module then the vertex-source pairs of (G, M) are pairwise equivalent.*

Moreover, if H is a group and N is an indecomposable FH-module such that (G, M) is equivalent to (H, N) then the vertex-source pairs of (G, M) and (H, N) are pairwise equivalent.

Proof. Let $V \leqslant G$ be a vertex, and let S be a V-source of M. Then the set of all vertices of M is given as $\{{}^gV \mid g \in G\}$ and, for a given $g \in G$, the set of ${}^gV$-sources of M (up to isomorphism) is $\{{}^{hg}S \mid h \in N_G({}^gV)\}$. For $g \in G$ and $h \in N_G({}^gV)$, let

$$\kappa : V \longrightarrow {}^gV = {}^{hg}V, \qquad x \longmapsto {}^{hg}x = hgxg^{-1}h^{-1}$$

be the associated conjugation homomorphism, and let $\psi : S \longrightarrow {}^{hg}S$, $m \longmapsto hg \otimes m$. Then for all $x \in V$ and $m \in S$, we have

$$\kappa(x) \cdot \psi(m) = \big({}^{hg}x\big) \cdot (hg \otimes m) = hg \otimes (x \cdot m) = \psi(x \cdot m),$$

hence the pairs (V, S) and $({}^gV, {}^{hg}S)$ are equivalent via (κ, ψ). Since every vertex-source pair of (G, M) is equivalent to one of the pairs $({}^gV, {}^{hg}S)$, this shows that all vertex-source pairs of (G, M) belong to the same equivalence class.

Moreover, if (G,M) and (H,N) are equivalent via (φ,ψ) then $\psi(\alpha(g)\cdot m)=\beta(\varphi(g))\cdot\psi(m)$ for all $g\in G$ and $m\in M$, where α and β are the representations associated with M and N, respectively. From this we infer that $\varphi(V)$ is a vertex of the FH-module N having $\psi(S)$ as a $\varphi(V)$-source, and that (V,S) is equivalent to $(\varphi(V),\psi(S))$ via $(\varphi|_V,\psi|_S)$. $\square$

Remark 2.8. We remark that, given G and an indecomposable FG-module M, specifying a vertex V as a subgroup of G amounts to restricting to those vertex-source pairs of shape $(V,?)$, henceforth only allowing for isomorphisms of the form $(\mathrm{id}_V,?)$. The above argument now shows that these vertex-source pairs are given by the FV-modules $\{{}^hS \mid h\in N_G(V)/VC_G(V)\}$, where S is one of the V-sources. Thus we possibly do not obtain the full $\mathrm{Aut}(V)$-orbit of S, but only see its orbit under $N_G(V)/VC_G(V)\leqslant \mathrm{Aut}(V)$, as the following example shows:

Example 2.9. Let $p:=2$, let $G:=\mathfrak{S}_6$, and let $M:=D^{(5,1)}$ be the natural simple $F\mathfrak{S}_6$-module of F-dimension 4. Then, by [24], the vertices of $D^{(5,1)}$ are the Sylow 2-subgroups of $\mathfrak{S}_6$. Let $P_6:=P_4\times P_2\cong D_8\times C_2$, where $P_4=\langle(1,2),(1,3)(2,4)\rangle$ and $P_2:=\langle(5,6)\rangle$. Then P_6 is a Sylow 2-subgroup of $\mathfrak{S}_6$ and, by [24], the restriction $S:=\mathrm{Res}^{\mathfrak{S}_6}_{P_6}(D^{(5,1)})$ is indecomposable, thus every P_6-source of $D^{(5,1)}$ is isomorphic to S. Since $N_{\mathfrak{S}_6}(P_6)=P_6$, in view of Proposition 2.7 we have to show that the $\mathrm{Aut}(P_6)$-orbit of S consists of more than a single isomorphism class of FP_6-modules.

Let $\varphi\in\mathrm{Aut}(P_6)$ be the involutory automorphism given by fixing $P_4=\langle(1,2),(1,3)(2,4)\rangle$ and mapping $(5,6)$ to $(1,2)(3,4)(5,6)$. Since $\mathrm{Res}^{\mathfrak{S}_6}_{\mathfrak{S}_4}(D^{(5,1)})$ is the natural permutation $F\mathfrak{S}_4$-module, there is an F-basis of S with respect to which the elements of P_4 are mapped to the associated permutation matrices, while

$$(5,6)\longmapsto\begin{bmatrix} . & 1 & 1 & 1\\ 1 & . & 1 & 1\\ 1 & 1 & . & 1\\ 1 & 1 & 1 & . \end{bmatrix}\quad\text{and}\quad \varphi\big((5,6)\big)=(1,2)(3,4)(5,6)\longmapsto\begin{bmatrix} 1 & 1 & . & 1\\ 1 & 1 & 1 & .\\ . & 1 & 1 & 1\\ 1 & . & 1 & 1 \end{bmatrix}.$$

It can be checked, for example with the help of the computer algebra system MAGMA [2], that the FP_6-modules S and ${}^{\varphi}S$ are not isomorphic.

2.10. *Source algebras.* Let G be a group, and let B be a block of FG. Let further P be a p-group such that the defect groups of B are isomorphic to P. Then we have an embedding of groups $f:P\longrightarrow G$ such that $f(P)$ is a defect group of B. The block B is an indecomposable $F[G\times G]$-module with vertex $\Delta f(P)$ and trivial source. Moreover, there is an indecomposable direct summand M of $\mathrm{Res}^{G\times G}_{f(P)\times G}(B)$ with vertex $\Delta f(P)$, where M is unique up to isomorphism and conjugation in $N_G(f(P))$.

So there is a primitive idempotent $i\in B^{f(P)}$ such that $M=iB$, where i is unique up to taking associates in $B^{f(P)}$ and conjugates under the action of $N_G(f(P))$. We call i and M, respectively, a *source idempotent* and a *source module* of B, respectively; as a general reference see [33, Ch. 38]. The embedding f gives rise to the group homomorphism

$$\alpha_{f,i}:P\longrightarrow(iBi)^{\times},\qquad g\longmapsto if(g)i,$$

which turns $(P,\alpha_{f,i},iBi)$ into an interior P-algebra. Note that $\alpha_{f,i}$ is injective, by [33, Exc. 38.2]. We call an interior algebra that is isomorphic to $(P,\alpha_{f,i},iBi)$ in $\mathcal{A}$ a *source algebra* of B.

Proposition 2.11. *Let G be a group, and let B be a block of FG. Then the source algebras of B are pairwise isomorphic in $\mathcal{A}$.*

Moreover, if $\psi:G\longrightarrow G'$ is a group isomorphism and if $B':=\psi(B)$ is the block of FG' obtained by extending ψ to FG then the source algebras of B and B' are pairwise isomorphic in $\mathcal{A}$.

Proof. Let P be a p-group isomorphic to the defect groups of B, let $f:P\longrightarrow G$ be an embedding such that $f(P)\leqslant G$ is a defect group of B, and let $\alpha_{f,i}:P\longrightarrow(iBi)^{\times}$, $g\longmapsto if(g)i$ be the associated group homomorphism, where $i \in B^{f(P)}$ is a source idempotent. Moreover, let $\varphi : Q \longrightarrow P$ be a group isomorphism, and let $f' : Q \longrightarrow G$ be an embedding such that $f'(Q) \leqslant G$ is a defect group of B, with associated group homomorphism $\alpha_{f',j} : Q \longrightarrow (jBj)^\times$, where $j \in B^{f'(Q)}$ is a source idempotent. Note that, hence, there is some $h \in G$ such that $f'(Q) = {}^h f(P)$, and the idempotents ${}^h i = hih^{-1}$ and j are associate in $B^{f'(Q)}$. To show that $(P, \alpha_{f,i}, iBi)$ is isomorphic to $(Q, \alpha_{f',j}, jBj)$ in $\mathcal{A}$ we proceed in three steps:

(i) We first consider the particular case where $Q = P$, $\varphi = \mathrm{id}$, and $f' = f$, and let $j \in B^{f(P)}$ be a source idempotent that is associate to i. Then there is some $a \in (B^{f(P)})^\times$ such that $j = {}^a i = aia^{-1}$, and $\kappa : iBi \longrightarrow jBj$, $x = ixi \longmapsto {}^a(ixi) = j({}^a x)j$ is an isomorphism of algebras. Hence, we obtain the group homomorphism

$$\kappa \circ \alpha_{f,i} : P \longrightarrow (jBj)^\times, \qquad g \longmapsto {}^a\big(if(g)i\big) = j\big(af(g)a^{-1}\big)j = jf(g)j,$$

that is, $\kappa \circ \alpha_{f,i} = \alpha_{f',j}$. Moreover, $\kappa(\alpha_{f,i}(g)) = {}^a(if(g)i) = jf(g)j = \alpha_{f',j}(g)$, for all $g \in P$, shows that the interior P-algebras $(P, \alpha_{f,i}, iBi)$ and $(P, \alpha_{f',j}, jBj)$ are isomorphic via (id_P, κ).

(ii) Next, let still $\varphi = \mathrm{id}$, let $h \in G$ be arbitrary with associated conjugation automorphism $G \longrightarrow G$, $g \longmapsto {}^h g = hgh^{-1}$, and let $f' : P \longrightarrow G$, $g \longmapsto hf(g)h^{-1}$ be the associated conjugated embedding. Since $j \in B^{f'(P)}$ is a source idempotent, by (i) we may assume that $j = {}^h i$. This yields the isomorphism of algebras $\gamma : iBi \longrightarrow jBj$, $x = ixi \longmapsto {}^h(ixi) = j({}^h x)j$ and, associated to f', the group homomorphism

$$\alpha_{f',j} = \gamma \circ \alpha_{f,i} : P \longrightarrow (jBj)^\times, \qquad g \longmapsto {}^h\big(if(g)i\big) = j\big(hf(g)h^{-1}\big)j.$$

Moreover, $\gamma(\alpha_{f,i}(g)) = {}^h(if(g)i) = j(hf(g)h^{-1})j = \alpha_{f',j}(g)$, for all $g \in P$, shows that the interior P-algebras $(P, \alpha_{f,i}, iBi)$ and $(P, \alpha_{f',j}, jBj)$ are isomorphic via (id_P, γ).

(iii) We finally consider the general case of a group isomorphism $\varphi : Q \longrightarrow P$ and an embedding $f' : Q \longrightarrow G$ as above. By (ii) we may assume that $f'(Q) = f(P)$. Hence, there is a group automorphism $\rho : f(P) \longrightarrow f(P)$ such that $f \circ \varphi = \rho \circ f'$. Thus, replacing φ by $\varphi' := (f^{-1} \circ \rho^{-1} \circ f) \circ \varphi$ we get $f \circ \varphi' = f'$. So we may assume that $f \circ \varphi = f'$. Moreover, by (ii) we may assume that $j = i \in B^{f(P)}$. Hence we have the associated group homomorphisms

$$\alpha_{f,i} : P \longrightarrow (iBi)^\times, \qquad g \longmapsto if(g)i \quad \text{and} \quad \alpha_{f',i} : Q \longrightarrow (iBi)^\times, \qquad h \longmapsto if'(h)i = if\big(\varphi(h)\big)i.$$

Moreover, $\alpha_{f',i}(h) = if(\varphi(h))i = \alpha_{f,i}(\varphi(h))$, for all $h \in Q$, which shows that the interior algebras $(Q, \alpha_{f',i}, iBi)$ and $(P, \alpha_{f,i}, iBi)$ are isomorphic via $(\varphi, \mathrm{id}_{iBi})$. This proves the first statement.

