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The symmetric and unimodal expansion of Eulerian polynomials via continued fractions

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ABSTRACT

This paper was motivated by a conjecture of Brändén [P. Brändén, Actions on permutations and unimodality of descent polynomials, European J. Combin. 29 (2) (2008) 514–531] about the divisibility of the coefficients in an expansion of generalized Eulerian polynomials, which implies the symmetric and unimodal property of the Eulerian numbers. We show that such a formula with the conjectured property can be derived from the combinatorial theory of continued fractions. We also discuss an analogous expansion for the corresponding formula for derangements and prove a (p, q)-analogue of the fact that the (-1)-evaluation of the enumerator polynomials of permutations (resp. derangements) by the number of excedances gives rise to tangent numbers (resp. secant numbers). The (p, q)-analogue unifies and generalizes our recent results [H. Shin, J. Zeng, The q-tangent and q-secant numbers via continued fractions, European J. Combin. 31 (7) (2010) 1689–1705] and that of Josuat-Vergès [M. Josuat-Vergés, A q-enumeration of alternating permutations, European J. Combin. 31 (7) (2010) 1892–1906].

1. Introduction

The Eulerian polynomials $A_n(t)$ can be defined by

$$\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{1-t}{e^{(t-1)x}-t}.$$

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Let $\mathfrak{S}_n$ be the set of permutations on $[n] = \{1, \ldots, n\}$. For $\sigma = \sigma(1) \ldots \sigma(n) \in \mathfrak{S}_n$, the entry $i \in [n]$ is called an *excedance* (position) of σ if $i < \sigma(i)$ and the number of *excedances* of σ is denoted by $\operatorname{exc}\sigma$. It is a classical result (cf. [9]) that the Eulerian polynomials have the following combinatorial interpretation

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{exc}\sigma} = A(n,0) + A(n,1)t + \cdots + A(n,n-1)t^{n-1},$$

with $A(n,k) = \#\{\sigma \in \mathfrak{S}_n : \operatorname{exc}\sigma = k\}$ and the expansion

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-1-2k}, \tag{1}$$

where $a_{n,k}$ are nonnegative integers with known combinatorial interpretations. Recall that a sequence of real numbers $a_0, a_1, \ldots, a_d$ is said to be symmetric if $a_i = a_{d-i}$ for $i = 0, \ldots \lfloor d/2 \rfloor$ and said to be unimodal if there exists an index $1 \le j \le d$ such that $a_0 \le a_1 \le \cdots \le a_{j-1} \le a_j \ge a_{j+1} \ge \cdots \ge a_d$. Note that the expansion (1) enables to derive immediately the *symmetry* and *unimodality* of the Eulerian numbers $\{A(n,k)\}_{0 \le k \le n-1}$. Foata and Strehl [10] studied the combinatorial aspect of the expansion (1) via a group acting on the symmetric groups. In 2008, generalizing Foata and Strehl's action, Brändén [4] gave a (p,q)-refinement of (1) and conjectured that the corresponding polynomial coefficient $a_{n,k}(p,q)$ has a factor $(p+q)^k$ for all $0 \le k \le \lfloor (n-1)/2 \rfloor$ (see [4, Conjecture 10.3]). In this paper we shall give a new approach to such an expansion with a proof of his conjecture as a bonus (cf. Theorem 2).

Next we shall study the derangement counterpart of (1). Recall that a permutation $\sigma \in \mathfrak{S}_n$ is a *derangement* if it has no fixed points, i.e., $\sigma(i) \neq i$ for all $i \in [n]$. Let $\mathfrak{D}_n$ be the set of *derangements* in $\mathfrak{S}_n$. The derangement analogue of the Eulerian polynomials (see [9,5]) is defined by

$$B_n(t) = \sum_{\sigma \in \mathfrak{D}_n} t^{\operatorname{exc}\sigma} = B(n,1)t + B(n,2)t^2 + \cdots + B(n,n-1)t^{n-1},$$

with $B(n,k) = \#\{\sigma \in \mathfrak{D}_n : \operatorname{exc}\sigma = k\}$. The generating function for $B_n(x)$ reads as follows (see [9,5,15]):

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{1-t}{e^{tx} - te^x}.$$

By analytical method, one can show (cf. [3,18]) that there are non negative integers $b_{n,k} \in \mathbb{Z}$ such that

$$B_n(t) = \sum_{k=1}^{\lfloor n/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k}. \tag{2}$$

However, no combinatorial interpretation for $b_{n,k}$ seems to be known hitherto. A special case of our results (cf. Corollary 9 and Theorem 11) will provide a combinatorial interpretation for the coefficients $b_{n,k}$.

Another interesting feature of (1) and (2) is that they generalize the well-known relation between the (-1)-evaluation of $A_n(t)$ and $B_n(t)$ and the Euler numbers E_n (see [9]):

$$\begin{aligned} &A_{2n}(-1) = 0, \qquad A_{2n+1}(-1) = (-1)^n E_{2n+1}, \\ &B_{2n}(-1) = (-1)^n E_{2n}, \qquad B_{2n+1}(-1) = 0, \end{aligned} \tag{3}$$

where the Euler numbers $\{E_n\}_{n \ge 0}$ are defined by $\sum_{n \ge 0} E_n \frac{x^n}{n!} = \tan x + \sec x$. It follows that $a_{2n+1,n} = E_{2n+1}$ and $b_{2n,n} = E_{2n}$. We will prove a (p,q)-analogue of the formulas (3) (cf. Theorem 10), which unifies the recent results in [14,16].

Our main tool is the combinatorial theory of continued fractions due to Flajolet [8] and bijections due to Françon–Viennot, Foata–Zeilberger and Biane, see [12,11,8,2,6,16] between permutations and Motzkin paths.

2. Definitions and main results

Given a permutation $\sigma \in \mathfrak{S}_n$, the entry $i \in [n]$ is called a *descent* of σ if $i < n$ and $\sigma(i) > \sigma(i+1)$. Also the entry $i \in [n]$ is called a *weak excedance* (resp. *drop*) of σ if $i \le \sigma(i)$ (resp. $i > \sigma(i)$). Denote the number of descents (resp. weak excedances, drops) in σ by $\operatorname{des}\sigma$ (resp. $\operatorname{wex}\sigma$, $\operatorname{drop}\sigma$). It is well known [9] that the statistics des and exc have the same distribution on $\mathfrak{S}_n$. Since $\operatorname{drop}\sigma = \operatorname{exc}\sigma^{-1}$ for any $\sigma \in \mathfrak{S}_n$, we see that the Eulerian polynomial has the following two interpretations

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{des}\sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\operatorname{drop}\sigma}. \tag{4}$$

The two above interpretations of Eulerian polynomials give rise then two possible extensions: one uses linear statistics and the other one uses cyclic statistics. We need some more definitions.

Definition 1. For $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = \sigma(n+1) = 0$. Then any entry $\sigma(i)$ ($i \in [n]$) can be classified according to one of the four cases:

- a *peak* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;
- a *valley* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double ascent* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double descent* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) > \sigma(i+1)$.

Let $\operatorname{peak}^*\sigma$ (resp. $\operatorname{valley}^*\sigma$, $\operatorname{da}^*\sigma$, $\operatorname{dd}^*\sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in σ. Clearly we have $\operatorname{peak}^*\sigma = \operatorname{valley}^*\sigma + 1$.

For $\sigma \in \mathfrak{S}_n$, the statistic $(31\text{-}2)\sigma$ (resp. $(13\text{-}2)\sigma$) is the number of pairs (i, j) such that $2 \le i < j \le n$ and $\sigma(i-1) > \sigma(j) > \sigma(i)$ (resp. $\sigma(i-1) < \sigma(j) < \sigma(i)$). Similarly, the statistic $(2\text{-}13)\sigma$ (resp. $(2\text{-}31)\sigma$) is the number of pairs (i, j) such that $1 \le i < j \le n-1$ and $\sigma(j+1) > \sigma(i) > \sigma(j)$ (resp. $\sigma(j+1) < \sigma(i) < \sigma(j)$). Introduce the generalized Eulerian polynomial defined by

$$A_n(p, q, t, u, v, w) := \sum_{\sigma \in \mathfrak{S}_n} p^{(2\text{-}13)\sigma} q^{(31\text{-}2)\sigma} t^{\operatorname{des}\sigma} u^{\operatorname{da}^*\sigma} v^{\operatorname{dd}^*\sigma} w^{\operatorname{valley}^*\sigma}. \tag{5}$$

Let $\mathfrak{S}_{n,k}$ be the subset of permutations $\sigma \in \mathfrak{S}_n$ with exactly k valleys and without double descents. Define the polynomial

$$a_{n,k}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k}} p^{(2\text{-}13)\sigma} q^{(31\text{-}2)\sigma}. \tag{6}$$

Note that the coefficient $a_{2n+1,n}(p, q)$ is the (p, q)-analogue of tangent number, i.e., $E_{2n+1}(p, q)$ in [16, Eq. (14)].

