MATHEMATICS

NOTE ON POINTS AND SYMPLECTA IN THE METASYMPLECTIC GEOMETRY

BY

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In [2], Professor FREUDENTHAL investigated the geometric properties of exceptional simple Lie groups of type $\ulcorner F_4, E_6, E_7, E_8 \urcorner$, namely the metasymplectic geometry. In this note, the relations: interwoven, hinged, between symplecta in the metasymplectic geometry will be interpreted by the nilpotency of the elements in $\mathfrak{R}_1$, and a condition of a point to be half incident with a symplecton will be given.

Throughout this note, we follow the definitions and notations in [2, also cf. 3], but the points $\langle \Phi_1, \Phi_2 \rangle$ are considered for Φ_1, Φ_2 in $\mathfrak{W}_4$ such that $[\Phi_1, \Phi_2]=0$ [cf. 2, § 37].

1. For Φ_1, Φ_2 in $\mathfrak{R}_4$, by definition in [2, p. 462 (35.3)]

$$\langle \Phi_1, \Phi_2 \rangle \Phi^* = \tfrac{1}{2}(\tilde{\Phi}_1 \tilde{\Phi}_2 + \tilde{\Phi}_2 \tilde{\Phi}_1)\Phi^* - \tfrac{1}{2}(\Phi_1, \Phi^*)\Phi_2 - \tfrac{1}{2}(\Phi_2, \Phi^*)\Phi_1 + \alpha(\Phi_1, \Phi_2)\Phi^*,$$

where $\alpha = \ulcorner \frac{6}{25}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \urcorner (= \varepsilon_2 \varepsilon_3^{-1}$ in [2]). Then, we obtain easily the following relations:

$$(1.1) \qquad \langle \Phi_1, \Phi_2 \rangle \Phi_i = (\alpha - 1)(\Phi_1, \Phi_2)\Phi_i \text{ for } \Phi_i \in \mathfrak{W}_4, \; i = 1, 2,$$

$$(1.2) \qquad \langle \Phi_1, \Phi_2 \rangle \tilde{\Phi}_1 \Phi_2 = (\alpha - 1)(\Phi_1, \Phi_2)\tilde{\Phi}_1 \Phi_2 \text{ for } \Phi_1, \Phi_2 \in \mathfrak{W}_4.$$

After a direct calculation, using [2, (35.4) and § 40], and above relations, we obtain

$$(1.3) \qquad 4\langle \Phi_1, \Phi_2 \rangle^2 \Phi^* = (\Phi_1, \Phi_2)\Psi - 3(\tilde{\Phi}_1 \Phi_2, \Phi^*)\tilde{\Phi}_1 \Phi_2 \text{ for all } \Phi^* \text{ in } \mathfrak{R}_4,$$

where

$$\Psi = (\tfrac{7}{2} - 2\alpha)((\Phi_1, \Phi^*)\Phi_2 + (\Phi_2, \Phi^*)\Phi_1) - (\tfrac{1}{2} - 2\alpha)(\tilde{\Phi}_1 \tilde{\Phi}_2 + \tilde{\Phi}_2 \tilde{\Phi}_1)\Phi^* + 4\alpha A \Phi^*.$$

In [2, § 42], it is proved that $[\Phi_1, \Phi_2]=0$ implies $\langle \Phi_1, \Phi_2 \rangle^2 = 0$. Conversely, assume that $\langle \Phi_1, \Phi_2 \rangle^2 = 0$, then from (1.1) we have $(\Phi_1, \Phi_2) = 0$, and (1.3) shows that $[\Phi_1, \Phi_2] = 0$. Thus, the following proposition follows.

Proposition 1.1: For symplecta Φ_1, Φ_2, $\langle \Phi_1, \Phi_2 \rangle^2 = 0$ is equivalent to $[\Phi_1, \Phi_2] = 0$ (interwoven).

Proposition 1.2: For Φ_1, Φ_2 in $\mathfrak{W}_4$, the following conditions are equivalent each other:

(a) $\langle \Phi_1, \Phi_2 \rangle^3 = 0$,

(b) $\langle \Phi_1, \Phi_2 \rangle^k = 0$ for some integer $k \geq 3$,

(c) $(\Phi_1, \Phi_2) = 0$ (hinged),

(d) $\langle \Phi_1, \Phi_2 \rangle^2 \Phi^* = -\frac{3}{4}(\tilde{\Phi}_1 \Phi_2, \Phi^*) \tilde{\Phi}_1 \Phi_2$ for all Φ^* in $\mathfrak{R}_4$.