Let $\psi : G \longrightarrow G'$ be a group isomorphism, which extends to an F-algebra isomorphism $\psi : FG \longrightarrow FG'$, and let $B' := \psi(B)$. Then, letting $f' := \psi \circ f : P \longrightarrow G'$, we conclude that $f'(P)$ is a defect group of B', and $i' := \psi(i) \in (B')^{f'(P)}$ is a source idempotent of B'. Thus we get $(P, \alpha_{f',i'}, i'B'i')$ as a source algebra associated with B', where

$$\alpha_{f',i'} = \psi \circ \alpha_{f,i} : P \longrightarrow \big(i'B'i'\big)^\times = \psi(iBi)^\times, \qquad g \longmapsto i'f'(g)i' = \psi\big(if(g)i\big).$$

Then we have $\psi(\alpha_{f,i}(g)) = \alpha_{f',i'}(g)$, for all $g \in P$, that is, $(P, \alpha_{f,i}, iBi)$ and $(P, \alpha_{f',i'}, i'B'i')$ are isomorphic in $\mathcal{A}$ via $(\mathrm{id}_P, \psi|_{iBi})$. Since, by what we have shown above, every source algebra of B is $\mathcal{A}$-isomorphic to $(P, \alpha_{f,i}, iBi)$ and every source algebra of B' is $\mathcal{A}$-isomorphic to $(P, \alpha_{f',i'}, i'B'i')$, this completes the proof of the proposition. $\square$

Remark 2.12. We remark that specifying a defect group P of the block B as a subgroup of G amounts to keeping the embedding $f : P \longrightarrow G$ fixed, and thus to restricting to the source algebras of shape $(P, \alpha_{f,i}, iBi)$, for some $i \in B^{f(P)}$, and to isomorphisms of the form (id_P, Φ). The above argument now shows that the isomorphisms Φ realized in G are precisely those of the form $\Phi = \gamma \circ \kappa$, where $\kappa : iBi \longrightarrow iBi$, $x \longmapsto {}^a x$ is the inner automorphism of iBi induced by some $a \in (iBi)^\times$, and where $\gamma : iBi \longrightarrow jBj$, $x \longmapsto {}^h x$ is induced by the conjugation automorphism $G \longrightarrow G$ afforded by some

$h \in N_G(f(P))$, where $j := {}^h i$. Hence possibly not all elements of the isomorphism class of $(P, \alpha_{f,i}, iBi)$ are realized as source algebras in this strict sense, as the following example shows:

Example 2.13. Let $p := 3$ and let $G = P = \langle z \rangle \cong C_3$ be the cyclic group of order 3; hence FP is a local F-algebra. Letting $f = \mathrm{id}_P : P \longrightarrow P$, *the* source algebra of $B = FP$ (by necessarily taking $i := 1_{FP}$) is given as $(P, \alpha_{\mathrm{id}_P}, FP)$. Thus $(P, \alpha_{\mathrm{id}_P}, FP)$ is the only interior algebra in its isomorphism class that is actually realized in the above strict sense.

We describe all interior algebras $(P, ?, FP)$ isomorphic to $(P, \alpha_{\mathrm{id}_P}, FP)$ in $\mathcal{A}$, that is, all source algebras of FP in the sense of 2.10: note first that in this particular case any group automorphism of P can be extended uniquely to an algebra automorphism of FP, so that any isomorphism $(P, \alpha_{\mathrm{id}_P}, FP) \longrightarrow (P, ?, FP)$ is of the form (id_P, Φ), where $\Phi \in \mathrm{Aut}(FP)$ is an algebra automorphism of FP.

Letting $y := 1 - z \in FP$, hence $y^2 = 1 + z + z^2$, the F-basis $\{1, y, y^2\}$ is adjusted to the radical series $FP = J^0(FP) > J^1(FP) > J^2(FP) > J^3(FP) = \{0\}$ of FP, and it can be checked, for example with the help of the computer algebra system GAP [14], that, with respect to this basis, we have

$$\mathrm{Aut}(FP) \cong \left\{ \Phi_{a,b} := \begin{bmatrix} 1 & . & . \\ . & a & . \\ . & b & a^2 \end{bmatrix} \in \mathrm{GL}_3(F) \;\middle|\; a \in F^\times,\ b \in F \right\}.$$

Hence we have $\Phi_{a,b}(z) = (1 - a - b) + (a - b)z - bz^2$; in particular, we have $\mathrm{id}_{FP} = \Phi_{1,0}$, and the non-trivial automorphism of P, mapping $z \longmapsto z^2$, extends to $\Phi_{-1,-1}$. Thus the interior algebras looked for are given as $(P, \Phi_{a,b} \circ \alpha_{\mathrm{id}_P}, FP)$, where $(P, \alpha_{\mathrm{id}_P}, FP) = (P, \Phi_{1,0} \circ \alpha_{\mathrm{id}_P}, FP)$.

Finally note that this does not encompass all possible embeddings $P \longrightarrow (FP)^\times$: since $-z - z^2 \in (FP)^\times$ has order 3, there is an embedding of groups $\beta : P \longrightarrow (FP)^\times$, $z \longmapsto -z - z^2$, which extends to the unitary algebra endomorphism $\Phi_{0,1}$ of FP, which is not an automorphism. Anyway, this gives rise to the interior algebra (P, β, FP), which is not isomorphic to $(P, \alpha_{\mathrm{id}_P}, FP)$ in $\mathcal{A}$, hence is not a source algebra of FP.

3. Reducing Feit's Conjecture to Puig's Conjecture

We have now prepared the language to state Feit's Conjecture on sources of simple modules over group algebras as well as Puig's Conjecture on source algebras of blocks precisely. We will then prove the reduction theorem relating these conjectures, which we will use extensively throughout this paper.

3.1. *Source algebras vs vertex-source pairs.* (a) The relation between source algebras, in the sense of 2.10, and vertex-source pairs, in the sense of Definition 2.6, is given as follows: let G be a group, let B be a block of FG, let $f : P \longrightarrow G$ be an embedding such that $f(P) \leqslant G$ is a defect group of B, and let $(P, \alpha_{f,i}, iBi)$ be a source algebra of B. Then, by [33, Prop. 38.2], we have a Morita equivalence between the algebras B and iBi, in the language of Remark 2.5 given by the restriction functor

$$\mathrm{Res}_\Psi : B\text{-}\mathbf{mod} \longrightarrow iBi\text{-}\mathbf{mod}$$

with respect to the natural embedding of algebras $\Psi : iBi \longrightarrow B$.

Suppose that M is an indecomposable FG-module belonging to the block B. Then the Morita correspondent of M in iBi is $\mathrm{Res}_\Psi(M) = iM$. Moreover, restricting iM along $\alpha_{f,i}$, we get an FP-module $\mathrm{Res}_{\alpha_{f,i}}(iM)$, which is, in general, decomposable. By [33, Prop. 38.3], the vertex-source pairs of (G, M) are precisely the vertex-source pairs $(Q, ?)$ of the indecomposable direct summands of the FP-module $\mathrm{Res}_{\alpha_{f,i}}(iM)$ such that $|Q|$ is maximal.

(b) We show that proceeding like this to determine the vertex-source pairs of (G, M) is independent of the particular choice of a source algebra: let (D, α, A) be any source algebra of B. Hence, by

Proposition 2.11, there is an isomorphism $(\varphi, \Phi) : (D, \alpha, A) \longrightarrow (P, \alpha_{f,i}, iBi)$ in $\mathcal{A}$. By Remark 2.5, we have an equivalence

$$\mathrm{Res}_{\Phi} : iBi\text{-}\mathbf{mod} \longrightarrow A\text{-}\mathbf{mod}.$$

Letting $N := \mathrm{Res}_{\Phi}(iM)$, we infer that the pairs $(P, \mathrm{Res}_{\alpha_{f,i}}(iM))$ and $(D, \mathrm{Res}_{\alpha}(N))$ are equivalent. Hence, by Lemma 2.4, the equivalence classes of pairs (P, M') where M' is a direct summand of the FP-module iM coincide with the equivalence classes of pairs (D, N') where N' is a direct summand of the FD-module N. Moreover, if the pairs (P, M') and (D, N') are equivalent then, by Proposition 2.7, their vertex-source pairs are pairwise equivalent.

In conclusion, to find the vertex-source pairs of (G, M), we may go over from $(P, \alpha_{f,i}, iBi)$ to an arbitrary source algebra (D, α, A) by considering the module N instead, and check the above maximality condition by varying over the pairs (D, N').

Definition 3.2. Let $\mathcal{G}$ be a set of groups, and let Q be a p-group.

(a) We define $\mathcal{V}_{\mathcal{G}}(Q)$ to be the set of all equivalence classes of pairs (Q, L) where L is an indecomposable FQ-module such that (Q, L) is a vertex-source pair of some pair (G, M), where G is a group in $\mathcal{G}$ and M is a *simple* FG-module. In the case that $\mathcal{G}$ is the set of all (finite) groups, we also write $\mathcal{V}(Q)$ rather than $\mathcal{V}_{\mathcal{G}}(Q)$.

(b) We say that $\mathcal{G}$ has the *vertex-bounded-defect property* with respect to Q if there is an integer $c_{\mathcal{G}}(Q)$ such that, for every pair (Q, L) in $\mathcal{V}_{\mathcal{G}}(Q)$ and for every pair (G, M) consisting of a group G in $\mathcal{G}$ and a *simple* FG-module M having (Q, L) as a vertex-source pair, M belongs to a block of FG having defect groups of order at most $c_{\mathcal{G}}(Q)$.

Remark 3.3. The vertex-bounded-defect property, by [11], holds in the case where Q is cyclic, with $c(Q) = |Q|$, including the case $Q = \{1\}$, covering all blocks of finite representation type. But it does indeed not hold in general, where, in particular, in the realm of blocks of tame representation type there are prominent counterexamples:

Let $p = 2$. For the groups $\{\mathrm{PSL}_2(q) \mid q \equiv 1 \pmod 4\}$, the Sylow 2-subgroups are isomorphic to the dihedral group $D_{(q-1)_2}$, where $(q-1)_2$ denotes the 2-part of $q-1$. Also, there is a simple $F[\mathrm{PSL}_2(q)]$-module in the principal block having dimension $(q-1)/2$ and whose vertices, by [10], are isomorphic to the Klein four-group $V_4 \cong C_2 \times C_2$. Moreover, for the groups $\{\mathrm{SL}_2(q) \mid q \equiv 1 \pmod 4\}$, consisting of the universal covering groups of groups above, the Sylow 2-subgroups are isomorphic to the generalized quaternion group $\mathfrak{Q}_{2(q-1)_2}$, and the inflations of the above simple $F[\mathrm{PSL}_2(q)]$-modules to $F[\mathrm{SL}_2(q)]$ have vertices isomorphic to the quaternion group $\mathfrak{Q}_8$. Finally, for the groups $\{\mathrm{GU}_2(q) \mid q \equiv 1 \pmod 4\}$ the Sylow 2-subgroups are isomorphic to the semidihedral group $\mathrm{SD}_{4(q-1)_2}$, and the identification $\mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$ shows that there is a simple $F[\mathrm{GU}_2(q)]$-module in the principal block having dimension $q-1$ whose vertices are isomorphic to V_4. (Alternatively, for the groups $\{\mathrm{PSL}_3(q) \mid q \equiv 3 \pmod 4\}$, the Sylow 2-subgroups are isomorphic to the semidihedral group $\mathrm{SD}_{2(q+1)_2}$, and there is a simple $F[\mathrm{PSL}_3(q)]$-module in the principal block having dimension $q(q+1)$ whose vertices, by [9], are isomorphic to V_4.)

From these cases we also obtain blocks of wild representation type violating the vertex-bounded-defect property, for example by taking direct products. Hence the question arises for which defect groups P or groups $\mathcal{G}$ one might expect the vertex-bounded-defect property to hold. In particular, the following is in [34] attributed to Puig:

Question 3.4. If p is odd, does then $\mathcal{G}$ always have the vertex-bounded-defect property with respect to Q?

We can now state Feit's and Puig's Conjectures, and prove the reduction theorem.

Conjecture 3.5. *(See Feit [12].) Let $\mathcal{G}$ be a set of groups (which might, in particular, be the set of all groups), let Q be a p-group, and let $\mathcal{V}_{\mathcal{G}}(Q)$ denote the set of equivalence classes of vertex-source pairs introduced in Definition 3.2. Then $\mathcal{V}_{\mathcal{G}}(Q)$ is finite.*

In consequence of Lemma 2.3, we can reformulate Feit's Conjecture equivalently also in the following way:

Conjecture 3.6. *Let $\mathcal{G}$ be a set of groups, and let Q be a p-group. Then there are, up to isomorphism in $\mathcal{A}$, only finitely many interior algebras (Q, α, E_L), where $E_L = \mathrm{End}_F(L)$ for an indecomposable FQ-module L with corresponding representation $\alpha : Q \longrightarrow E_L^\times$, such that (Q, L) is a vertex-source pair of some pair (G, M), where G is a group in $\mathcal{G}$ and M is a simple FG-module.*

Conjecture 3.7 *(Puig). Let $\mathcal{G}$ be a set of groups (which might, in particular, be the set of all groups), and let P be a p-group. Then there are only finitely many $\mathcal{A}$-isomorphism classes of interior algebras of p-blocks of groups in $\mathcal{G}$ whose defect groups are isomorphic to P.*

As for the origin of this conjecture, see [33, Conj. 38.5], and the comment on [33, p. 340].

Theorem 3.8. *Let $\mathcal{G}$ be a set of groups satisfying the vertex-bounded-defect property with respect to any p-group. Suppose that Puig's Conjecture* 3.7 *holds true for $\mathcal{G}$. Then Feit's Conjecture* 3.5 *is true for $\mathcal{G}$ as well.*

Proof. Let Q be a p-group, and let $c_{\mathcal{G}}(Q)$ be the integer appearing in Definition 3.2. Then there are finitely many (mutually non-isomorphic) p-groups $R_1, \ldots, R_n$ such that, whenever $G \in \mathcal{G}$ and M is a simple FG-module with vertex isomorphic to Q, the defect groups of the block containing M are isomorphic to one of the groups in $\{R_1, \ldots, R_n\}$.