Theorem 2. *We have the expansion formula*

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)(tw)^k (u + vt)^{n-1-2k}. \tag{7}$$

Moreover, for all $0 \le k \le \lfloor (n-1)/2 \rfloor$*, the following divisibility holds*

$$(p + q)^k \mid a_{n,k}(p, q). \tag{8}$$

Remark. Brändén [4] used the convention $\sigma(0) = \sigma(n+1) = n+1$ for the definition of *peak* (resp. *valleys*, *double ascent*, *double descent*). Let peak_B, valley_B, da_B, dd_B be the corresponding statistics. To see how the above theorem implies his conjecture we define the *reverse* and *complement* transformations on the permutations $\sigma \in \mathfrak{S}_n$ by

$$\sigma \mapsto \sigma^r = \sigma(n)\sigma(n-1)\ldots\sigma(1),$$
$$\sigma \mapsto \sigma^c = (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n)).$$

Clearly the reverse-complement transformation $\sigma \mapsto \sigma^{rc} := (\sigma^r)^c$ satisfies

$$\begin{aligned}&(\mathrm{des}, \mathrm{peak}^*, \mathrm{valley}^*, \mathrm{da}^*, \mathrm{dd}^*, (2\text{-}13), (31\text{-}2))\sigma \\ &\quad = (\mathrm{des}, \mathrm{valley}_B, \mathrm{peak}_B, \mathrm{da}_B, \mathrm{dd}_B, (13\text{-}2), (2\text{-}31))\sigma^{rc}.\end{aligned} \tag{9}$$

Thus, setting $t = u = v = 1$ in Theorem 2, we have

$$\sum_{\sigma\in\mathfrak{S}_n} p^{(13\text{-}2)\sigma} q^{(2\text{-}31)\sigma} w^{\mathrm{peak}_B\,\sigma} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} a_{n,k}(p,q) 2^{n-1-2k} w^k. \tag{10}$$

Comparing two coefficients of w^k in both sides of (10), we recover Brändén's result [4, (5.1)]:

$$a_{n,k}(p,q) = 2^{-n+1+2k} \sum_{\substack{\pi\in\mathfrak{S}_n \\ \mathrm{peak}_B\,\pi = k}} p^{(13\text{-}2)\pi} q^{(2\text{-}31)\pi}. \tag{11}$$

Moreover (8) confirms his conjecture [4, Conjecture 10.3].

Definition 3. For $\sigma \in \mathfrak{S}_n$, a value $x = \sigma(i)$ $(i \in [n])$ is called

- a *cyclic peak* if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;
- a *cyclic valley* if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$;
- a *double excedance* if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$;
- a *double drop* if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;
- a *fixed point* if $x = \sigma(x)$.

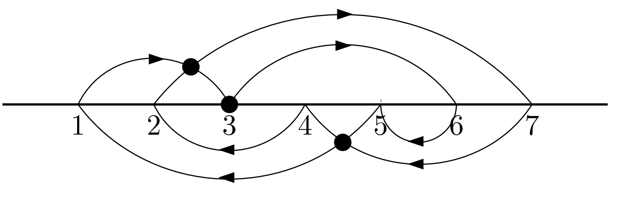
Let *cpeak*σ (resp. *cvalley*σ, cdaσ, cddσ, fixσ) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in σ.

For a permutation $\sigma \in \mathfrak{S}_n$ the crossing and nesting numbers are defined by

$$\mathrm{cros}\sigma = \#\{(i,j) \in [n]\times[n] : (i < j \le \sigma(i) < \sigma(j)) \vee (i > j > \sigma(i) > \sigma(j))\}, \tag{12}$$

$$\mathrm{nest}\sigma = \#\{(i,j) \in [n]\times[n] : (i < j \le \sigma(j) < \sigma(i)) \vee (i > j > \sigma(j) > \sigma(i))\}. \tag{13}$$

As in [7], we can illustrate these statistics by a permutation diagram. Given $\sigma \in \mathfrak{S}_n$, we put the numbers from 1 to n on a line and draw an edge from i to $\sigma(i)$ above the line if i is a weak excedance, and below the line otherwise. For example, the permutation diagram of $\sigma = 3\,762\,154 = \begin{pmatrix}1&2&3&4&5&6&7\\3&7&6&2&1&5&4\end{pmatrix}$ is as follows:



$\sigma = 3762154$

Clearly, there are three crossings $(1 < 2 < \sigma(1) < \sigma(2))$, $(1 < 3 \le \sigma(1) < \sigma(3))$, $(7 > 4 > \sigma(7) > \sigma(4))$ and three nestings $(2 < 3 < \sigma(3) < \sigma(2))$, $(5 > 4 > \sigma(4) > \sigma(5))$, $(7 > 6 > \sigma(6) > \sigma(7))$, thus cros$\sigma$ = nest$\sigma = 3$.

Definition 4. Given a permutation $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = 0$ and $\sigma(n+1) = n+1$. The corresponding number of peaks, valleys, double ascents, and double descents of permutation $\sigma \in \mathfrak{S}_n$ is denoted by peak σ, valley σ, da σ, and dd σ, respectively. Moreover, a double ascent $\sigma(i)$ of σ $(i \in [n])$ is said to be a *foremaximum* if $\sigma(i)$ is a *left-to-right maximum* of σ, i.e., $\sigma(j) < \sigma(i)$ for all $1 \le j < i$. Denote the number of foremaxima of σ by fmaxσ.

For instance, $\mathrm{da}(42\,157\,368) = 3$, but $\mathrm{da}^*(42\,157\,368) = 2$ and $\mathrm{fmax}(42\,157\,368) = 2$. Note that by the above definition we have $\mathrm{peak}\,\sigma = \mathrm{valley}\,\sigma$ for any $\sigma \in \mathfrak{S}_n$.

Theorem 5. *There is a bijection Φ on $\mathfrak{S}_n$ such that for all $\sigma \in \mathfrak{S}_n$ we have*

$$(\mathrm{nest}, \mathrm{cros}, \mathrm{drop}, \mathrm{cda}, \mathrm{cdd}, cvalley, \mathrm{fix})\sigma = (\text{2-31}, \text{31-2}, \mathrm{des}, \mathrm{da} - \mathrm{fmax}, \mathrm{dd}, \mathrm{valley}, \mathrm{fmax})\Phi(\sigma).$$

Since $\mathrm{wex} = n - \mathrm{drop}$, we derive immediately from Theorems 2 and 5 the following (p, q)-analogue of (1).

Corollary 6. *We have*

$$\sum_{\sigma \in \mathfrak{S}_n} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} t^{\mathrm{wex}\,\sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q) t^{k+1} (1+t)^{n-1-2k}. \tag{14}$$

Consider the common enumerator polynomial

$$B_n(p, q, t, u, v, w, y) := \sum_{\sigma \in \mathfrak{S}_n} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} t^{\mathrm{drop}\,\sigma} u^{\mathrm{cda}\sigma} v^{\mathrm{cdd}\sigma} w^{cvalley\sigma} y^{\mathrm{fix}\sigma} \tag{15}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} p^{(\text{2-31})\sigma} q^{(\text{31-2})\sigma} t^{\mathrm{des}\,\sigma} u^{\mathrm{da}\,\sigma - \mathrm{fmax}\sigma} v^{\mathrm{dd}\,\sigma} w^{\mathrm{valley}\,\sigma} y^{\mathrm{fmax}\sigma}. \tag{16}$$

When $u = 0$ and $t = v = 1$ we can write

$$B_n(p, q, 1, 0, 1, w, y) := \sum_{k,j \geq 0} b_{n,k,j}(p, q) w^k y^d, \tag{17}$$

where $b_{n,k,j}(p, q)$ is a polynomial in p and q with non negative integral coefficients. In order to give a combinatorial interpretation for $b_{n,k,j}(p, q)$ we introduce some more definitions. Let $\mathfrak{S}_{n,k,j}$ denote the subset of all the permutations $\sigma \in \mathfrak{S}_n$ with exactly k cyclic valleys, d fixed points, and without double excedance, and let $\mathfrak{S}^*_{n,k,j}$ denote the subset of all permutations $\sigma \in \mathfrak{S}_n$ with exactly k valleys and j double ascents, which are all foremaxima. We derive from Theorem 5 the following result.