Let $k \in \{1, \ldots, n\}$. Then, by Puig's Conjecture, there are, up to isomorphism in $\mathcal{A}$, only finitely many interior R_k-algebras occurring as source algebras of p-blocks for groups in $\mathcal{G}$ with defect groups isomorphic to R_k. Denote by $\{(R_k, \alpha_{k,1}, A_{k,1}), \ldots, (R_k, \alpha_{k,l_k}, A_{k,l_k})\}$ a transversal for these isomorphism classes.

Let further $r \in \{1, \ldots, l_k\}$, and choose representatives $\{M_{k,r,1}, \ldots, M_{k,r,d_{k,r}}\}$ for the isomorphism classes of simple $A_{k,r}$-modules. Via restriction along $\alpha_{k,r}$ we get FR_k-modules $\mathrm{Res}_{\alpha_{k,r}}(M_{k,r,1}), \ldots, \mathrm{Res}_{\alpha_{k,r}}(M_{k,r,d_{k,r}})$. For each $i \in \{1, \ldots, d_{k,r}\}$ we determine a vertex-source pair $(Q_{k,r,i}, S_{k,r,i})$ of an indecomposable direct summand of $\mathrm{Res}_{\alpha_{k,r}}(M_{k,r,i})$ such that $|Q_{k,r,i}|$ is maximal. So this gives rise to the finite set of pairs

$$\mathcal{V} := \bigcup_{k=1}^{n} \bigcup_{r=1}^{l_k} \bigcup_{i=1}^{d_{k,r}} \{(Q_{k,r,i}, S_{k,r,i})\}.$$

Consequently, by [33, Prop. 38.3], any vertex-source pair of some pair (G, M), with $G \in \mathcal{G}$ and M a simple FG-module, is equivalent to one of the pairs in the finite set $\mathcal{V}$. Hence $\mathcal{V}_{\mathcal{G}}(Q)$ is finite, proving Feit's Conjecture. $\square$

To prove Feit's Conjecture for the groups listed in the main theorem, we are going to apply Theorem 3.8. In order to do so, we will show that each of these sets satisfies the vertex-bounded-defect property with respect to any p-group; this will be done by giving explicit bounds as in the next theorem, whose proof will be broken up into several steps in subsequent sections.

Theorem 3.9. *Let Q be a p-group, let G be a finite group possessing a simple FG-module M belonging to a block with defect group isomorphic to P, and having vertices isomorphic to Q. Then the following hold:*

(a) *If $G = \mathfrak{S}_n$ then $|P| \leqslant |Q|!$.*
(b) *If $G = \mathfrak{A}_n$ and $p = 2$ then $|P| \leqslant (|Q| + 2)!/2$.*
(c) *If $G \in \{\widetilde{\mathfrak{S}}_n, \widehat{\mathfrak{S}}_n\}$ and $p \geqslant 3$ then $|P| \leqslant |Q|!$.*
(d) *If $G = \mathfrak{B}_n$ and $p \geqslant 3$ then $|P| \leqslant |Q|!$.*
(e) *If $G = \mathfrak{B}_n$ and $p = 2$ then $|P| \leqslant |Q| \cdot \log_2(|Q|)!$.*
(f) *If $G = \mathfrak{D}_n$ and $p = 2$ then $|P| \leqslant |Q| \cdot (\log_2(|Q|) + 1)!$.*

Proof. (a) follows from [6, Thm. 5.1].
(b) is proved in Proposition 4.7.
(c) is proved in Proposition 5.2.
(d) is proved in Proposition 6.5.
(e) and (f) are proved in 6.3. □

Using Theorems 3.8 and 3.9, we are now in a position to prove our main result:

Theorem 3.10. *Feit's Conjecture holds for the following groups*:

$$\{\mathfrak{S}_n\}_{n\geqslant 1},\quad \{\mathfrak{A}_n\}_{n\geqslant 1},\quad \{\widetilde{\mathfrak{S}}_n\}_{n\geqslant 1},\quad \{\widehat{\mathfrak{S}}_n\}_{n\geqslant 1},\quad \{\widetilde{\mathfrak{A}}_n\}_{n\geqslant 1},\quad \{\mathfrak{B}_n\}_{n\geqslant 2},\quad \{\mathfrak{D}_n\}_{n\geqslant 4}.$$

Proof. (i) Let $\mathcal{G}=\{\mathfrak{S}_n\}_{n\geqslant 1}$. Then Puig's Conjecture holds for $\mathcal{G}$, by work of Puig [30] and Scopes [32]. Moreover, $\mathcal{G}$ has the vertex-bounded-defect property with respect to any p-group, by Theorem 3.9(a). Hence Feit's Conjecture holds, by Theorem 3.8.

(ii) Let $\mathcal{G}=\{\mathfrak{A}_n\}_{n\geqslant 1}$. Suppose first that $p\geqslant 3$, and let E be a simple $F\mathfrak{A}_n$-module with vertex-source pair (Q,L). Then there is a simple $F\mathfrak{S}_n$-module D such that $E\mid \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D)$. Furthermore, (Q,L) is also a vertex-source pair of D. Hence we have $\mathcal{V}_{\mathcal{G}}(Q)\subseteq \mathcal{V}_{\{\mathfrak{S}_n\}}(Q)$, and we are done using (i).

Let now $p=2$. Then Puig's Conjecture holds for $\mathcal{G}$, by work of Kessar [21]. Moreover, $\mathcal{G}$ has the vertex-bounded-defect property with respect to any p-group, by Theorem 3.9(b). Hence Feit's Conjecture holds, by Theorem 3.8.

(iii) Let $\mathcal{G}=\{\widetilde{\mathfrak{S}}_n\}_{n\geqslant 1}$, where we may argue identically for $\mathcal{G}=\{\widehat{\mathfrak{S}}_n\}_{n\geqslant 1}$. Suppose first that $p\geqslant 3$. Then Puig's Conjecture holds for $\mathcal{G}$, by work of Kessar [19]. Moreover, $\mathcal{G}$ has the vertex-bounded-defect property with respect to any p-group, by Theorem 3.9(c). Hence Feit's Conjecture holds, by Theorem 3.8.

Let now $p=2$, and let D be a simple $F\widetilde{\mathfrak{S}}_n$-module. Since $Z:=\langle z\rangle\leqslant Z(\widetilde{\mathfrak{S}}_n)$, in the notation of 5.1, is a normal 2-subgroup of $\widetilde{\mathfrak{S}}_n$, it acts trivially on D. Thus there is a simple $F\mathfrak{S}_n$-module $\overline{D}$ such that $D=\mathrm{Inf}_Z^{\widetilde{\mathfrak{S}}_n}(\overline{D})$, where Inf denotes the inflation from $F\mathfrak{S}_n$-modules to $F\widetilde{\mathfrak{S}}_n$-modules via the normal subgroup $Z\trianglelefteqslant\widetilde{\mathfrak{S}}_n$. If (Q,L) is a vertex-source pair of D then $Z\leqslant Q$ and $\overline{Q}:=Q/Z$ is a vertex of $\overline{D}$. Moreover, there is an indecomposable $F\overline{Q}$-module $\bar{L}$ such that $L\cong \mathrm{Inf}_Z^Q(\bar{L})$ and such that $(\overline{Q},\bar{L})$ is a vertex-source pair of $\overline{D}$, see [23, Prop. 2.1] and [16, Prop. 2]. Hence we have $|\mathcal{V}_{\mathcal{G}}(Q)|\leqslant|\mathcal{V}_{\{\mathfrak{S}_n\}}(\overline{Q})|$, and we are done by (i).

(iv) Let $\mathcal{G}=\{\widetilde{\mathfrak{A}}_n\}_{n\geqslant 1}$. Letting again first $p\geqslant 3$, we may argue as in (ii) to show that $\mathcal{V}_{\mathcal{G}}(Q)\subseteq \mathcal{V}_{\{\widetilde{\mathfrak{S}}_n\}}(Q)$, and we are done using (iii). Moreover, letting $p=2$, since $Z\leqslant\widetilde{\mathfrak{A}}_n$, again using the notation of 5.1, we may argue as in (iii) to show that $|\mathcal{V}_{\mathcal{G}}(Q)|\leqslant|\mathcal{V}_{\{\mathfrak{A}_n\}}(\overline{Q})|$, and we are done using (ii).

(v) Let $\mathcal{G}=\{\mathfrak{B}_n\}_{n\geqslant 2}$. Then Puig's Conjecture holds for $\mathcal{G}$, by work of Kessar [20]. Moreover, $\mathcal{G}$ has the vertex-bounded-defect property with respect to any p-group, by Theorem 3.9 (d) and (e). Hence Feit's Conjecture holds, by Theorem 3.8.

(vi) Let $\mathcal{G}=\{\mathfrak{D}_n\}_{n\geqslant 4}$. Again suppose first that $p\geqslant 3$. Then we may argue as in (ii) to show that $\mathcal{V}_{\mathcal{G}}(Q)\subseteq \mathcal{V}_{\{\mathfrak{B}_n\}}(Q)$, and we are done using (v). Moreover, letting $p=2$, Puig's Conjecture holds for $\mathcal{G}$, by work of Kessar [20], and $\mathcal{G}$ has the vertex-bounded-defect property by Theorem 3.9(f). Hence Feit's Conjecture holds, by Theorem 3.8. □

Remark 3.11. We remark that the list of groups in Theorem 3.10 in particular encompasses all infinite series of real reflection groups, except the groups of type $I_2(m)$, that is, the dihedral groups D_{2m}, where $m\geqslant 3$. We give a direct proof that Feit's Conjecture holds for $\mathcal{G}=\{D_{2m}\}_{m\geqslant 3}$ as well; note that, since D_{2m} is soluble, this also follows from much more general work of Puig [29,31]:

Let first p be odd. Then $D_{2m}\cong C_m:C_2$ has a normal cyclic Sylow p-subgroup C_{m_p}, where m_p denotes the p-part of m. Hence, by [11], any simple FD_{2m}-module has the normal subgroup C_{m_p} as its vertex, and is thus a trivial-source module. Hence Feit's Conjecture holds for $\mathcal{G}$. Note that, by [33, Thm. 45.12], the source algebras of the blocks in question are isomorphic to FC_{m_p} or $F[C_{m_p}:C_2]\cong FD_{2m_p}$ as interior C_{m_p}-algebras, thus Puig's Conjecture holds for $\mathcal{G}$ as well.

Let now $p=2$, and let D be a simple FD_{2m}-module. Then there are two cases: if D is relatively C_m-projective then D has a normal subgroup C_{m_2} as its vertex, and is thus a trivial-source module. If D is not relatively C_m-projective then its restriction to $C_m \trianglelefteq D_{2m}$ is simple, hence one-dimensional, implying again that D is a trivial-source module. Hence Feit's Conjecture holds for $\mathcal{G}$. Note that, since D_{2m} is 2-nilpotent, by [33, Prop. 49.13] the blocks in question are nilpotent, hence, by Puig's Theorem [33, Thm. 50.6], their source algebras are isomorphic to $\mathrm{End}_F(iD) \otimes_F FP$, where i denotes a source idempotent, and the defect groups in the two cases are $P = C_{m_2}$ and $P = D_{2m_2}$, respectively; thus, since D is a trivial-source module, we, moreover, conclude that iD is the trivial FP-module, hence the source algebras are isomorphic to FP as interior P-algebras, so that Puig's Conjecture holds for $\mathcal{G}$ as well.

Thus it remains to prove Theorem 3.9. To do so, we will often argue along the lines of [6], where Theorem 3.9(a) has already been established. The key to this line of reasoning is the following:

Remark 3.12. By a *Brauer pair* of a group G we understand a pair (P, b) where P is a p-subgroup of G and b is a block of $F[PC_G(P)]$. Recall that the *Brauer correspondent* b^G, a block of FG, is defined, and if $B = b^G$ then we call (P, b) a *Brauer B-pair*. Moreover, in the case that P is a defect group of the block b we call (P, b) a *self-centralizing* Brauer pair.

Let now $\mathcal{G}$ be a set of groups, and let Q be a p-group. We say that $\mathcal{G}$ has the *strongly-bounded-defect property* with respect to Q if there is an integer $d_{\mathcal{G}}(Q)$ such that, for every group G in $\mathcal{G}$, the Brauer correspondent $(b_Q)^G$ of any self-centralizing Brauer pair (Q, b_Q) of G has defect groups of order at most $d_{\mathcal{G}}(Q)$.

By Knörr's Theorem [22], given a block B of G, a self-centralizing Brauer B-pair exists, in particular, in the case where Q is a vertex of some simple FG-module M belonging to the block B. Hence to prove the vertex-bounded-defect property of $\mathcal{G}$ with respect to a p-group Q, it suffices to show the strongly-bounded-defect property of $\mathcal{G}$ with respect to Q, and we infer $c_{\mathcal{G}}(Q) \leqslant d_{\mathcal{G}}(Q)$. We remark that the converse of Knörr's Theorem does not hold, see for example Example 4.8, but, to the authors' knowledge, there are no general results known towards a characterization of those self-centralizing Brauer pairs whose first components actually occur as vertices of simple modules.

Actually, we prove the strongly-bounded-defect property in the cases (a)–(c) of Theorem 3.9, while for the cases (d)–(f) we are content with the weaker vertex-bounded-defect property.

Before we proceed, we give a lemma needed later, relating Brauer correspondence to covering of blocks. It should be well known, but we have not been able to find a suitable reference.

Lemma 3.13. *Let G be a finite group, and let $H \trianglelefteq G$. Moreover, let $Q \leqslant H$ be a p-subgroup, let (Q, b) be a Brauer pair of H, that is, b is a block of $F[QC_H(Q)]$, and let $\tilde{b}$ be a block of $F[QC_G(Q)]$ covering b. Then the Brauer correspondent $\tilde{b}^G$ of $\tilde{b}$ in G covers the Brauer correspondent b^H of b in H.*

Proof. By Passman's Theorem [25, Thm. 5.5.5], we have to show that

$$\omega_{\tilde{b}^G}\left(h^{G+}\right) = \omega_{b^H}\left(h^{G+}\right) \quad \text{for all } h \in H,$$

where the ω's are the associated central characters, h^G is the G-conjugacy class of $h \in H$, and M^+ denotes the sum over any subset $M \subseteq G$. By definition of the Brauer correspondence, and by Passman's Theorem again, for all $h \in H$, we have

$$\begin{aligned}\omega_{b^H}\left(h^{G+}\right) &= \omega_b\left(\left(h^G \cap QC_H(Q)\right)^+\right) = \omega_b\left(\left(h^G \cap QC_G(Q)\right)^+\right)\\ &= \omega_{\tilde{b}}\left(\left(h^G \cap QC_G(Q)\right)^+\right) = \omega_{\tilde{b}^G}\left(h^{G+}\right),\end{aligned}$$

proving the lemma. $\square$

4. The alternating groups $\mathfrak{A}_n$

We proceed to prove the bound given in Theorem 3.9 for the alternating groups. We begin by fixing our notation for the Sylow p-subgroups of the symmetric and alternating groups, respectively; for later use we do this for arbitrary p. Then we focus on the case $p = 2$, collect the necessary facts about the self-centralizing Brauer pairs of the alternating groups, and use this to finally prove the desired bound.

4.1. *Sylow p-subgroups.* (a) We will use the following convention for denoting the Sylow p-subgroups of $\mathfrak{S}_n$ and $\mathfrak{A}_n$, respectively. Let $\mathfrak{S}_n$ act on the set $\{1, \dots, n\}$. Suppose first that $n = p^m$, for some $m \in \mathbb{N}$. Moreover, let $C_p := \langle (1, 2, \dots, p) \rangle$, and set $P_1 := 1$, $P_p := C_p$ and $P_{p^{i+1}} := P_{p^i} \wr C_p = \{(x_1, x_2, \dots, x_p; \sigma) \mid x_1, \dots, x_p \in P_{p^i},\ \sigma \in C_p\}$ for $i \geqslant 1$. As usual, for any $i \in \mathbb{N}_0$, we view P_{p^i} as a subgroup of $\mathfrak{S}_{p^i}$ in the obvious way. Then, by [18, 4.1.22, 4.1.24], P_{p^m} is a Sylow p-subgroup of $\mathfrak{S}_{p^m}$, and is generated by the following elements, where $j = 1, \dots, m$:

$$g_j := \prod_{k=1}^{p^{j-1}} \bigl(k, k + p^{j-1}, k + 2p^{j-1}, \dots, k + (p-1)p^{j-1}\bigr). \tag{2}$$

For instance, if $p = 2$ then P_8 is generated by $g_1 = (1, 2)$, $g_2 = (1, 3)(2, 4)$, and $g_3 = (1, 5)(2, 6)(3, 7)(4, 8)$.

(b) Next let $n \in \mathbb{N}$ be divisible by p, with p-adic expansion $n = \sum_{j=1}^{s} \alpha_j p^{i_j}$, for some $s \geqslant 1$, $i_1 > \dots > i_s \geqslant 1$, and $1 \leqslant \alpha_j \leqslant p - 1$ for $j = 1, \dots, s$. By [18, 4.1.22, 4.1.24], $P_n := \prod_{j=1}^{s} \prod_{l_j=1}^{\alpha_j} P_{p^{i_j}, l_j}$ is then a Sylow p-subgroup of $\mathfrak{S}_n$. Here, the direct factor $P_{p^{i_1}, 1}$ is acting on $\{1, \dots, p^{i_1}\}$, $P_{p^{i_1}, 2}$ is acting on $\{p^{i_1} + 1, \dots, 2p^{i_1}\}$, and so on. If, finally, $n \geqslant p + 1$ is not divisible by p then we set $P_n := P_r$ where $r < n$ is maximal with $p \mid r$. So, in any case, P_n is a Sylow p-subgroup of $\mathfrak{S}_n$.

(c) We will examine the case $p = 2$ in more detail, as this will be of particular importance for our subsequent arguments. As above, suppose that n is even, with 2-adic expansion $n = \sum_{j=1}^{s} 2^{i_j}$, for some $s \geqslant 2$ and $i_1 > i_2 > \dots > i_s \geqslant 1$. Letting $n_j := 2^{i_j}$, we get $P_n = \prod_{j=1}^{s} P_{n_j}$, where P_{n_j} is understood to be acting on the set

$$\Omega_j := \left\{ \left(\sum_{l=1}^{j-1} n_l \right) + 1, \dots, \sum_{l=1}^{j} n_l \right\},$$

for $j = 1, \dots, s$. The corresponding generating set for P_{n_j} given by (2) will be denoted by $\{g_{1,j}, \dots, g_{i_j,j}\}$, for $j = 1, \dots, s$. So if, for instance, $n = 14 = 8 + 4 + 2$ then $P_n = P_{14}$ is generated by $g_{1,1} = (1, 2)$, $g_{2,1} = (1, 3)(2, 4)$, $g_{3,1} = (1, 5)(2, 6)(3, 7)(4, 8)$, $g_{1,2} = (9, 10)$, $g_{2,2} = (9, 11)(10, 12)$, and $g_{1,3} = (13, 14)$.

(d) We now set $Q_n := P_n \cap \mathfrak{A}_n$, so that Q_n is a Sylow p-subgroup of the alternating group $\mathfrak{A}_n$. If $p > 2$ then clearly $Q_n = P_n$. Thus, suppose again that $p = 2$. If $n = 2$ then $Q_n = Q_2 = 1$. If $n = 2^m$, for some $m \geqslant 2$, then, by (2), we obtain the following generators for Q_n:

$$h_1 := (1, 2)\bigl(2^{m-1} + 1, 2^{m-1} + 2\bigr); \qquad h_j := g_j, \quad \text{for } j = 2, \dots, m. \tag{3}$$

For clearly $Q := \langle h_1, \dots, h_m \rangle \leqslant Q_n$, and $Q \langle (1, 2) \rangle = \langle (1, 2) \rangle Q = P_n$. Thus $Q = Q_n$.

If $n > 4$ is even but not a power of 2 then we again consider the 2-adic expansion $n = \sum_{j=1}^{s} 2^{i_j}$, for some $s \geqslant 2$ and some $i_1 > \dots > i_s \geqslant 1$. Then the following elements generate Q_n:

$$\begin{aligned} h_{1,j} &:= g_{1,s} g_{1,j}, \quad \text{for } j = 1, \dots, s-1; \\ h_{k,j} &:= g_{k,j}, \quad \text{for } j = 1, \dots, s \text{ and } k = 2, \dots, i_j. \end{aligned} \tag{4}$$

Namely, these elements generate a subgroup Q of Q_n such that $Q\langle g_{1,s}\rangle = \langle g_{1,s}\rangle Q = P_n$. For instance, $Q_{14} = P_{14} \cap \mathfrak{A}_{14} = (P_8 \times P_4 \times P_2) \cap \mathfrak{A}_{14}$ is generated by the elements $h_{1,1} = (1,2)(13,14)$, $h_{1,2} = (9,10)(13,14)$, $h_{2,1} = (1,3)(2,4)$, $h_{3,1} = (1,5)(2,6)(3,7)(4,8)$, and $h_{2,2} = (9,11)(10,12)$.

For the remainder of this section, let $p = 2$.

Proposition 4.2. *Let $n = \sum_{j=1}^{s} 2^{i_j} \geqslant 2$ be the 2-adic expansion of n, where $s \in \mathbb{N}$ and $i_1 > \cdots > i_s \geqslant 1$, and let again $n_j := 2^{i_j}$ for $j = 1, \ldots, s$.*

(a) *If $n \equiv 0 \pmod 4$ then*

$$C_{\mathfrak{S}_n}(Q_n) = C_{\mathfrak{A}_n}(Q_n) = Z(Q_n) = \begin{cases} Q_4, & \text{if } n = 4, \\ Z(P_n) = Z(P_{n_1}) \times \cdots \times Z(P_{n_s}), & \text{if } n > 4. \end{cases}$$

(b) *If $n \equiv 2 \pmod 4$ then $i_s = 1$, and*

$$C_{\mathfrak{S}_n}(Q_n) = Z(P_n) = Z(P_{n_1}) \times \cdots \times Z(P_{n_s}) = Z(Q_n) \times P_2 = C_{\mathfrak{A}_n}(Q_n) \times P_2.$$

Proof. We may assume that $n \geqslant 4$. Then $\Omega_1, \ldots, \Omega_s$ are the orbits of P_n on $\{1, \ldots, n\}$, as well as the orbits of Q_n on $\{1, \ldots, n\}$, where Ω_j is as above in 4.1(c). Since $|\Omega_1| > \cdots > |\Omega_s|$, the Q_n-sets $\Omega_1, \ldots, \Omega_s$ are pairwise non-isomorphic. For $j = 1, \ldots, s$, let $\omega_j \in \Omega_j$, and set $R_j := \mathrm{Stab}_{Q_n}(\omega_j)$. Then Ω_j is as Q_n-set isomorphic to Q_n/R_j, and we have the following group isomorphism, see [6, La. 4.3]:

$$\prod_{j=1}^{s} N_{Q_n}(R_j)/R_j \longrightarrow C_{\mathfrak{S}_n}(Q_n).$$

In particular, $C_{\mathfrak{S}_n}(Q_n)$ is a 2-group and, hence, so is $Q_n C_{\mathfrak{S}_n}(Q_n)$. Thus there is some $g \in \mathfrak{S}_n$ such that ${}^g(Q_n C_{\mathfrak{S}_n}(Q_n)) \leqslant P_n$. In particular, we have ${}^g Q_n \leqslant P_n \cap \mathfrak{A}_n = Q_n$, that is, $g \in N_{\mathfrak{S}_n}(Q_n)$. Hence we have $g \in N_{\mathfrak{S}_n}(C_{\mathfrak{S}_n}(Q_n))$ as well, implying $C_{\mathfrak{S}_n}(Q_n) \leqslant P_n$, and thus $C_{\mathfrak{S}_n}(Q_n) = C_{P_n}(Q_n)$. So it suffices to show that

$$C_{P_n}(Q_n) = \begin{cases} Q_4, & \text{if } n = 4, \\ Z(P_n), & \text{if } n \neq 4, \end{cases}$$

since then we also get

$$C_{\mathfrak{A}_n}(Q_n) = Z(Q_n) = \begin{cases} Q_4, & \text{if } n = 4, \\ Z(P_n), & \text{if } 4 < n \equiv 0 \pmod 4, \\ Z(P_{n_1}) \times \cdots \times Z(P_{n_{s-1}}), & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

The statement for $n = 4$ is clear. Next suppose that $n = 2^m$, for some $m \geqslant 3$. We argue with induction on m, and show that $Z(P_n) = C_{P_n}(Q_n)$. For $m = 3$ this is immediately checked to be true, so that we may now suppose that $m > 3$. We consider P_n again as the wreath product $P_{2^{m-1}} \wr C_2 = \{(x_1, x_2; \sigma) \mid x_1, x_2 \in P_{2^{m-1}},\ \sigma \in C_2\}$. Let $x := (x_1, x_2; \sigma) \in C_{P_n}(Q_n)$, so that, for each $y := (y_1, y_2; \pi) \in Q_n$, we have

$$\begin{aligned} (x_1 y_{\sigma(1)}, x_2 y_{\sigma(2)}; \sigma\pi) &= (x_1, x_2; \sigma)(y_1, y_2; \pi) = (y_1, y_2; \pi)(x_1, x_2; \sigma) \\ &= (y_1 x_{\pi(1)}, y_2 x_{\pi(2)}; \pi\sigma). \end{aligned} \tag{5}$$

Setting $y_1 := y_2 := 1$ and $\pi := (1,2)$, Eq. (5) yields $x_1 = x_2$. Next we set $\pi := 1$, $y_1 := 1$, and $1 \neq y_2 \in Q_{2^{m-1}}$. Then (5) this time implies $x_1 y_{\sigma(1)} = x_1$ and $x_1 y_{\sigma(2)} = y_2 x_1$. Therefore $\sigma = 1$ and

$x_1 \in C_{P_{2^{m-1}}}(Q_{2^{m-1}})$. Thus, by induction, $x_1 \in Z(P_{2^{m-1}})$. Consequently, $x = (x_1, x_1; 1) \in Z(P_n)$, and we have $C_{P_n}(Q_n) = Z(P_n) \leqslant Q_n$.