Corollary 7. *We have*

$$b_{n,k,d}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k,j}} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} = \sum_{\sigma \in \mathfrak{S}^*_{n,k,j}} p^{(\text{2-31})\sigma} q^{(\text{31-2})\sigma}. \tag{18}$$

In particular, when $j = 0$, we obtain

$$b_{n,k,0}(p, q) = \sum_{\sigma \in \mathfrak{D}_{n,k,0}} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} = \sum_{\sigma \in \mathfrak{D}^*_{n,k,0}} p^{(\text{2-31})\sigma} q^{(\text{31-2})\sigma}. \tag{19}$$

Recall (see [16]) that a *coderangement* is a permutation without foremaximum. Let $\mathfrak{D}^*_n$ be the subset of $\mathfrak{S}_n$ consisting of coderangements, that is, $\mathfrak{D}^*_n = \{\sigma \in \mathfrak{S}_n : \mathrm{fmax}\sigma = 0\}$. For example, we have $\mathfrak{D}^*_4 = \{2143, 3142, 3241, 4123, 4132, 4213, 4231, 4312, 4321\}$. Thus, $\mathfrak{D}_{n,k,0}$ is the subset of derangements $\sigma \in \mathfrak{D}_n$ with exactly k cyclic valleys, and without double excedance, and $\mathfrak{D}^*_{n,k,0}$ is the subset of coderangements $\sigma \in \mathfrak{D}^*_n$ with exactly k valleys and without double ascents. The following is our main result about the polynomial $B_n(p, q, t, u, v, w, y)$.

Theorem 8. *We have the following decomposition*

$$B_n(p, q, t, u, v, w, y) = \sum_{j=0}^{n} y^j \sum_{k=0}^{\lfloor (n-j)/2 \rfloor} b_{n,k,j}(p, q)(tw)^k (qu + tv)^{n-j-2k}. \tag{20}$$

For any $\sigma \in \mathfrak{D}_n$, since $\mathrm{exc}\,\sigma = n - \mathrm{drop}\,\sigma$, setting $y = 0$ in Theorem 8 we obtain immediately the following (p, q)-analogue of (2).

Corollary 9. *We have*

$$\sum_{\sigma\in\mathfrak{D}_n} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} t^{\mathrm{exc}\,\sigma} = \sum_{k=0}^{\lfloor n/2\rfloor} b_{n,k,0}(p,q) t^k (1+qt)^{n-2k}. \tag{21}$$

Note that the coefficient $b_{2n,n,0}(p,q)$ is the (p,q)-analogue of *secant number*, namely, $E_{2n}(p,q)$ in [16, Eq. (15)]. A permutation $\sigma \in \mathfrak{S}_n$ is said to be a *(falling) alternating permutation* if $\sigma(1) > \sigma(2), \sigma(2) < \sigma(3), \sigma(3) > \sigma(4)$, etc. Let $\mathfrak{A}_n$ be the set of (falling) alternating permutations on $[n]$. It is a folklore result that the cardinality of $\mathfrak{A}_n$ is the Euler number E_n. We derive from Corollaries 6, 7 and 9 the following (p,q)-analogue of (3).

Theorem 10. *For $n \geq 1$, we have*

$$\sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{wex}\,\sigma} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} = \begin{cases} 0 & \text{if } n \text{ is even},\\ (-1)^{\frac{n+1}{2}} \sum_{\sigma\in\mathfrak{A}_n} p^{(2\text{-}13)\sigma} q^{(31\text{-}2)\sigma} & \text{if } n \text{ is odd}; \end{cases}$$

and

$$\sum_{\sigma\in\mathfrak{D}_n} (-1/q)^{\mathrm{exc}\,\sigma} p^{\mathrm{nest}\sigma} q^{\mathrm{cros}\sigma} = \begin{cases} (-1/q)^{\frac{n}{2}} \sum_{\sigma\in\mathfrak{A}_n} p^{(2\text{-}31)\sigma} q^{(31\text{-}2)\sigma} & \text{if } n \text{ is even},\\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

Remark. The $p = 1$ case of Theorem 10 was proved by Josuat-Vergès [14, Eqs. (5)–(6)]. Since the *inversion* number of any $\sigma \in \mathfrak{S}_n$ can be written as $\mathrm{drop}\,\sigma + \mathrm{cros}\sigma + 2\mathrm{nest}\sigma$ (cf. [16]), the $p = q^2$ case of Theorem 10 was used by Shin and Zeng [16, Eqs. (12)–(13)] to derive a q-analogue of (3) using the *inversion number* q-analogue of Eulerian polynomials.

For any permutation $\sigma \in \mathfrak{S}_n$, we denote by *cyc*$\sigma$ the number of its cycles. Define

$$c_{n,k}(\beta) = \sum_{\sigma\in\mathfrak{D}_n(k)} \beta^{\mathrm{cyc}\sigma}, \tag{22}$$

where $\mathfrak{D}_n(k)$ is the subset of derangements in $\mathfrak{D}_n$ with exactly k cyclic valleys and without cyclic double descents. Clearly $c_{n,0}(\beta) = 0$ for all $n \geq 1$. The following result gives another generalization of the expansion (2).

Theorem 11. *We have*

$$\sum_{\sigma\in\mathfrak{D}_n} \beta^{\mathrm{cyc}\sigma} t^{\mathrm{exc}\,\sigma} = \sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(\beta) t^k (1+t)^{n-2k}. \tag{23}$$

The rest of this paper is organized as follows. In Sections 3–6 we shall prove Theorems 2, 5, 8 and 11, respectively. In Section 7 we give two variations of the unimodal and symmetric expansion of Eulerian polynomials and one for its derangement analogue. Finally we conclude with an open problem related to the descent polynomial of involutions.

3. Proof of Theorem 2

A Motzkin path of length n is a sequence of points $(s_0, \dots, s_n)$ in the plan $\mathbb{N} \times \mathbb{N}$ such that $s_0 = (0,0)$, $s_i - s_{i-1} = (1,0), (1,\pm 1)$ and $s_n = (n,0)$. Denote by $\mathfrak{M}_n$ the set of Motzkin paths of length $n \geq 1$. We shall call a step (s_{i-1}, s_i) East, North-East, South-East, respectively, if $s_i - s_{i-1} = (1,0)$, $s_i - s_{i-1} = (1,1)$, $s_i - s_{i-1} = (1,-1)$. The *height* of the step (s_{i-1}, s_i) is the ordinate of s_{i-1}.

Given a Motzkin path γ, if we weight each East (resp. North-East, South-East) step of height i by a_i (resp. b_i and c_i) and define the weight of γ by the product of its step weights, denoted by $w(\gamma)$, then

$$1+\sum_{n\geq 1}\sum_{\gamma\in\mathfrak{M}_n} w(\gamma)x^n = \cfrac{1}{1-b_0x-\cfrac{a_0c_1x^2}{1-b_1x-\cfrac{a_1c_2x^2}{\cdots}}}. \tag{24}$$

It is convenient to use two kinds of horizontal steps, say, *blue* and *red*. Let $\mathfrak{M}'_n$ be the set of *colored Motzkin paths* of length n. A *Laguerre history* of length n is a couple $(\gamma,(p_1,\ldots,p_n))$, where γ is a colored Motzkin path of length n and $(p_1,\ldots,p_n)$ is a sequence such that $0\leq p_i\leq v(s_{i-1},s_i)$, where $v(s_{i-1},s_i)=k$ if $s_{i-1}=(i-1,k)$. Denote by $\mathfrak{H}_n$ the set of Laguerre histories of length n.

Let $\sigma\in\mathfrak{S}_n$, the refinements of three generalized patterns are defined by

$$\begin{aligned}
(31\text{-}2)_k\sigma &= \#\left\{i : i+1<j \text{ and } \sigma(i+1)<\sigma(j)=k<\sigma(i)\right\},\\
(2\text{-}31)_k\sigma &= \#\left\{i : j<i-1 \text{ and } \sigma(i)<\sigma(j)=k<\sigma(i-1)\right\},\\
(2\text{-}13)_k\sigma &= \#\left\{i : j<i-1 \text{ and } \sigma(i-1)<\sigma(j)=k<\sigma(i)\right\}.
\end{aligned}$$

We clearly have $(31\text{-}2)\sigma=\sum_{k=1}^n(31\text{-}2)_k\sigma$, $(2\text{-}31)\sigma=\sum_{k=1}^n(2\text{-}31)_k\sigma$, and $(2\text{-}13)\sigma=\sum_{k=1}^n(2\text{-}13)_k\sigma$. Two numbers $l_k=(31\text{-}2)_k\sigma$ and $r_k=(2\text{-}31)_k\sigma$ are called the *left embracing numbers* and *right embracing numbers* of $k\in[n]$ in σ.