Now let $n > 4$ with $s \geqslant 2$. We show that also in this case $C_{P_n}(Q_n) = Z(P_n) = Z(P_{n_1}) \times \cdots \times Z(P_{n_s})$. For this, let $x \in C_{P_n}(Q_n)$, and write $x = x_1 \cdots x_s$ for appropriate $x_j \in P_{n_j}$ and $j = 1, \ldots, s$. Since $Q_{n_1} \times \cdots \times Q_{n_j} \leqslant Q_n$, we deduce that $x_j \in C_{P_{n_j}}(Q_{n_j})$, for $j = 1, \ldots, s$. Hence, by what we have just proved above, $x_j \in Z(P_{n_j})$ if $i_j > 2$. Moreover, $x_j \in Z(Q_4) = Q_4$ if $i_j = 2$, and clearly $x_j \in Z(P_2) = P_2$ if $i_j = 1$. Suppose that there is some $j \in \{1, \ldots, s\}$ with $i_j = 2$. Then $j \in \{s-1, s\}$. We need to show that $x_j \in Z(P_4)$. Assume that this is not the case. In the notation of 4.1, we may then suppose that $x_j = h_{i_j,j} = g_{2,j}$. But this leads to the contradiction $x(g_{1,j}g_{1,s})x^{-1} = x_j g_{1,j} x_j \cdot g_{1,s} \neq g_{1,j}g_{1,s}$ if $j = s-1$, and to the contradiction $x(g_{1,1}g_{1,j})x^{-1} = g_{1,1} \cdot x_j g_{1,j} x_j \neq g_{1,1}g_{1,j}$ if $j = s$. Thus also $x_j \in Z(P_4)$, and we have shown that $Z(P_n) \leqslant C_{P_n}(Q_n) \leqslant Z(P_{n_1}) \times \cdots \times Z(P_{n_s}) = Z(P_n)$. $\square$

4.3. *The* 2-*Blocks of* $\mathfrak{A}_n$. (a) Recall from [18, 6.1.21] that each block B of $F\mathfrak{S}_n$ can be labelled combinatorially by some integer $w \geqslant 0$ and a 2-regular partition κ of $n - 2w$. We call w the 2-*weight* of B, and κ the 2-*core* of B. Moreover, by [18, Thm. 6.2.39], the defect groups of B are in $\mathfrak{S}_n$ conjugate to $P_{2w} \leqslant \mathfrak{S}_{2w} \leqslant \mathfrak{S}_n$.

(b) The following relationships between blocks of $F\mathfrak{S}_n$ and $F\mathfrak{A}_n$ are well known; see for instance [27]: for any partition λ of n, we denote its conjugate partition by λ'. That is, the Young diagram $[\lambda']$ of λ' is obtained by transposing the Young diagram $[\lambda]$.

Suppose now that B is a block of $F\mathfrak{S}_n$ of weight w and with 2-core κ. Denote the corresponding block idempotent of $F\mathfrak{S}_n$ by e_B. The 2-core κ is a triangular partition, so that $\kappa = \kappa'$. If $w \geqslant 1$ then e_B is a block idempotent of $F\mathfrak{A}_n$, and if $w = 0$ then $e_B = e_{B'} + e_{B''}$ for $\mathfrak{S}_n$-conjugate blocks $B' \neq B''$ of $F\mathfrak{A}_n$ of defect 0; note that for $w = 1$ this also yields a (single) block of $F\mathfrak{A}_n$ of defect 0. Hence, the *weight* of any block of $F\mathfrak{A}_n$ is understood to be the weight w of the covering block of $F\mathfrak{S}_n$, and the associated defect groups are in $\mathfrak{A}_n$ conjugate to $Q_{2w} \leqslant \mathfrak{A}_{2w} \leqslant \mathfrak{A}_n$.

(c) Let B be a block of $F\mathfrak{A}_n$ of weight $w \geqslant 0$. Let further (Q, b_Q) be a self-centralizing Brauer B-pair. Then, by [25, Thm. 5.5.21], there is a defect group P of B such that $Z(P) \leqslant C_P(Q) \leqslant Q \leqslant P$. Replacing (Q, b_Q) by a suitable $\mathfrak{A}_n$-conjugate, we may assume that $P = Q_{2w}$. If w is even then, by Proposition 4.2, $Z(Q_{2w})$ has no fixed points on $\{1, \ldots, 2w\}$, hence, in this case, Q acts fixed point freely on precisely $2w$ points; note that this also holds for $w = 0$. If w is odd then, by Proposition 4.2 again, $Z(Q_{2w})$ has exactly the two fixed points $\{2w-1, 2w\}$ on $\{1, \ldots, 2w\}$, hence, in this case Q acts fixed point freely precisely on either $\{1, \ldots, 2w-2\}$ or $\{1, \ldots, 2w\}$, that is, on $2x$ points, where $w - 1 \leqslant x \leqslant w$; note that for $w = 1$ we have $x = 0$.

The following example shows that even if we restrict ourselves to Brauer pairs arising from vertices of simple modules we have to deal with both cases $w - 1 \leqslant x \leqslant w$:

Example 4.4. Suppose that $n = 2m$, for some odd integer $m \geqslant 3$, and consider the simple $F\mathfrak{S}_n$-module $D^{(m+1,m-1)}$ labelled by the partition $(m+1, m-1)$ of n. This is the *basic spin* $F\mathfrak{S}_n$-module, belonging to the principal block of $F\mathfrak{S}_n$, which has weight m. Since $n \equiv 2 \pmod 4$, the restriction $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n}(D^{(m+1,m-1)}) =: E^{(m+1,m-1)}$ is simple, by [1], and thus belongs to the principal block B_0 of $F\mathfrak{A}_n$. By [7, Thm. 7.2], $E^{(m+1,m-1)}$ has common vertices with the basic spin $F\mathfrak{S}_{n-1}$-module $D^{(m,m-1)}$. Therefore, the vertices of $E^{(m+1,m-1)}$ are conjugate to subgroups of Q_{n-2} and have, in particular, fixed points on $\{1, \ldots, n\}$, while Q_n acts of course fixed point freely. This shows that there is indeed a self-centralizing Brauer B_0-pair (Q, b_Q) of $\mathfrak{A}_n$, where Q arises as a vertex of a simple $F\mathfrak{A}_n$-module and such that Q has strictly more fixed points on $\{1, \ldots, n\}$ than the associated defect group Q_n of its Brauer correspondent $b_Q^{\mathfrak{A}_n} = B_0$.

The next theorem is motivated by the results of [6], where the self-centralizing Brauer pairs of the symmetric groups are examined, for which, using the above notation, we necessarily have $x = w$. We pursue the analogy to the case of the symmetric groups as far as possible, the treatment being reminiscent of the exposition in [26, Sect. 1].

Theorem 4.5. *Let (Q, b_Q) be a self-centralizing Brauer pair of $\mathfrak{A}_n$.*

(a) *Let $\Omega \subseteq \{1, \dots, n\}$ be such that Q acts fixed point freely on Ω and fixes $\{1, \dots, n\} \setminus \Omega$ pointwise. Then we have $C_{\mathfrak{A}(\Omega)}(Q) = Z(Q)$.*

(b) *Let P be a defect group of the Brauer correspondent $B := b_Q^{\mathfrak{A}_n}$ of b_Q in $\mathfrak{A}_n$ such that $C_P(Q) \leqslant Q \leqslant P$, and let $\widehat{\Omega} \subseteq \{1, \dots, n\}$ be such that P acts fixed point freely on $\widehat{\Omega}$ and fixes $\{1, \dots, n\} \setminus \widehat{\Omega}$ pointwise. Then we even have $C_{\mathfrak{A}(\widehat{\Omega})}(Q) = Z(Q)$.*

Proof. Since (Q, b_Q) is self-centralizing, the block b_Q of $F[QC_{\mathfrak{A}_n}(Q)]$ has defect group Q. Let $w \geqslant 0$ be the weight of $B = b_Q^{\mathfrak{A}_n}$. By the observations made in 4.3, we have $2w - 2 \leqslant 2x = |\Omega| \leqslant 2w = |\widehat{\Omega}|$, and we may suppose that $\Omega = \{1, \dots, 2x\}$ and $\widehat{\Omega} = \{1, \dots, 2w\}$, that is, $Q \leqslant \mathfrak{A}_{2x} \leqslant \mathfrak{A}_{2w}$ and $P = Q_{2w} \leqslant \mathfrak{A}_{2w}$. We have $C_{\mathfrak{S}_n}(Q) = C_{\mathfrak{S}_{2x}}(Q) \times \mathfrak{S}_{n-2x}$, and thus also $QC_{\mathfrak{S}_n}(Q) = QC_{\mathfrak{S}_{2x}}(Q) \times \mathfrak{S}_{n-2x}$. Consider the following chain of normal subgroups

$$QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x} \trianglelefteq QC_{\mathfrak{A}_n}(Q) \trianglelefteq QC_{\mathfrak{S}_n}(Q) = QC_{\mathfrak{S}_{2x}}(Q) \times \mathfrak{S}_{n-2x}. \tag{6}$$

Since $|QC_{\mathfrak{S}_n}(Q) : QC_{\mathfrak{A}_n}(Q)| \leqslant 2$, by [25, Cor. 5.5.6] there is a unique block $\tilde{b}_Q$ of $F[QC_{\mathfrak{S}_n}(Q)]$ covering b_Q. In particular, $(Q, \tilde{b}_Q)$ is a (not necessarily self-centralizing) Brauer pair of $\mathfrak{S}_n$. We may write $\tilde{b}_Q = \tilde{b}_0 \otimes \tilde{b}_1$, for some block $\tilde{b}_0$ of $F[QC_{\mathfrak{S}_{2x}}(Q)]$ and some block $\tilde{b}_1$ of $F\mathfrak{S}_{n-2x}$. Since Q has no fixed points on $\{1, \dots, 2x\}$, by [26, Prop. 1.2, Prop. 1.3] we conclude that $F[QC_{\mathfrak{S}_{2x}}(Q)]$ has only one block, that is, the principal one. Therefore, each block of $F[QC_{\mathfrak{A}_{2x}}(Q)]$ is covered by the principal block $\tilde{b}_0$ of $F[QC_{\mathfrak{S}_{2x}}(Q)]$. Hence all blocks of $F[QC_{\mathfrak{A}_{2x}}(Q)]$ are conjugate in $QC_{\mathfrak{S}_{2x}}(Q)$. But then also $F[QC_{\mathfrak{A}_{2x}}(Q)]$ has only one block, that is, the principal block b_0.

Moreover, b_Q covers some block $b_0 \otimes b_1$ of $F[QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x}]$, where b_1 is a block of $F\mathfrak{A}_{n-2x}$, and, by [25, Cor. 5.5.6] again, b_Q is the unique block of $F[QC_{\mathfrak{A}_n}(Q)]$ covering $b_0 \otimes b_1$. The defect groups of $b_0 \otimes b_1$ are in $QC_{\mathfrak{A}_n}(Q)$ conjugate to subgroups of Q, by Fong's Theorem [25, Cor. 5.5.16]. Hence b_1 has to be a block of defect 0. Thus, since $\tilde{b}_1$ covers b_1 we infer from 4.3 that $\tilde{b}_1$ is a block of weight $\tilde{w} = 0$ or $\tilde{w} = 1$, that is, of defect 0 or 1, respectively. Moreover, since b_0 is the principal block of $F[QC_{\mathfrak{A}_{2x}}(Q)]$, we infer that $Q \in \mathrm{Syl}_2(QC_{\mathfrak{A}_{2x}}(Q))$.

Thus we may summarize the properties of the relevant blocks of the subgroups in (6) as follows:

$b_0 \otimes b_1$		b_Q	$\tilde{b}_Q =$	$\tilde{b}_0 \otimes \tilde{b}_1$
defect Q	defect 0	defect Q		weight $\tilde{w} \in \{0, 1\}$
principal			principal	

To show (a), assume that $QC_{\mathfrak{A}_{2x}}(Q)$ is not a 2-group, so that there is some $1 \neq g \in QC_{\mathfrak{A}_{2x}}(Q)$ of odd order. Thus $g \in C_{\mathfrak{A}_{2x}}(Q)$, and we denote the conjugacy class of g in $QC_{\mathfrak{A}_{2x}}(Q)$ by C. Since $Q \in \mathrm{Syl}_2(QC_{\mathfrak{A}_{2x}}(Q))$, we also have $Q \in \mathrm{Syl}_2(C_{QC_{\mathfrak{A}_{2x}}(Q)}(g))$. In particular, Q is a defect group of the conjugacy class C and of the conjugacy class $\{1\} \neq C$. Hence, from [13, Cor. IV.4.17] we infer that $F[QC_{\mathfrak{A}_{2x}}(Q)]$ has two blocks of maximal defect, contradicting the fact that the principal block is the only block of $F[QC_{\mathfrak{A}_{2x}}(Q)]$. Hence $QC_{\mathfrak{A}_{2x}}(Q)$ is a 2-group, and from this we finally deduce that $QC_{\mathfrak{A}_{2x}}(Q) = Q$, that is, $C_{\mathfrak{A}_{2x}}(Q) = Z(Q)$, proving (a).

To show (b) we may suppose that $x = w - 1$, hence we have $n \geqslant 2x + 2$. We show that in this case we have

$$QC_{\mathfrak{A}_n}(Q) = QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x},$$

from which we get $C_{\mathfrak{A}_{2w}}(Q) = QC_{\mathfrak{A}_{2x}}(Q) = Z(Q)$. So assume, for a contradiction, that $QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x} < QC_{\mathfrak{A}_n}(Q)$. We consider again the chain of subgroups (6), where $[QC_{\mathfrak{S}_n}(Q) : (QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x})] \leqslant 4$. Since $QC_{\mathfrak{A}_n}(Q) \leqslant \mathfrak{A}_n$ but $\mathfrak{S}_{n-2x} \nleqslant \mathfrak{A}_n$, we get $[QC_{\mathfrak{S}_n}(Q) : QC_{\mathfrak{A}_n}(Q)] = 2$, and hence $[QC_{\mathfrak{A}_n}(Q) : (QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x})] = 2$.

Since b_Q has defect group Q, from Fong's Theorem [25, Thm. 5.5.16] we conclude that the inertial group of $b_0 \otimes b_1$ is given as $T_{QC_{\mathfrak{A}_n}(Q)}(b_0 \otimes b_1) = QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x}$. Thus $b_0 \otimes b_1$ is not $QC_{\mathfrak{A}_n}(Q)$-invariant, hence is not $QC_{\mathfrak{S}_n}(Q)$-invariant either. Since b_0 is the principal block of $F[QC_{\mathfrak{A}_{2x}}(Q)]$, this implies that b_1 is not $\mathfrak{S}_{n-2x}$-invariant, from which, by 4.3, we infer that $\tilde{b}_1$ has weight $\tilde{w} = 0$, that is, $\tilde{b}_1$ has defect 0.

Let $\widetilde{B} := \tilde{b}_Q^{\mathfrak{S}_n}$ be the Brauer correspondent of $\tilde{b}_Q = \tilde{b}_0 \otimes \tilde{b}_1$ in $\mathfrak{S}_n$. Since $\tilde{b}_1$ has weight $\tilde{w} = 0$ and Q acts fixed point freely on the set Ω of cardinality $2x$, we conclude from [26, Thm. 1.7] that $\widetilde{B}$ is a block of weight x. But, by 4.3, the block of $F\mathfrak{S}_n$ covering B, has weight w, contradicting Lemma 3.13. Thus we have $QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x} = QC_{\mathfrak{A}_n}(Q)$, proving (b). □

Remark 4.6. A closer analysis of the arguments in the proof of Theorem 4.5 yields a somewhat more precise description of the self-centralizing Brauer pairs of $\mathfrak{A}_n$, which at least helps to exclude certain subgroups from being the first component of a self-centralizing Brauer pair.

It turns out that there are only the cases listed below, all of which, by Example 4.8, actually occur. We give the results without proofs, since we do not use these facts later on. Note that, by Brauer's Third Main Theorem [25, Thm. 5.6.1], b_Q is the principal block if and only if its Brauer correspondent $b_Q^{\mathfrak{A}_n}$ is, which in turn is equivalent to $n \in \{2w, 2w+1\}$.

(a) Let first $n \in \{2x, 2x+1\}$, thus we have $x = w$, and hence $QC_{\mathfrak{A}_n}(Q) = QC_{\mathfrak{A}_{2x}}(Q)$ anyway. Moreover, b_Q is principal, hence, in particular, is $QC_{\mathfrak{S}_n}(Q)$-invariant. It turns out that always $\tilde{w} = 0$, and that both cases $d := [C_{\mathfrak{S}_{2x}}(Q) : C_{\mathfrak{A}_{2x}}(Q)] \in \{1, 2\}$ occur.

(b) Now let $n \geqslant 2x+2$. Then it turns out that $w = x + \tilde{w}$ and that

$$d := \left[C_{\mathfrak{S}_{2x}}(Q) : C_{\mathfrak{A}_{2x}}(Q)\right] = \left[QC_{\mathfrak{A}_n}(Q) : \left(QC_{\mathfrak{A}_{2x}}(Q) \times \mathfrak{A}_{n-2x}\right)\right] \in \{1, 2\}.$$

Moreover, only the following cases occur:

w odd	$d = 1$	$\tilde{w} = 1$	b_Q is $QC_{\mathfrak{S}_n}(Q)$-invariant
	$d = 2$	$\tilde{w} = 0$	b_Q is $QC_{\mathfrak{S}_n}(Q)$-invariant
w even	$d = 1$	$\tilde{w} = 0$	b_Q is not $QC_{\mathfrak{S}_n}(Q)$-invariant
	$d = 2$	$\tilde{w} = 0$	b_Q is $QC_{\mathfrak{S}_n}(Q)$-invariant

Note that b_Q is principal if and only if $n \in \{2x+2, 2x+3\}$ where x is even and $d = 1$.

We are now prepared to prove the bound given in Theorem 3.9.

Proposition 4.7. *Let B be a block of $F\mathfrak{A}_n$ with defect group P. Let further (Q, b_Q) be a self-centralizing Brauer B-pair. Then we have*

$$|P| \leqslant \frac{(|Q|+2)!}{2}.$$

Proof. Let w be the weight of B. By Theorem 4.5, there is a subset Ω of $\{1, \ldots, n\}$ with $2w - 2 \leqslant 2x = |\Omega| \leqslant 2w$, where x is as in 4.3, such that Q acts fixed point freely on Ω and fixes $\{1, \ldots, n\} \setminus \Omega$ pointwise. Moreover, $C_{\mathfrak{A}(\Omega)}(Q) = Z(Q)$.

Assume there are Q-orbits $\Omega' = \{\omega'_1, \ldots, \omega'_m\}$ and $\Omega'' = \{\omega''_1, \ldots, \omega''_m\}$ on Ω that are isomorphic as Q-sets. Then we may suppose that there is an isomorphism of Q-sets mapping ω'_i to ω''_i, for $i = 1, \ldots, m$. Since, by our assumption, Q acts fixed point freely on Ω, we deduce that $m \geqslant 2$ is even, and therefore the permutation $(\omega'_1, \omega''_1) \cdots (\omega'_m, \omega''_m)$ is contained in $C_{\mathfrak{A}(\Omega)}(Q)$. But, on the other hand, the elements in $Z(Q)$ have to fix every Q-orbit, so that $(\omega_1, \omega'_1) \cdots (\omega_m, \omega'_m) \notin Z(Q)$, a contradiction.

Hence we deduce that $\Omega = \biguplus_{i=1}^{k} \Omega_i$ consists of pairwise non-isomorphic Q-orbits. For $j = 1, \ldots, k$, let $\omega_j \in \Omega_j$, and set $R_j := \mathrm{Stab}_Q(\omega_j)$, where we may choose notation such that $|R_1| \leqslant \ldots \leqslant |R_k|$. Then, by [6, La. 4.3], we have the group isomorphism

$$\varphi : \prod_{j=1}^{k} N_Q(R_j)/R_j \longrightarrow C_{\mathfrak{S}(\Omega)}(Q),$$

which can be described as follows: for $i = 1, \ldots, k$, let $\pi_i : Q \longrightarrow \mathfrak{S}(\Omega_i) \cap Q$ be the canonical projection, and let $Z_i := C_{\mathfrak{S}(\Omega_i)}(\pi_i(Q))$. Then $\varphi(N_Q(R_i)/R_i) = Z_i$, for $i = 1, \ldots, k$. Hence $C_{\mathfrak{S}(\Omega)}(Q) = Z_1 \times \cdots \times Z_k$, and $Z(Q) = C_{\mathfrak{A}(\Omega)}(Q) = (Z_1 \times \cdots \times Z_k) \cap \mathfrak{A}(\Omega)$. Since Q acts fixed point freely on Ω, we have $R_i < Q$, thus $N_Q(R_i) > R_i$, in particular $|Z_i| \geqslant 2$. Since $(Z_2 \times \cdots \times Z_k) \cap \mathfrak{A}(\Omega) \leqslant Z(Q) \leqslant Q$ acts trivially on Ω_1, we have $(Z_2 \times \cdots \times Z_k) \cap \mathfrak{A}(\Omega) \leqslant R_1$. Moreover, as we have just mentioned, $|Z_2 \times \cdots \times Z_k| \geqslant 2^{k-1}$, so that for $k \geqslant 2$ we have $|R_1| \geqslant |(Z_2 \times \cdots \times Z_k) \cap \mathfrak{A}(\Omega)| \geqslant 2^{k-2}$. Therefore, for $k \geqslant 2$ we get

$$|\Omega| = \sum_{i=1}^{k} |Q : R_i| \leqslant \frac{k|Q|}{|R_1|} \leqslant \frac{k|Q|}{2^{k-2}}.$$

Thus, for $k \geqslant 4$ we infer $|\Omega| \leqslant |Q|$. It remains to consider the cases $k \leqslant 3$: if $k = 1$ then we have $|\Omega| \leqslant |Q|$ anyway. If $k = 2$ then we have $|\Omega| \leqslant |Q|$, except if Ω_1 is the regular Q-orbit, that is, $R_1 = \{1\}$, and hence $Z_1 \cong Q$. Now $Z_1 \cap \mathfrak{A}(\Omega) \leqslant R_2$ entails

$$\frac{|Q|}{2} \leqslant \left|Z_1 \cap \mathfrak{A}(\Omega)\right| \leqslant |R_2| \leqslant \frac{|Q|}{2},$$

thus $|R_2| = |Q|/2$, hence $|Z_2| = 2$, implying $|\Omega| = |Q| + 2$. Moreover, we have $|Z_1 \cap \mathfrak{A}(\Omega)| = |Q|/2$, hence $Z_1 \nleqslant \mathfrak{A}(\Omega)$. Since Z_1 acts regularly on Ω_1 and fixes Ω_2 pointwise, we infer that Z_1 contains an $|\Omega_1|$-cycle, that is, $Z_1 \cong Q$ is cyclic. Note that we have $|Q| \geqslant 4$, implying that $2x = |\Omega| = |Q| + 2 \equiv 2 \pmod 4$, that is, x is odd, hence, by 4.3, we infer that $w = x$.

Next we observe that if $R_1 = \{1\}$ then $k \geqslant 2$ forces $k = 2$, since $1 = |R_1| \geqslant |(Z_2 \times \cdots \times Z_k) \cap \mathfrak{A}(\Omega)| \geqslant 2^{k-2}$. Hence, if $k = 3$ then we have $|\Omega| \leqslant |Q|$ except if $|R_1| = |R_2| = 2$. Thus from $|(Z_2 \times Z_3) \cap \mathfrak{A}(\Omega)| \geqslant 2$ and $|(Z_1 \times Z_3) \cap \mathfrak{A}(\Omega)| \geqslant 2$ we deduce $R_1 = (Z_2 \times Z_3) \cap \mathfrak{A}(\Omega) \leqslant Z(Q)$ and $R_2 = (Z_1 \times Z_3) \cap \mathfrak{A}(\Omega) \leqslant Z(Q)$, showing $|Z_1| = |Q|/2 = |Z_2|$. This yields

$$|Q| \geqslant \left|(Z_1 \times Z_2 \times Z_3) \cap \mathfrak{A}(\Omega)\right| \geqslant \frac{|Q|}{2} \cdot \frac{|Q|}{2} \cdot 2 \cdot \frac{1}{2} = \frac{|Q|^2}{4},$$

hence $|Q| \leqslant 4$. Therefore we have $Q \cong V_4$, and $|\Omega_1| = |\Omega_2| = |\Omega_3| = 2$, thus $2x = |\Omega| = 6 = |Q| + 2$. Note that $x = 3$, by 4.3, implies that $w = x = 3$ as well.