We need a modified version of Françon–Viennot's bijection $\Psi_{FV}:\mathfrak{S}_n\to\mathfrak{H}_{n-1}$ defined by the following: for $\sigma\in\mathfrak{S}_n$ we construct the Laguerre history $(s_0,\ldots,s_{n-1},p_1,\ldots,p_{n-1})$, where $s_0=(0,0)$ and the step (s_{i-1},s_i) is North-East, South-East, East blue and East red if i is a valley, peak, double ascent, or double descent, respectively; while $p_i=(2\text{-}13)_i\sigma$ for $i=1,\ldots,n-1$. If h_i is the height of (s_{i-1},s_i), i.e., $s_{i-1}=(i-1,h_i)$, then $(2\text{-}13)_i\sigma+(31\text{-}2)_i\sigma=h_i$. Since $\sigma(0)=\sigma(n+1)=0$, so n must be a peak and valley $\sigma=$ peak $\sigma-1$. Thus $(s_0,\ldots,s_{n-1},p_1,\ldots,p_{n-1})$ is a Laguerre history of length $n-1$ and

$$w(\sigma)=t^{\mathrm{ER}\gamma+\mathrm{NE}\gamma}u^{\mathrm{EB}\gamma}v^{\mathrm{ER}\gamma}w^{\mathrm{NE}\gamma}\prod_{i=1}^{n-1}p^{p_i}q^{h_i-p_i},$$

where $\mathrm{NE}\gamma$, $\mathrm{EB}\gamma$, and $\mathrm{ER}\gamma$ are the number of North-East steps, East blue steps, and East red steps of γ. Therefore,

$$A_n(p,q,t,u,v,w)=\sum_{\gamma\in\mathfrak{M}'_{n-1}}t^{\mathrm{ER}\gamma+\mathrm{NE}\gamma}u^{\mathrm{EB}\gamma}v^{\mathrm{ER}\gamma}w^{\mathrm{NE}\gamma}\prod_{i=1}^{n-1}[h_i+1]_{p,q}, \tag{25}$$

where $[n]_{p,q}=(p^n-q^n)/(p-q)$. Given a Motzkin path γ, weight each step at height k by

$$a_k:=tw[k+1]_{p,q},\qquad b_k:=(u+tv)[k+1]_{p,q},\qquad c_k:=[k+1]_{p,q} \tag{26}$$

if the step is North-East, East, and South-East, respectively, and the weight of γ is defined to be the product of the step weights. Then the last sum amounts to sum over all the Motzkin paths of length $n-1$ with respect to (26), i.e.,

$$A_n(p,q,t,u,v,w)=\sum_{\gamma\in\mathfrak{M}_{n-1}}w(\gamma). \tag{27}$$

It follows from (24) that the Eq. (27) is equivalent to the following continued fraction expansion

$$\sum_{n\geq 1}A_n(p,q,t,u,v,w)x^{n-1} = \cfrac{1}{1-(u+tv)[1]_{p,q}x-\cfrac{[1]_{p,q}[2]_{p,q}twx^2}{1-(u+tv)[2]_{p,q}x-\cfrac{[2]_{p,q}[3]_{p,q}twx^2}{\cdots}}}. \tag{28}$$

For $0\leq k\leq\lfloor(n-1)/2\rfloor$, let $a_{n,k}(p,q,t,u,v)$ be the coefficient of w^k in $A_n(p,q,t,u,v,w)$, i.e.,

$$A_n(p,q,t,u,v,w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p,q,t,u,v) w^k. \tag{29}$$

Substituting $x \leftarrow \frac{x}{(u+tv)}$ and $w \leftarrow \frac{w(u+tv)^2}{t}$ in (28), we obtain

$$\sum_{n\geq 1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{a_{n,k}(p,q,t,u,v)}{t^k (u+tv)^{n-1-2k}} w^k x^{n-1} = \cfrac{1}{1-[1]_{p,q}x - \cfrac{[1]_{p,q}[2]_{p,q}wx^2}{1-[2]_{p,q}x - \cfrac{[2]_{p,q}[3]_{p,q}wx^2}{1-[3]_{p,q}x - \cfrac{[3]_{p,q}[4]_{p,q}wx^2}{\cdots}}}}. \tag{30}$$

Since the right-hand side of the above identity is free of variables t, u, and v, the coefficient of $w^k x^{n-1}$ in the left-hand side is a polynomial in p and q with nonnegative integral coefficients. If we denote this coefficient by

$$P_{n,k}(p,q) := \frac{a_{n,k}(p,q,t,u,v)}{t^k(u+tv)^{n-1-2k}},$$

then $P_{n,k}(p,q) = a_{n,k}(p,q,1,1,0) = a_{n,k}(p,q,0,1)$. On the other hand, comparing (29) and (6), we see that $a_{n,k}(p,q) = a_{n,k}(p,q,0,1)$. Thus $P_{n,k}(p,q) = a_{n,k}(p,q)$. This proves (5). Finally, as $(p+q) \mid [n]_{p,q}[n+1]_{p,q}$ for all $n \geq 1$, each w appears with a factor $(p+q)$ in the right-hand side of (30), and the polynomial $P_{n,k}(p,q)$ is divisible by $(p+q)^k$. □

4. Proof of Theorem 5

In our previous paper [16, Theorem 8], a bijection $\Phi : \mathfrak{S}_n \to \mathfrak{S}_n$ was constructed such that

$$(\text{2-31}, \text{31-2}, \text{des}, \text{fmax})\sigma = (\text{nest}, \text{cros}, \text{drop}, \text{fix})\Phi(\sigma) \quad \forall \sigma \in \mathfrak{S}_n.$$

In this section we first recall the bijection Φ and then show that the same bijection satisfies

$$(\text{da} - \text{fmax}, \text{dd}, \text{valley})\sigma = (\text{cda}, \text{cdd}, \textit{cvalley})\Phi(\sigma) \quad \forall \sigma \in \mathfrak{S}_n. \tag{31}$$

Let $\sigma = \sigma(1)\dots\sigma(n)$ be a permutation of $[n]$, an *inversion top number* (resp. *inversion bottom number*) of a letter i in the word σ is the number of occurrences of inversions of form (i,j) (resp (j,i)) in σ. We now construct $\tau = \Phi(\sigma)$ in such a way that

$$(\text{2-31})_k \sigma = \text{nest}_k \tau \quad \forall k = 1, \dots, n.$$

Given a permutation σ, we first construct two biwords, $\binom{f}{f'}$ and $\binom{g}{g'}$, and then form the biword $\tau = \binom{f \ \ g}{f' \ \ g'}$ by concatenating f and g, and f' and g', respectively. The word f is defined as the subword of descent bottoms in σ, ordered increasingly, and g is defined as the subword of nondescent bottoms in σ, also ordered increasingly. The word f' is the permutation on descent tops in σ such that the inversion bottom number of each letter a in f' is the right embracing number of a in σ. Similarly, the word g' is the permutation on nondescent tops in σ such that the inversion top number of each letter b in g' is the right embracing number of b in σ. Rearranging the columns of τ', so that the bottom row is in increasing order, we obtain the permutation $\tau = \Phi(\sigma)$ as the top row of the rearranged bi-word.