Consequently, in any case we get $|\Omega| \leqslant |Q|$ or, in the two exceptions, $|\Omega| = |Q| + 2$ and $x = w$. This implies $2w \leqslant |Q| + 2$ and, since $P \leqslant \mathfrak{A}_{2w}$, we get $|P| \leqslant \frac{(|Q|+2)!}{2}$. $\square$

Example 4.8. We give a few examples, found with the help of the computer algebra system GAP [14], showing that all the cases listed in Remark 4.6 actually occur. In particular, the exceptional cases detected in the proof of Proposition 4.7, namely Q cyclic with two orbits of lengths $|Q|$ and 2, as well as $Q \cong V_4$ with three orbits of length 2 each, occur for the principal block of $\mathfrak{A}_6$.

(a) The principal blocks of $\mathfrak{A}_4$ and of $\mathfrak{A}_5$ both have weight $w = 2$, their defect groups are abelian and conjugate to $Q_4 \leqslant \mathfrak{A}_4$ and, in each case, up to conjugacy, there is a unique self-centralizing Brauer pair:

Q $(n \in \{4, 5\},\ w = 2)$		$\lvert Q\rvert$	$\lvert Z(Q)\rvert$	x	d
$\langle (1,2)(3,4), (1,3)(2,4) \rangle$	$\cong V_4$	4	4	2	1

The non-principal blocks of $\mathfrak{A}_7$ and of $\mathfrak{A}_{10}$ both have weight $w = 2$, their defect groups are abelian and conjugate to $Q_4 \leqslant \mathfrak{A}_4$ and, in each case, up to conjugacy, there are two self-centralizing Brauer pairs:

$Q \quad (n \in \{7, 10\},\ w = 2)$		$\lvert Q \rvert$	$\lvert Z(Q) \rvert$	x	d
$\langle (1,2)(3,4), (1,3)(2,4) \rangle$	$\cong V_4$	4	4	2	1
$\langle (1,2)(3,4), (1,3)(2,4) \rangle$	$\cong V_4$	4	4	2	1

(b) The principal blocks of $\mathfrak{A}_6$ and of $\mathfrak{A}_7$, and the non-principal block of $\mathfrak{A}_9$ all have weight $w = 3$, their defect groups are conjugate to $Q_6 \leqslant \mathfrak{A}_6$ and, in each case, up to conjugacy, there are four self-centralizing Brauer pairs:

$Q \quad (n \in \{6, 7, 9\},\ w = 3)$		$\lvert Q \rvert$	$\lvert Z(Q) \rvert$	x	d
$\langle (1,2)(3,4), (1,3)(2,4) \rangle$	$\cong V_4$	4	4	2	1
$\langle (1,2)(3,4), (3,4)(5,6) \rangle$	$\cong V_4$	4	4	3	2
$\langle (1,2)(3,4), (1,3,2,4)(5,6) \rangle$	$\cong C_4$	4	4	3	2
$\langle (1,2)(5,6), (1,3)(2,4) \rangle$	$\cong D_8$	8	2	3	2

(c) The non-principal block of $\mathfrak{A}_{11}$ has weight $w = 4$, its defect groups are conjugate to $Q_8 \leqslant \mathfrak{A}_8$ and, up to conjugacy, there are thirty-three self-centralizing Brauer pairs. We do not mention all of them, but just one concluding the list of cases in Remark 4.6:

$Q \quad (n = 11,\ w = 4)$		$\lvert Q \rvert$	$\lvert Z(Q) \rvert$	x	d
...					
$\langle (1,2)(3,4), (1,2)(5,6), (5,6)(7,8) \rangle$	$\cong C_2^3$	8	8	4	2
...					

Remark 4.9. With the help of the computer algebra system MAGMA [2], and using the techniques described in [8], it can be shown that all the 2-subgroups listed above actually occur as vertices of suitable simple modules, with the exception of the cyclic group in (b), of course, and the group given in (c).

We also point out a mistake in [6, Cor. 6.3(iii)], where the 2-groups $Q \leqslant \mathfrak{S}_n$ of order 4 occurring as vertices of simple $F\mathfrak{S}_n$-modules were classified (up to $\mathfrak{S}_n$-conjugation). In fact, the case where $Q = P_2 \times P_2$ and $w = 2$ cannot occur.

5. The double covers of $\mathfrak{S}_n$ and $\mathfrak{A}_n$

We begin by recalling the group presentations of the double covers of the symmetric and alternating groups, as well as the necessary facts about their blocks. Then we immediately proceed to prove the bound given in Theorem 3.9.

5.1. *Notation.* (a) Let $n \geqslant 1$, and consider the group $\widetilde{\mathfrak{S}}_n := \langle z, t_1, \ldots, t_{n-1} \rangle$ with relations

$$\begin{aligned} z^2 &= 1, \\ zt_i &= t_i z, \quad \text{for } i = 1, \ldots, n-1, \\ t_i^2 &= z, \quad \text{for } i = 1, \ldots, n-1, \quad (*) \\ t_i t_j &= z t_j t_i, \quad \text{for } |i - j| > 1, \\ (t_i t_{i+1})^3 &= z, \quad \text{for } i = 1, \ldots, n-2; \quad (**) \end{aligned}$$

in particular, we have $\widetilde{\mathfrak{S}}_1 := \langle z \rangle \cong C_2$. Note that also $\widetilde{\mathfrak{S}}_n \leqslant \widetilde{\mathfrak{S}}_{n+1}$, for $n \geqslant 1$. Via $\theta : \widetilde{\mathfrak{S}}_n \longrightarrow \mathfrak{S}_n$, $t_i \longmapsto (i, i+1)$, we obtain a group epimorphism with central kernel $\langle z \rangle$.

Replacing the relations $(*)$ by $t_i^2 = 1$, for $i = 1, \ldots, n-1$, and the relations $(**)$ by $(t_i t_{i+1})^3 = 1$, for $i = 1, \ldots, n-2$, we get an isoclinic group $\widehat{\mathfrak{S}}_n$, which also is a central extension of $\mathfrak{S}_n$; we have $\widetilde{\mathfrak{S}}_n \ncong \widehat{\mathfrak{S}}_n$ if and only if $1 \neq n \neq 6$. In the case where $n \geqslant 4$, the groups $\widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{S}}_n$ are the Schur representation groups of the symmetric group $\mathfrak{S}_n$. Whenever we have a subgroup H of $\mathfrak{S}_n$, we denote its preimage under θ by $\widetilde{H}$, and similarly its preimage in $\widehat{\mathfrak{S}}_n$ is denoted by $\widehat{H}$.

In particular, for $H = \mathfrak{A}_n$, we get $\widetilde{\mathfrak{A}}_n \trianglelefteq \widetilde{\mathfrak{S}}_n$ and $\widehat{\mathfrak{A}}_n \trianglelefteq \widehat{\mathfrak{S}}_n$, where we actually always have $\widetilde{\mathfrak{A}}_n \cong \widehat{\mathfrak{A}}_n$, and $|\widetilde{\mathfrak{S}}_n : \widetilde{\mathfrak{A}}_n| = |\widehat{\mathfrak{S}}_n : \widehat{\mathfrak{A}}_n| = 2$, for $n \geqslant 2$. If $n \geqslant 4$ and $6 \neq n \neq 7$ then $\widetilde{\mathfrak{A}}_n$ is the universal covering group of the alternating group $\mathfrak{A}_n$. Since we have no distinction between $\widetilde{\mathfrak{A}}_n$ and $\widehat{\mathfrak{A}}_n$ anyway, and since it will turn out that all observations for $\widetilde{\mathfrak{S}}_n$ immediately translate to $\widehat{\mathfrak{S}}_n$, we from now on confine ourselves to investigating $\widetilde{\mathfrak{A}}_n \trianglelefteq \widetilde{\mathfrak{S}}_n$.

(b) We list the known facts concerning the block theory of $\widetilde{\mathfrak{A}}_n$ and $\widetilde{\mathfrak{S}}_n$ we will need, where we from now on suppose that $p \geqslant 3$, for the remainder of this section. Each faithful block B of $F\widetilde{\mathfrak{S}}_n$ can be labelled combinatorially by some integer $w \geqslant 0$, called the *p-bar weight* of B, and a 2-regular partition κ of $n - pw$, called the *p-bar core of B*; for details we refer to [17, Appendix 10]. Given the p-bar weight w of the block B, by [27, Thm. (1.3)], the defect groups of B are the $\widetilde{\mathfrak{S}}_n$-conjugates of the Sylow p-subgroups of $\widetilde{\mathfrak{S}}_{pw}$. The latter in turn are via θ mapped to Sylow p-subgroups of $\mathfrak{S}_{pw}$.

Arguing along the lines of [6], we now have:

Proposition 5.2. *Let $p \geqslant 3$, and let B be a faithful block of $F\widetilde{\mathfrak{S}}_n$ with defect group P. Let further (Q, b_Q) be a self-centralizing Brauer B-pair. Then we have*

$$|P| \leqslant |Q|!.$$

Proof. Let w be the p-bar weight of B. Then P is conjugate to a Sylow p-subgroup of $\widetilde{\mathfrak{S}}_{pw}$, and $\theta(P)$ is in $\mathfrak{S}_n$ conjugate to a Sylow p-subgroup of $\mathfrak{S}_{pw}$. Thus we have $\theta(P) =_{\mathfrak{S}_n} P_{pw}$, so that there is a subset Ω of $\{1, \ldots, n\}$ with $|\Omega| = pw$ and such that $\theta(P)$ acts fixed point freely on Ω and fixes $\{1, \ldots, n\} \setminus \Omega$ pointwise. Moreover, by [25, Thm. 5.5.21], we may assume that

$$Z(P) \leqslant C_P(Q) \leqslant Q \leqslant P. \tag{7}$$

In particular, since $\theta|_P$ is injective, we infer that $Z(\theta(P)) = \theta(Z(P)) \cong Z(P)$ acts fixed point freely on Ω as well. By (7), we have $Z(\theta(P)) \leqslant \theta(Q) \leqslant \theta(P)$, hence $\theta(Q)$ acts fixed point freely on Ω and fixes $\{1, \ldots, n\} \setminus \Omega$ pointwise. That is, $\theta(Q) \leqslant \mathfrak{S}(\Omega)$ and $Q \leqslant \widetilde{\mathfrak{S}}(\Omega)$. By [3, Prop. 3.8 e], we further have

$$C_{\widetilde{\mathfrak{S}}(\Omega)}(Q) = Z(Q) \times Z\big(\widetilde{\mathfrak{S}}(\Omega)\big) = Z(Q) \times \langle z \rangle.$$

Since Q is a p-group, this implies

$$C_{\mathfrak{S}(\Omega)}\big(\theta(Q)\big) = \theta\big(C_{\widetilde{\mathfrak{S}}(\Omega)}(Q)\big) = \theta\big(Z(Q)\big) = Z\big(\theta(Q)\big),$$

thus, applying [6, Thm. 5.1], we get $|P| = |\theta(P)| \leqslant |\mathfrak{S}(\Omega)| = |\Omega|! \leqslant |\theta(Q)|! = |Q|!$. $\square$

6. The Weyl groups $\mathfrak{B}_n$ and $\mathfrak{D}_n$

We begin by recalling the description of the Weyl groups $\mathfrak{B}_n$ and $\mathfrak{D}_n$ of type B_n and D_n, respectively. Then we recall the necessary facts about their blocks, distinguishing the cases p even and p odd, in order to immediately proceed to prove the bound given in Theorem 3.9.

6.1. *Notation.* Let $C_2 := \langle (1, 2) \rangle$ be the group of order 2, and for $n \in \mathbb{N}$ set

$$\mathfrak{B}_n := C_2 \wr \mathfrak{S}_n = \big\{ (x_1, \ldots, x_n; \pi) \bigm| x_1, \ldots, x_n \in C_2,\ \pi \in \mathfrak{S}_n \big\}.$$

We define $\mathfrak{S}_n^* := \{(1,\dots,1;\pi) \mid \pi \in \mathfrak{S}_n\} \leqslant \mathfrak{B}_n$, and denote the usual group isomorphism $\mathfrak{S}_n^* \longrightarrow \mathfrak{S}_n$ by φ. Moreover, whenever U is a subgroup of $\mathfrak{S}_n$, we denote by $U^* \leqslant \mathfrak{S}_n^*$ its image under φ^{-1}. Let also $H := \{(x_1,\dots,x_n;1) \mid x_1,\dots,x_n \in C_2\} \trianglelefteq \mathfrak{B}_n$ be the base group of $\mathfrak{B}_n$; so H is isomorphic to a direct product of n copies of C_2, and we have $\mathfrak{B}_n = H\mathfrak{S}_n^*$. We will identify $\mathfrak{B}_n$ with a subgroup of $\mathfrak{S}_{2n}$, in the usual way by the primitive action. Furthermore, let $\mathfrak{D}_n := \mathfrak{B}_n \cap \mathfrak{A}_{2n}$. For $n \geqslant 2$, the group $\mathfrak{B}_n$ is isomorphic to the *Weyl group of type* B_n, and for $n \geqslant 4$, the group $\mathfrak{D}_n$ is isomorphic to the *Weyl group of type* D_n; see [18, 4.1.33].

Remark 6.2. As has been pointed out by the referee, Feit's Conjecture for the Weyl groups $\mathfrak{B}_n$ and $\mathfrak{D}_n$ can also be deduced directly from a more general theorem on semidirect products with abelian kernel; we will state and prove this theorem in Section 7 below. We will, however, treat the groups $\mathfrak{B}_n$ and $\mathfrak{D}_n$ separately, in order to stick to our general strategy for proving Feit's Conjecture by relating it to Puig's Conjecture via Theorem 3.8.

6.3. *The case* $p = 2$. (a) Suppose first that $p = 2$. Since H is a normal 2-subgroup of $\mathfrak{B}_n$ such that $C_{\mathfrak{B}_n}(H) = H$ is a 2-group, it follows from [25, Thm. 5.2.8] (see also [25, Exc. 5.2.10]) that $F\mathfrak{B}_n$ has only the principal block. Moreover, H acts trivially on every simple $F\mathfrak{B}_n$-module. Thus if Q is a vertex of a simple $F\mathfrak{B}_n$-module then $H \leqslant Q$. In particular, we have $|Q| \geqslant 2^n$, hence $n \leqslant \log_2(|Q|)$. Therefore, if P is a Sylow 2-subgroup of $\mathfrak{B}_n$ then

$$|P| \leqslant |\mathfrak{B}_n| = 2^n \cdot n! \leqslant |Q| \cdot \log_2\bigl(|Q|\bigr)!.$$

(b) To deal with $\mathfrak{D}_n$, note that $\mathfrak{D}_n = \mathfrak{B}_n \cap \mathfrak{A}_{2n} = (H \cap \mathfrak{A}_{2n})\mathfrak{S}_n^*$. An argument analogous to the one used in Part (a) above shows that also $F\mathfrak{D}_n$ has only the principal block and, whenever Q is a vertex of a simple $F\mathfrak{D}_n$-module, $H \cap \mathfrak{A}_{2n} \leqslant Q$. In particular, we have $|Q| \geqslant 2^{n-1}$, hence $n \leqslant \log_2(|Q|) + 1$. Therefore, if P is a Sylow 2-subgroup of $\mathfrak{D}_n$ then we deduce

$$|P| \leqslant |\mathfrak{D}_n| = 2^{n-1} \cdot n! \leqslant |Q| \cdot \bigl(\log_2\bigl(|Q|\bigr) + 1\bigr)!.$$

6.4. *The case* $p \geqslant 3$. Let now $p \geqslant 3$, for the remainder of this section. We briefly recall the well-known structure of the defect groups of the blocks of $F\mathfrak{B}_n$. By the Theorem of Fong–Reynolds [25, Thm. 5.5.10] applied to the base group $C_2^n \cong H \trianglelefteq \mathfrak{B}_n$, see also [28] or [18, Ch. 4], the blocks of $F\mathfrak{B}_n$ are parametrized by pairs (κ, w), with $\kappa = (\kappa_0, \kappa_1)$ and $w = (w_0, w_1)$, and where, for $i = 0, 1$, the partition κ_i is the p-core of some partition of $n_i := |\kappa_i| + pw_i$ such that $n = n_0 + n_1$. Moreover, the inertial group of the block $B(\kappa, w)$ is given as $T_{\mathfrak{B}_n}(B(\kappa, w)) := (C_2 \wr \mathfrak{S}_{n_0}) \times (C_2 \wr \mathfrak{S}_{n_1}) = C_2 \wr (\mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1}) \leqslant \mathfrak{B}_n$.

Note that every p-subgroup of $\mathfrak{B}_n$ is conjugate to a subgroup of $\mathfrak{S}_n^*$. If P is a defect group of the block $B(\kappa, w)$ then P is conjugate to a Sylow p-subgroup $P_{pw_0}^* \times P_{pw_1}^*$ of $\mathfrak{S}_{pw_0}^* \times \mathfrak{S}_{pw_1}^* \leqslant \mathfrak{S}_{n_0}^* \times \mathfrak{S}_{n_1}^* \leqslant \mathfrak{S}_n^*$. Note that $\varphi(P_{pw_i}^*) = P_{pw_i} \leqslant \mathfrak{S}_{pw_i} \leqslant \mathfrak{S}_{n_i}$, for $i = 0, 1$, is a defect group of the block of $\mathfrak{S}_{n_i}$ parametrized by κ_i.

Proposition 6.5. *Let* $p \geqslant 3$*, let* D *be a simple* $F\mathfrak{B}_n$*-module belonging to a block with defect group* P*, and let* Q *be a vertex of* D*. Then we have*

$$|P| \leqslant |Q|!.$$

Proof. Let $B(\kappa, w)$ be the block in question, and $n_i := |\kappa_i| + pw_i$, for $i = 0, 1$. Hence we may assume that $Q \leqslant P = P_{pw_0}^* \times P_{pw_1}^* \leqslant \mathfrak{S}_{n_0}^* \times \mathfrak{S}_{n_1}^* \leqslant \mathfrak{S}_n^*$. Let $T := T_{\mathfrak{B}_n}(B(\kappa, w)) = (C_2 \wr \mathfrak{S}_{n_0}) \times (C_2 \wr \mathfrak{S}_{n_1}) \leqslant \mathfrak{B}_n$ be the inertial group associated with $B(\kappa, w)$. Then the simple $F\mathfrak{B}_n$-modules belonging to $B(\kappa, w)$ can be described as follows (see [18, Sect. 4.3], and [28]):

Let F be the trivial FC_2-module, and let E be the non-trivial simple FC_2-module. Then the outer tensor product $F^{\otimes n_0} \otimes_F E^{\otimes n_1}$ naturally becomes an FT-module. Letting D_i, for $i = 0, 1$, be a simple $F\mathfrak{S}_{n_i}$-module in the block parametrized by κ_i, the tensor product $D_0 \otimes_F D_1$ becomes an

$F[\mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1}]$-module with respect to the outer-tensor-product action. Inflating with respect to the base group $C_2^{n_0} \times C_2^{n_1} \trianglelefteq T$ yields the simple FT-module $\mathrm{Inf}_{C_2^{n_0} \times C_2^{n_1}}^{T}(D_0 \otimes_F D_1)$. Then inducing the ordinary tensor product

$$M := \left(F^{\otimes n_0} \otimes_F E^{\otimes n_1}\right) \otimes_F \mathrm{Inf}_{C_2^{n_0} \times C_2^{n_1}}^{T}(D_0 \otimes_F D_1),$$

which is a simple FT-module, to $\mathfrak{B}_n$ we get a simple $F\mathfrak{B}_n$-module $\mathrm{Ind}_T^{\mathfrak{B}_n}(M)$ belonging to $B(\kappa, w)$. Conversely, every simple $F\mathfrak{B}_n$-module belonging to the block $B(\kappa, w)$ arises in such a way.

Hence, $D \cong \mathrm{Ind}_T^{\mathfrak{B}_n}(M)$, for suitably chosen D_0 and D_1. Let $Q_i \leqslant \mathfrak{S}_{n_i}$ be a vertex of D_i, for $i = 0, 1$, where we may assume that $Q_i \leqslant P_{pw_i}$. Hence $Q_0 \times Q_1 \leqslant \mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1}$ is a vertex of the outer tensor product $D_0 \otimes_F D_1$. Since F and E are projective FC_2-modules, letting $Q_i^* := \varphi^{-1}(Q_i)$, it follows from [23] that $Q_0^* \times Q_1^* \leqslant \mathfrak{S}_{n_0}^* \times \mathfrak{S}_{n_1}^* \leqslant \mathfrak{S}_n^* \leqslant \mathfrak{B}_n$ is a vertex of $\mathrm{Ind}_T^{\mathfrak{B}_n}(M) \cong D$.

By [6] we have $pw_i \leqslant |Q_i|$, and hence

$$|P| = \left|P_{pw_0}^*\right| \cdot \left|P_{pw_1}^*\right| \leqslant (pw_0)! \cdot (pw_1)! \leqslant |Q_0|! \cdot |Q_1|! \leqslant \left(|Q_0| \cdot |Q_1|\right)! = |Q|!. \qquad \square$$

7. Semidirect products

As was pointed out by the referee, Feit's conjecture can be proven for general semidirect products with abelian kernel, provided it holds for the complements occurring and all their subgroups. We proceed to state and prove this.

7.1. *Simple modules of semidirect products.* Suppose that G is a semidirect product $G = H \rtimes_\alpha U$ of an abelian group H with a group U, with respect to a group homomorphism $\alpha : U \longrightarrow \mathrm{Aut}(H)$. We recall the well-known construction of the simple FG-modules from those of subgroups of G, which is a consequence of Clifford's Theorem; for a proof we refer to [4, Thm. 11.1, Thm. 11.20, Exc. 11.13].

Suppose that E is a simple FH-module, which is, in particular, one-dimensional, since H is abelian. Let further $T_G(E)$ be the inertial group of E in G; thus $T_G(E) = H \rtimes_\alpha (U \cap T_G(E))$, where we denote the restriction of α to $U \cap T_G(E)$ by α again. Then we can extend E to a simple $FT_G(E)$-module, which we denote by E again, by letting

$$(hu) \cdot x := hx \quad \left(h \in H,\ u \in U \cap T_G(E),\ x \in E\right);$$

to see that this indeed yields an $FT_G(E)$-module just note that we have

$${}^{u}h \cdot x = h \cdot x \quad \left(h \in H,\ u \in U \cap T_G(E),\ x \in E\right).$$

Set $T_{U,\alpha}(E) := U \cap T_G(E)$, hence we have $T_G(E)/H \cong T_{U,\alpha}(E)$, and let E' be a simple $FT_{U,\alpha}(E)$-module. Then the inflation $\mathrm{Inf}_H^{T_G(E)}(E')$ is a simple $FT_G(E)$-module, as is the tensor product $E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E')$. Moreover, the induction $D(E, E') := \mathrm{Ind}_{T_G(E)}^{G}(E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E'))$ is a simple FG-module.

Now, let $\mathcal{E}$ be a transversal for the isomorphism classes of simple FH-modules. Then, as E varies over $\mathcal{E}$, and E' varies over a transversal for the isomorphism classes of simple $FT_{U,\alpha}(E)$-modules, $D(E, E')$ varies over a transversal for the isomorphism classes of simple FG-modules.

With the above notation, we have the following

Theorem 7.2. *Let H be an abelian group, let $\mathcal{U}$ be a set of groups, and let*

$$H \rtimes \mathcal{U} := \left\{H \rtimes_\alpha U \,\middle|\, U \in \mathcal{U},\ \alpha : U \longrightarrow \mathrm{Aut}(H)\right\}.$$

Suppose that Feit's Conjecture holds for

$$\mathcal{T}(H \rtimes \mathcal{U}) := \left\{T_{U,\alpha}(E) \,\middle|\, U \in \mathcal{U},\ \alpha : U \longrightarrow \mathrm{Aut}(H),\ E \in \mathcal{E}\right\}.$$

Then Feit's Conjecture holds for $H \rtimes \mathcal{U}$ as well.

Proof. Let $G = H \rtimes_\alpha U$ be a group in $H \rtimes \mathcal{U}$, and let D be a simple FG-module. As we have just seen in 7.1, there are a simple FH-module E and a simple $FT_{U,\alpha}(E)$-module E' such that

$$D \cong \mathrm{Ind}_{T_G(E)}^{G}\bigl(E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E')\bigr).$$

Since $\dim_F(E) = 1$, it is a trivial-source module, and moreover tensoring with E is a vertex- and source-preserving auto-equivalence of the module category of $T_G(E)$. Hence every vertex of $\mathrm{Inf}_H^{T_G(E)}(E')$ is also vertex of $E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E')$, and every source of $\mathrm{Inf}_H^{T_G(E)}(E')$ is also a source of $E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E')$. Moreover, $\mathrm{Ind}_{T_G(E)}^{G}(E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E'))$ has some indecomposable direct summand that has a vertex and an associated source in common with $E \otimes_F \mathrm{Inf}_H^{T_G(E)}(E')$. Thus, since D is, in particular, indecomposable, the vertex-source pairs of $\mathrm{Inf}_H^{T_G(E)}(E')$ and those of D coincide.

So, suppose that Q is a vertex of $\mathrm{Inf}_H^{T_G(E)}(E')$, and L is a Q-source. Then, by [23, Prop. 2.1] and [16, Prop. 2], we deduce that QH/H is a vertex of E' and that $L = \mathrm{Res}_Q^{QH}(\mathrm{Inf}_H^{QH}(\bar{L}))$, for some QH/H-source $\bar{L}$ of E'.

Consequently, given any p-group Q, the above arguments imply

$$\bigl|\mathcal{V}_{H \rtimes \mathcal{U}}(Q)\bigr| \leqslant \bigl|\mathcal{V}_{\mathcal{T}(H \rtimes \mathcal{U})}(QH/H)\bigr|.$$

Since the latter cardinality is finite by our hypothesis, the assertion of the theorem follows. $\square$

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