Example. Let $\sigma = 412796583$, with right embracing numbers 1, 0, 0, 2, 0, 1, 1, 0, 0. Then

$$\binom{f}{f'} = \begin{pmatrix} 1 & 3 & 5 & 6 \\ 8 & 4 & 6 & 9 \end{pmatrix}, \qquad \binom{g}{g'} = \begin{pmatrix} 2 & 4 & 7 & 8 & 9 \\ 1 & 2 & 7 & 5 & 3 \end{pmatrix},$$

$$\begin{aligned} \tau' &= \begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 & 6 & 2 & 4 & 7 & 8 & 9 \\ 8 & 4 & 6 & 9 & 1 & 2 & 7 & 5 & 3 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 4 & 9 & 3 & 8 & 5 & 7 & 1 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \end{aligned}$$

$\sigma \in \mathfrak{S}_4$	$\tau = \Phi(\sigma)$	des σ	(31-2) σ	(2-31) σ	da σ − fmax σ	dd σ	valley σ	fmax σ
		drop τ	cros τ	nest τ	cda τ	cdd τ	cvalley τ	fix τ
1234	1234	0	0	0	0	0	0	4
1243	1243	1	0	0	0	0	1	2
1324	1324	1	0	0	0	0	1	2
1342	1432	1	0	1	0	0	1	2
1423	1342	1	1	0	1	0	1	1
1432	1423	2	0	0	0	1	1	1
2134	2134	1	0	0	0	0	1	2
2143	2143	2	0	0	0	0	2	0
2314	3214	1	0	1	0	0	1	2
2341	4231	1	0	2	0	0	1	2
2413	3241	1	1	1	1	0	1	1
2431	4213	2	0	1	0	1	1	1
3124	2314	1	1	0	1	0	1	1
3142	3421	2	1	1	0	0	2	0
3214	3124	2	0	0	0	1	1	1
3241	4321	2	0	2	0	0	2	0
3412	2431	1	1	1	1	0	1	1
3421	4132	2	0	1	0	1	1	1
4123	2341	1	2	0	2	0	1	0
4132	3412	2	2	0	0	0	2	0
4213	3142	2	1	0	1	1	1	0
4231	4312	2	1	1	0	0	2	0
4312	2413	2	1	0	1	1	1	0
4321	4123	3	0	0	0	2	1	0

Fig. 1. Illustration of Φ on $\mathfrak{S}_4$ with their statistics.

and thus $\Phi(\sigma) = \tau = 2\,4\,9\,3\,8\,5\,7\,1\,6$.

Let $\tau = \Phi(\sigma) \in \mathfrak{S}_n$. We enumerate the triple statistics (da σ − fmaxσ, dd σ, valley σ).

- If k is a double ascent of σ and not a foremaximum of σ, then k is a ascent top and also bottom. So k belongs to g and g'. By definition, $(31\text{-}2)_k\sigma > 0$. Hence, the column $\binom{k}{k}$ does not appear in $\binom{g}{g'}$, i.e., $\tau^{-1}(k) < k < \tau(k)$. So k is a cyclic double ascent of τ. Conversely, if $\tau^{-1}(k) < k < \tau(k)$, then the column k appears in g and g' and $(31\text{-}2)_k\sigma > 0$. It implies that k is a double ascent of σ and not a foremaximum of σ.
- If k is a double descent of σ, then k is a descent top and also bottom. So k belongs to f and f', then $\tau^{-1}(k) > k > \tau(k)$. Hence k is a cyclic double descent of τ. Conversely, if $\tau^{-1}(k) > k > \tau(k)$, then the column k appears in f and f'. It implies that k is a double descent of σ.
- If k is a valley of σ, then k is a descent bottom and also ascent top. So k belongs to f and g', then $\tau^{-1}(k) > k < \tau(k)$. Hence k is a cyclic valley of τ. Conversely, if $\tau^{-1}(k) > k < \tau(k)$, then the column k appears in f and g'. It implies that k is a valley of σ.

Thus (31) is established. The proof of Theorem 5 is then completed. □

We illustrate $\Phi : \mathfrak{S}_4 \to \mathfrak{S}_4$ with their statistics in Fig. 1.

5. Proof of Theorem 8

Let $\sigma \in \mathfrak{S}_n$, the refinements of crossing and nesting are defined by

$$\text{cros}_k\sigma = \#\{i \in [n] : (i < k \le \sigma(i) < \sigma(k)) \vee (i > k > \sigma(i) > \sigma(k))\},$$
$$\text{nest}_k\sigma = \#\{i \in [n] : (i < k \le \sigma(k) < \sigma(i)) \vee (i > k > \sigma(k) > \sigma(i))\}.$$

We clearly have cros$\sigma = \sum_{k=1}^n \text{cros}_k\sigma$ and nest$\sigma = \sum_{k=1}^n \text{nest}_k\sigma$.

Using Foata–Zeilberger's bijection $\Psi_{FZ} : \mathfrak{S}_n \to \mathfrak{H}_n$, we construct the Laguerre history $(s_0, \dots, s_n, p_1, \dots, p_n)$, where $s_0 = (0, 0)$ and the step (s_{i-1}, s_i) is North-East, South-East, East blue and East red if i is a cyclic valley, cyclic peak, cyclic double ascent (or fixed point), or cyclic double descent, respectively; while $p_i = \mathrm{nest}_i\sigma$ for $i = 1, \dots, n$. Then, we have

$$\mathrm{nest}_i\sigma + \mathrm{cros}_i\sigma = \begin{cases} h_i, & \text{if } (s_{i-1}, s_i) \text{ is North-East;} \\ h_i - 1, & \text{if } (s_{i-1}, s_i) \text{ is South-East;} \\ h_i, & \text{if } (s_{i-1}, s_i) \text{ is East blue;} \\ h_i - 1, & \text{if } (s_{i-1}, s_i) \text{ is East red.} \end{cases}$$

Thus $(s_0, \dots, s_n, p_1, \dots, p_n)$ is a Laguerre history of length n and

$$w(\sigma) = t^{\mathrm{ER}\gamma + \mathrm{NE}\gamma} u^{\mathrm{EB}\gamma} v^{\mathrm{ER}\gamma} w^{\mathrm{NE}\gamma} y^{\mathrm{EB}^*\gamma} q^{\mathrm{NE}\gamma + \mathrm{EB}\gamma} \prod_{i=1}^{n} p^{p_i} q^{h_i - 1 - p_i},$$

where $\mathrm{NE}\gamma$, $\mathrm{EB}\gamma$, and $\mathrm{ER}\gamma$ are the number of North-East steps, East blue steps, and East red steps of γ and $\mathrm{EB}^*\gamma$ is the number of East blue steps whose height is equal to p_i. Given a Motzkin path γ, weight each step at height h by

$$a_k := tw[h+1]_{p,q}, \qquad b_k := yp^h + (qu + tv)[h]_{p,q}, \qquad c_k := [h]_{p,q} \tag{32}$$

if the step is North-East, East, and South-East, respectively, and the weight of γ is defined to be the product of the step weights. Then the last sum amounts to sum over all the Motzkin paths of length $n - 1$ with respect to (32), i.e.,

$$B_n(p, q, t, u, v, w, y) = \sum_{\gamma \in \mathfrak{M}_n} w(\gamma). \tag{33}$$

By (24), we have the following continued fraction expansion:

$$\sum_{n \geq 0} B_n(p, q, t, u, v, w, y) x^n = \cfrac{1}{1 - b_0 x - \cfrac{a_0 c_1 x^2}{1 - b_1 x - \cfrac{a_1 c_2 x^2}{\cdots}}}, \tag{34}$$

where $a_h = tw[h+1]_{p,q}$, $b_h = yp^h + (qu + tv)[h]_{p,q}$, and $c_h = [h]_{p,q}$.

For $0 \leq k \leq \lfloor (n-d)/2 \rfloor$, let $b_{n,k,d}(p, q, t, u, v)$ be the coefficient of $w^k y^d$ in $B_n(p, q, t, u, v, w, y)$, that is,

$$B_n(p, q, t, u, v, w, y) = \sum_{d=0}^{n} \sum_{k=0}^{\lfloor (n-d)/2 \rfloor} b_{n,k,d}(p, q, t, u, v) w^k y^d. \tag{35}$$

Substituting $x \leftarrow \frac{x}{(qu+tv)}$, $w \leftarrow \frac{w(qu+tv)^2}{t}$, and $y \leftarrow (qu + tv)y$ in (34), we obtain

$$\sum_{n \geq 0} \sum_{d=0}^{n} \sum_{k=0}^{\lfloor (n-d)/2 \rfloor} \frac{b_{n,k,d}(p, q, u, v, t)}{t^k (qu + tv)^{n-2k-d}} w^k y^d x^n$$
$$= \cfrac{1}{1 - (y + [0]_{p,q})x - \cfrac{[1]_{p,q}^2 w x^2}{1 - (py + [1]_{p,q})x - \cfrac{[2]_{p,q}^2 w x^2}{1 - (p^2 y + [2]_{p,q})x - \cfrac{[3]_{p,q}^2 w x^2}{\cdots}}}}.$$

Since the right-hand side of the above identity is free of variables t, u, and v, the coefficient of $w^k y^d x^n$ in the left-hand side is a polynomial in p and q with nonnegative integral coefficients. Denote this polynomial by $Q_{n,k,d}(p, q)$, then

$$Q_{n,k,d}(p, q) := \frac{b_{n,k,d}(p, q, t, u, v)}{t^k (qu + tv)^{n-2k-d}}. \tag{36}$$

Hence $Q_{n,k,d}(p, q) = b_{n,k,d}(p, q, 1, 0, 1)$. By (15) and (35), we derive that $Q_{n,k,d}(p, q) = b_{n,k,d}(p, q)$, which is given by (18). This completes the proof. $\square$

Remark. Let $A_n(p,q,t) = A_n(p,q,t,1,1,1)$ and $B_n(p,q,t) = B_n(p,q,t,1,1,1,1)$. From (28) and (34) we derive

$$\sum_{n\geq 0} A_n(p,q,t)x^n = 1 + \cfrac{[1]_{p,q}x}{1-(1+t)[1]_{p,q}x - \cfrac{[1]_{p,q}[2]_{p,q}tx^2}{1-(1+t)[2]_{p,q}x - \cfrac{[2]_{p,q}[3]_{p,q}tx^2}{\cdots}}}, \tag{37}$$

and

$$\sum_{n\geq 0} B_n(p,q,t)x^n = \cfrac{1}{1-(t[0]_{p,q}+[1]_{p,q})x - \cfrac{[1]^2_{p,q}tx^2}{1-(t[1]_{p,q}+[2]_{p,q})x - \cfrac{[2]^2_{p,q}tx^2}{\cdots}}}. \tag{38}$$

In view of the following contraction formulas with $c_{2i-1} = [i]_{p,q}$ and $c_{2i} = t[i]_{p,q}$ for $i \geq 1$:

$$\begin{aligned}\cfrac{1}{1-\cfrac{c_1x}{1-\cfrac{c_2x}{\cdots}}} &= 1 + \cfrac{c_1x}{1-(c_1+c_2)x - \cfrac{c_2c_3x^2}{1-(c_3+c_4)x - \cfrac{c_4c_5x^2}{\cdots}}}\\ &= \cfrac{1}{1-c_1x - \cfrac{c_1c_2x^2}{1-(c_2+c_3)x - \cfrac{c_3c_4x^2}{\cdots}}},\end{aligned}$$

we see that

$$\sum_{n\geq 0} A_n(p,q,t)x^n = \sum_{n\geq 0} B_n(p,q,t)x^n = \cfrac{1}{1-\cfrac{c_1x}{1-\cfrac{c_2x}{1-\cfrac{c_3x}{\cdots}}}}, \tag{39}$$

where $c_{2i} = t[i]_q$ and $c_{2i-1} = [i]_q$. Hence $A_n(p,q,t) = B_n(p,q,t)$, i.e., the two triple statistics (2-13, 31-2, des) and (2-31, 31-2, des) are equidistributed on $\mathfrak{S}_n$. Although a bijection showing the latter equidistribution can be constructed by combining the known bijections from permutations to Motzkin paths, a direct bijective proof of the above equidistribution is desired.

6. Proof of Theorem 11

Consider the polynomial

$$C_n(\beta,t,u,v,w) := \sum_{\sigma\in\mathfrak{D}_n} \beta^{\mathrm{cyc}\,\sigma} t^{\mathrm{exc}\,\sigma} u^{\mathrm{cda}\sigma} v^{\mathrm{cdd}\sigma} w^{cvalley\sigma} = \sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(\beta,t,u,v)w^k. \tag{40}$$

From [17] we know that

$$\begin{aligned}&1 + \sum_{n\geq 1} C_n(\beta,t,u,v,w)x^n\\ &= \cfrac{1}{1-0(tu+v)x - \cfrac{1(\beta+0)twx^2}{1-1(tu+v)x - \cfrac{2(\beta+1)twx^2}{1-2(tu+v)x - \cfrac{3(\beta+2)twx^2}{\cdots}}}}.\end{aligned} \tag{41}$$

Substituting $x \leftarrow \frac{x}{(tu+v)}$ and $w \leftarrow \frac{w(tu+v)^2}{t}$ in (41), we get

$$1 + \sum_{n\geq 1}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{c_{n,k}(\beta,t,u,v)}{t^k(tu+v)^{n-2k}} w^k x^n = \cfrac{1}{1-\cfrac{1(\beta+0)wx^2}{1-x-\cfrac{2(\beta+1)wx^2}{1-2x-\cfrac{3(\beta+2)wx^2}{1-3x-\cfrac{4(\beta+3)wx^2}{\cdots}}}}}. \tag{42}$$

By the right-hand side we see that $c^*_{n,k}(\beta) := \frac{c_{n,k}(\beta,t,u,v)}{t^k(tu+v)^{n-2k}}$ must be a polynomial in β, which is independent of u, v, and t. Hence we have

$$C_n(\beta, t, u, v, w) = \sum_{k=0}^{\lfloor n/2\rfloor} c^*_{n,k}(\beta) t^k (tu+v)^{n-2k} w^k. \tag{43}$$

Thus $C_n(\beta, 1, 1, 0, w) = \sum_{k=0}^{\lfloor n/2\rfloor} c^*_{n,k}(\beta) w^k$. On the other side, by (22) and (49) we have $C_n(\beta, 1, 1, 0, w) = \sum_{k=0}^{\lfloor n/2\rfloor} c_{n,k}(\beta) w^k$. So $c_{n,k}(\beta) = c^*_{n,k}(\beta)$. Finally we get (23) by putting $u = v = w = 1$ in (43). $\square$

7. Two star variations

We show that the polynomial A_n, which is originally defined using *linear statistics*, has a new combinatorial interpretation by *cyclic statistics*. For $\sigma = \sigma(1)\dots\sigma(n) \in \mathfrak{S}_n$ and some integral function 'stat' on $\mathfrak{S}_n$, we define the *star* transformation from the permutation σ to the function $\sigma^* = \sigma^*(1)\dots\sigma^*(n)$ from $[n]$ to $\{0, \dots, n-1\}$ by

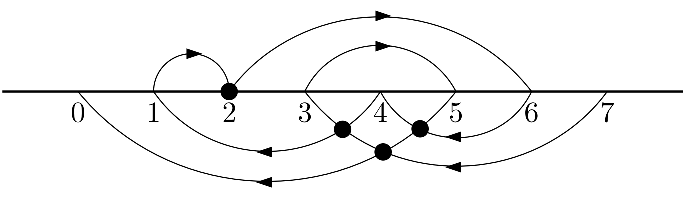
$$\sigma \mapsto \sigma^* = (\sigma(1) - 1)\dots(\sigma(n) - 1). \tag{44}$$

For any statistic stat on σ^* we can define the corresponding star statistic 'stat*' on σ by $\mathrm{stat}^*(\sigma) := \mathrm{stat}(\sigma^*)$. For instance, we can define the *star* cyclic statistics

$$\begin{aligned}
&\mathrm{fix}^*\sigma = \mathrm{fix}\sigma^* = \#\left\{i \in [n-1] : i = \sigma^*(i) = \sigma(i) - 1\right\},\\
&\mathrm{wex}^*\,\sigma = \mathrm{wex}\,\sigma^* = \#\left\{i \in [n-1] : i \le \sigma^*(i) = \sigma(i) - 1\right\} (= \mathrm{exc}\,\sigma),\\
&\mathrm{cros}^*\sigma \ = \ \mathrm{cros}\sigma^*\\
&\qquad\quad = \#\left\{(i,j) \in [n]\times[n] : (i < j \le \sigma^*(i) < \sigma^*(j)) \vee (i > j > \sigma^*(i) > \sigma^*(j))\right\},\\
&\mathrm{nest}^*\sigma \ = \ \mathrm{nest}\sigma^*\\
&\qquad\quad = \#\left\{(i,j) \in [n]\times[n] : (i < j \le \sigma^*(j) < \sigma^*(i)) \vee (i > j > \sigma^*(j) > \sigma^*(i))\right\},\\
&\mathrm{cdd}^*\sigma = \mathrm{cdd}\sigma^* = \#\left\{i \in [n-1] : (\sigma^*)^{-1}(i) > i > \sigma^*(i)\right\},\\
&\mathrm{cda}^*\sigma = \mathrm{cda}\sigma^* = \#\left\{i \in [n-1] : (\sigma^*)^{-1}(i) < i < \sigma^*(i)\right\},\\
&\mathit{cvalley}^*\sigma = \mathit{cvalley}\sigma^* = \#\left\{i \in [n-1] : (\sigma^*)^{-1}(i) > i < \sigma^*(i)\right\}.
\end{aligned}$$

Clearly, $\sigma^*(0)$ and $(\sigma^*)^{-1}(n)$ are neither defined nor needed.

Remark. The star crossing number $\mathrm{cros}^*(\sigma)$ of σ is different with the ordinary crossing number $\mathrm{cros}(\sigma)$ of σ. For example, for the previous example $\sigma = 3\,762\,154$, we have $\sigma^* = 2\,651\,043 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 5 & 1 & 0 & 4 & 3 \end{pmatrix}$.



$\sigma^* = 2651043$

Since the four pairs $(1 < 2 \le \sigma^*(1) < \sigma^*(2))$, $(6 > 5 > \sigma^*(6) > \sigma^*(5))$, $(7 > 4 > \sigma^*(7) > \sigma^*(4))$, and $(7 > 5 > \sigma^*(7) > \sigma^*(5))$ are only crossings in σ^*, $\mathrm{cros}^*(\sigma) = \mathrm{cros}(\sigma^*) = 4$. On the other hand, the three pairs $(2 < 3 \le \sigma^*(3) < \sigma^*(2))$, $(5 > 4 > \sigma^*(4) > \sigma^*(5))$, and $(7 > 6 > \sigma^*(6) > \sigma^*(7))$ are only nestings in σ^*, thus $\mathrm{nest}^*(\sigma) = \mathrm{nest}(\sigma^*) = 3$.

Theorem 12. *The two hextuple statistics* (2-13, 31-2, des, da*, dd*, valley*) *and* (nest*, [2]cros*, [2] drop* −1, [2][2]cda* + fix*, cdd*, *cvalley**) *are equidistributed on* $\mathfrak{S}_n$. *In other words, we have*

$$A_n(p,q,t,u,v,w)=\sum_{\sigma\in\mathfrak{S}_n} p^{\mathrm{nest}^*\sigma}q^{\mathrm{cros}^*\sigma}t^{\mathrm{drop}^*\sigma-1}u^{\mathrm{cda}^*\sigma+\mathrm{fix}^*\sigma}v^{\mathrm{cdd}^*\sigma}w^{cvalley^*\sigma}. \tag{45}$$

Proof. Using the bijection Φ in Section 4 (cf. Theorem 5), we can give a new bijection Ψ on $\mathfrak{S}_n$. Given a permutation $\sigma=\sigma(1)\dots\sigma(n)\in\mathfrak{S}_n$, let $\hat{\sigma}=(\sigma(1)+1)\dots(\sigma(n)+1)(1)\in\mathfrak{S}_{n+1}$ and consider the permutation $\tau=\Phi(\hat{\sigma})\in\mathfrak{S}_{n+1}$. Since the last element $\hat{\sigma}(n+1)$ of $\hat{\sigma}$ is 1, the first element $\tau(1)$ of τ should be $n+1$. So $\tau(2)\dots\tau(n+1)$ is a permutation on $[n]$ and we can define the permutation $\Psi(\sigma):=\tau(2)\dots\tau(n+1)\in\mathfrak{S}_n$. By Theorem 5, the bijection Φ satisfies

$$(\text{2-31, 31-2, des})\hat{\sigma}=(\text{nest, cros, drop})\Phi(\hat{\sigma}).$$

Since $(\text{2-13, 31-2, des})\sigma=(\text{2-31}+\text{des}-n,\ \text{31-2},\ \text{des}-1)\hat{\sigma}$ and

$$(\mathrm{nest}^*,\mathrm{cros}^*,\mathrm{drop}^*-1)\Psi(\sigma)=(\mathrm{nest}+\mathrm{drop}-n,\ \mathrm{cros},\ \mathrm{drop}-1)\Phi(\hat{\sigma}),$$

we have $(\text{2-13, 31-2, des})\sigma=(\mathrm{nest}^*,\mathrm{cros}^*,\mathrm{drop}^*-1)\Psi(\sigma)$. Hence, assuming $\sigma(0)=\sigma(n+1)=0$, the two hextuple statistics (2-13, 31-2, des, da*, dd*, valley*) and (nest*, cros*, [2] drop* −1, cdd*, cda* + fix*, *cvalley**) are equidistributed on $\mathfrak{S}_n$. □

Example. Given $\sigma=4\,1\,2\,7\,9\,6\,5\,8\,3$, we define $\hat{\sigma}=5\,2\,3\,8\,10\,7\,6\,9\,4\,1$ with right embracing numbers 1, 1, 1, 2, 0, 1, 1, 0, 0, 0. Then

$$\begin{pmatrix} f\\ f'\end{pmatrix}=\begin{pmatrix}1&2&4&6&7\\4&9&5&7&10\end{pmatrix},\qquad \begin{pmatrix} g\\ g'\end{pmatrix}=\begin{pmatrix}3&5&8&9&10\\2&3&8&6&1\end{pmatrix},$$

$$\begin{aligned}\tau' &= \begin{pmatrix} f & g\\ f' & g'\end{pmatrix}=\begin{pmatrix}1&2&4&6&7&3&5&8&9&10\\4&9&5&7&10&2&3&8&6&1\end{pmatrix}\\ &\to \begin{pmatrix}10&3&5&1&4&9&6&8&2&7\\1&2&3&4&5&6&7&8&9&10\end{pmatrix},\end{aligned}$$

and thus $\tau=\Phi(\hat{\sigma})=10\,3\,5\,1\,4\,9\,6\,8\,2\,7$. So, $\Psi(\sigma)=\tau(2)\dots\tau(10)=3\,5\,1\,4\,9\,6\,8\,2\,7$.

Let $\mathfrak{S}_n^*$ be the set of all sequences σ^* where $\sigma\in\mathfrak{S}_n$, that is, $\mathfrak{S}_n^*:=\{\sigma^*:\sigma\in\mathfrak{S}_n\}$. By definition (44), for all $\sigma\in\mathfrak{S}_n$, it holds that

$$(\mathrm{fix}^*,\mathrm{nest}^*,\mathrm{cros}^*,\mathrm{wex}^*,\mathrm{cdd}^*,\mathrm{cda}^*,cvalley^*)\sigma=(\mathrm{fix},\mathrm{nest},\mathrm{cros},\mathrm{wex},\mathrm{cdd},\mathrm{cda}\,cvalley)\sigma^*. \tag{46}$$

Note that, for a given $\sigma^*\in\mathfrak{S}_n^*$, the entries 0 and n are neither a cyclic valley nor a cyclic peak. We illustrate $\Psi:\mathfrak{S}_4\to\mathfrak{S}_4$ with their statistics in Fig. 2.

Given a $\sigma\in\mathfrak{S}_n$, the diagram of $\sigma^*\in\mathfrak{S}_n^*$ consists of cycles $i\to\sigma^*(i)\to\cdots\to i$ with $i\in[n]$, and the path $n\to\sigma^*(n)\to\cdots\to 0$. Let $cyc^*\sigma$ be the number of cycles in the diagram of σ^*. For example, for the previous example $\sigma=3\,762\,154$ and $\sigma^*=2\,651\,043$, we have $cyc^*(\sigma)=cyc(\sigma^*)=2$, since one cycle $1\to 2\to 6\to 4$ and one path $7\to 3\to 5\to 0$ exist in σ^*. Let $\mathfrak{S}_n(k)$ be the set of permutations $\sigma\in\mathfrak{S}_n$ with $cvalley^*\sigma=k$ and $\mathrm{cdd}^*\sigma=0$. Introduce the polynomial

$$d_{n,k}(\beta)=\sum_{\sigma\in\mathfrak{S}_n(k)}\beta^{cyc^*\sigma-\mathrm{fix}^*\sigma}. \tag{47}$$

Theorem 13. *We have*

$$\sum_{\sigma\in\mathfrak{S}_n}\beta^{cyc^*\sigma-\mathrm{fix}^*\sigma}t^{\mathrm{exc}\,\sigma}=\sum_{k=0}^{\lfloor (n-1)/2\rfloor}d_{n,k}(\beta)t^k(1+t)^{n-1-2k}. \tag{48}$$

Moreover, for all $k\geq 1$, the polynomial $d_{n,k}(\beta)$ has a factor $(\beta+1)$.

$\sigma \in \mathfrak{S}_4$	$\tau = \Psi(\sigma)$	$\tau^* \in \mathfrak{S}_4^*$	$\mathrm{des}\,\sigma$ / $\mathrm{drop}^*\,\tau - 1$ / $\mathrm{drop}\,\tau^* - 1$	$(31\text{-}2)\,\sigma$ / $\mathrm{cros}^*\,\tau$ / $\mathrm{cros}\,\tau^*$	$(2\text{-}13)\,\sigma$ / $\mathrm{nest}^*\,\tau$ / $\mathrm{nest}\,\tau^*$	$\mathrm{da}^*\,\sigma$ / $\mathrm{cda}^*\,\tau + \mathrm{fix}^*\,\tau$ / $\mathrm{cda}\,\tau^* + \mathrm{fix}\,\tau^*$	$\mathrm{dd}^*\,\sigma$ / $\mathrm{cdd}^*\,\tau$ / $\mathrm{cdd}\,\tau^*$	$\mathrm{valley}^*\,\sigma$ / $\mathrm{cvalley}^*\,\tau$ / $\mathrm{cvalley}\,\tau^*$
1234	2341	1230	0	0	0	3	0	0
1243	2314	1203	1	0	0	2	1	0
1324	2431	1320	1	0	1	1	0	1
1342	2143	1032	1	0	0	2	1	0
1423	2413	1302	1	1	0	1	0	1
1432	2134	1023	2	0	0	1	2	0
2134	3241	2130	1	0	1	1	0	1
2143	3214	2103	2	0	1	0	1	1
2314	4321	3210	1	0	2	1	0	1
2341	1342	0231	1	0	0	2	1	0
2413	4312	3201	1	1	1	1	0	1
2431	1324	0213	2	0	0	1	2	0
3124	3421	2310	1	1	1	1	0	1
3142	3124	2013	2	1	1	0	1	1
3214	4231	3120	2	0	2	0	1	1
3241	1432	0321	2	0	1	0	1	1
3412	3142	2031	1	1	0	1	0	1
3421	1243	0132	2	0	0	1	2	1
4123	3412	2301	1	2	0	1	0	1
4132	4123	3012	2	2	0	0	1	1
4213	4213	3102	2	1	1	0	1	1
4231	1423	0312	2	1	0	0	1	1
4312	3124	2013	2	1	0	0	1	1
4321	1234	0123	3	0	0	0	3	0

Fig. 2. Illustration of Ψ on $\mathfrak{S}_4$ with their statistics.

Proof. Let

$$D_n(\beta, t, u, v, w) := \sum_{\sigma \in \mathfrak{S}_n} \beta^{\mathrm{cyc}^*\sigma - \mathrm{fix}^*\sigma} t^{\mathrm{wex}^*\sigma} u^{\mathrm{cda}^*\sigma + \mathrm{fix}^*\sigma} v^{\mathrm{cdd}^*\sigma} w^{cvalley^*\sigma}. \tag{49}$$

By the same method in [17] to count the cycles, we obtain

$$\sum_{n\geq 1} D_n(\beta, t, u, v, w) x^{n-1} = \cfrac{1}{1 - 1(tu+v)x - \cfrac{1(\beta+1)twx^2}{1-2(tu+v)x - \cfrac{2(\beta+2)twx^2}{1-3(tu+v)x-\cfrac{3(\beta+3)twx^2}{\cdots}}}}. \tag{50}$$

Define the polynomial $d_{n,k}(\beta, t, u, v)$ to be the coefficients of w^k in $D_n(\beta, t, u, v, w)$:

$$D_n(\beta, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n,k}(\beta, t, u, v) w^k.$$

Substituting $x \leftarrow \frac{x}{(tu+v)}$ and $w \leftarrow \frac{w(tu+v)^2}{t}$ in (50), we get

$$\sum_{n\geq 1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{d_{n,k}(\beta, t, u, v)}{t^k (tu+v)^{n-1-2k}} w^k x^{n-1}$$

$$= \cfrac{1}{1 - x - \cfrac{(\beta+1)wx^2}{1-2x-\cfrac{2(\beta+2)wx^2}{1-3x-\cfrac{3(\beta+3)wx^2}{\cdots}}}}. \tag{51}$$

So $d^*_{n,k}(\beta) := \frac{d_{n,k}(\beta,t,u,v)}{t^k(tu+v)^{n-1-2k}}$ must be a polynomial in β, since the right-hand side of (51) is independent of u, v and t. Hence we have

$$D_n(\beta, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d^*_{n,k}(\beta) t^k (tu+v)^{n-1-2k} w^k. \tag{52}$$

To verify (47), taking $t = u = 1$ and $v = 0$ in (49) and (52) we obtain

$$D_n(\beta, 1, 1, 0, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n,k}(\beta) w^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d^*_{n,k}(\beta) w^k.$$

Thus $d_{n,k}(\beta) = d^*_{n,k}(\beta)$, and we get (48) by putting $u = v = w = 1$ in (52). Clearly, for all $k \geq 1$, the polynomial $d_{n,k}(\beta)$ has a factor $(\beta + 1)$ because of the presence of the term $(\beta + 1)w$ at the second row of the continued fraction on the right side of (51). $\square$

8. Concluding remarks

Consider the descent polynomial of involutions on $[n]$:

$$I_n(t) := \sum_{\sigma \in \mathfrak{I}_n} t^{\operatorname{des} \sigma} = I(n, 0) + I(n, 1)t + \cdots + I(n, n-1)t^{n-1}, \tag{53}$$

where $\mathfrak{I}_n$ is the subset of involutions in $\mathfrak{S}_n$. The sequence $\{I(n, k)\}_{0 \leq k \leq n-1}$ is known [13] to be *symmetric* and *unimodal*. Brenti had conjectured that the sequence $\{I(n, k)\}_{0 \leq k \leq n-1}$ is *log-concave*, which was later disproved by Barnabei et al. [1]. Hence the best result one could expect for $I_n(t)$ is the following expansion

$$I_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \alpha_{n,k} t^k (1+t)^{n-1-2k} \quad \text{with } \alpha_{n,k} \in \mathbb{N}. \tag{54}$$

This is Conjecture 4.1 in [13]. Unfortunately, the generating function of $I_n(t)$ does not have a nice continued fraction expansion as that for $A_n(t)$ or $B_n(t)$.

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Appendix

The first values of $a_{n,k}(p, q)$ are given by $a_{n,0}(p, q) = 1$ for $1 \leq n \leq 5$ and

$n \setminus k$	0	1	2
1	1		
2	1		
3	1	$p+q$	
4	1	$(p+q)(p+q+2)$	
5	1	$(p+q)[(p+q)^2+2(p+q)+3]$	$(p+q)^2(p^2+pq+q^2+1)$

For $j = 0, 1, 2$ the first values of the polynomials $b_{n,k,j}(p, q)$ are given as follows.

$j = 0$	$k = 0$	1	2	3
$n = 1$	0			
2	0	1		
3	0	1		
4	0	1	$(p+q)^2+1$	
5	0	1	$(p+q)^3+2(p+q)^2+2$	
6	0	1	$(p+q)^4+2(p+q)^3+3(p+q)^2+3$	$b_{6,3,0}(p,q)$

$j = 1$	$k = 0$	1	2
$n = 1$	1		
2	0		
3	0	$p+2$	
4	0	$2p+2$	
5	0	$3p+2$	$b_{5,2,1}(p,q)$

$j = 2$	$k = 0$	1	2
$n = 2$	1		
3	0		
4	0	p^2+2p+3	
5	0	$3p^2+4p+3$	
6	0	$6p^2+6p+3$	$b_{6,2,2}(p,q)$

where

$$b_{6,3,0}(p,q) = (p+q)^6 + (1-2pq)(p+q)^4 + (2+p^2q^2)(p+q)^2 + 1,$$

$$b_{5,2,1}(p,q) = (p^2+2p+2)q^2 + (2p^3+4p^2+4p)q + (p^4+2p^3+2p^2+2p+3),$$

$$b_{6,2,2}(p,q) = (p^4+2p^3+5p^2+4p+3)q^2 + (2p^5+4p^4+10p^3+8p^2+6p)q + (p^6+2p^5+5p^4+4p^3+6p^2+6p+6).$$

The first non-zero values of $c_{n,k}(\beta)$ $(2 \le n \le 7)$ are given by the following table.

$n \setminus k$	1	2	3
2	β		
3	β		
4	β	$\beta(3\beta+2)$	
5	β	$2\beta(5\beta+4)$	
6	β	$\beta(25\beta+22)$	$\beta(15\beta^2+30\beta+16)$
7	β	$4\beta(14\beta+13)$	$\beta(105\beta^2+238\beta+136)$.

The first values of $d_{n,k}(\beta)$ are given by $d_{n,0}(\beta) = 1$ for $1 \le n \le 7$ and

$n \setminus k$	0	1	2	3
1	1			
2	1			
3	1	$\beta+1$		
4	1	$4(\beta+1)$		
5	1	$11(\beta+1)$	$(\beta+1)(3\beta+5)$	
6	1	$26(\beta+1)$	$(\beta+1)(25\beta+43)$	
7	1	$57(\beta+1)$	$10(\beta+1)(13\beta+23)$	$(\beta+1)(15\beta^2+60\beta+61)$.

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