Proof: (b) → (c): Since $\alpha \neq 1$, if $\langle \Phi_1, \Phi_2 \rangle^k \Phi_i = 0$ for some positive integer k, then $(\Phi_1, \Phi_2) = 0$ from (1.1).

(c) → (d) follows from (1.3).

(c) → (a): Using (d) and (1.2) we have $\langle \Phi_1, \Phi_2 \rangle^3 \Phi^* = 0$ for all Φ^* in $\mathfrak{R}_4$.

(d) → (c): From assumption, (1.3) follows $(\Phi_1, \Phi_2)\Psi = 0$. By putting $\Phi^* = \Phi$ in this relation, $4(1-\alpha)^2(\Phi_1, \Phi_2)^2 \Phi_2 = 0$, hence $(\Phi_1, \Phi_2) = 0$.

Remark: In [2, § 42], it is proved that for Φ_1, Φ_2 in $\mathfrak{W}_4$ $[\Phi_1, \Phi_2] \in \mathfrak{W}_4$ is equivalent to $(\Phi_1, \Phi_2) = 0$, hence if $(\Phi_1, \Phi_2) = 0$, $\langle \Phi_1, \Phi_2 \rangle^2 \Phi^* \in \mathfrak{W}_4$ because of (d). Conversely, assume that $\langle \Phi_1, \Phi_2 \rangle^2 \Phi^* \in \mathfrak{W}_4$ for all Φ^* in $\mathfrak{R}_4$, then

$$(\langle \Phi_1, \Phi_2 \rangle^2 \Phi^*, \langle \Phi_1, \Phi_2 \rangle^2 \Phi^*) = 0,$$

especially

$$(\langle \Phi_1, \Phi_2 \rangle^2 \tilde{\Phi}_1 \Phi_2, \langle \Phi_1, \Phi_2 \rangle^2 \tilde{\Phi}_1 \Phi_2) = 0,$$

then, using (1.2),

$$(\alpha - 1)^4 (\Phi_1, \Phi_2)^4 (\tilde{\Phi}_1 \Phi_2, \tilde{\Phi}_1 \Phi_2) = 0,$$

i.e. $(\alpha-1)^4(\Phi_1, \Phi_2)^6 = 0$ and $(\Phi_1, \Phi_2) = 0$ follows. Thus, $(\Phi_1, \Phi_2) = 0$ if and only if $\langle \Phi_1, \Phi_2 \rangle^2 \Phi^* \in \mathfrak{W}_4$ for all Φ^* in $\mathfrak{R}_4$.

2. For symplecta $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ such that $[\Phi_1, \Phi_2] = 0$ and $[\Phi_3, \Phi_4] = 0$, put

$$f(\Phi_1, \Phi_2; \Phi_3, \Phi_4) = \tilde{\Phi}_3 \langle \Phi_1, \Phi_2 \rangle \Phi_4 + \tilde{\Phi}_4 \langle \Phi_1, \Phi_2 \rangle \Phi_3, \tag{2.1}$$

then f is symmetric with respect to Φ_1 and Φ_2, (Φ_3 and Φ_4), and anti-symmetric with respect to $\ulcorner\Phi_1, \Phi_2\urcorner$ and $\ulcorner\Phi_3, \Phi_4\urcorner$. Indeed, using $[\Phi_1, \Phi_2] = [\Phi_3, \Phi_4] = 0$ and Jacobi-associativity,

$$[\tilde{\Phi}_1 \tilde{\Phi}_2 \Phi_3, \Phi_4] + [\tilde{\Phi}_1 \tilde{\Phi}_2 \Phi_4, \Phi_3] = -\tilde{\Phi}_4 \tilde{\Phi}_1 \tilde{\Phi}_2 \Phi_3 - \tilde{\Phi}_3 \tilde{\Phi}_2 \tilde{\Phi}_1 \Phi_4$$

$$= -\tilde{\Phi}_1 \tilde{\Phi}_4 \tilde{\Phi}_2 \Phi_3 - \tilde{\Phi}_2 \tilde{\Phi}_3 \tilde{\Phi}_1 \Phi_4 = \tilde{\Phi}_1 \tilde{\Phi}_4 \tilde{\Phi}_3 \Phi_2 + \tilde{\Phi}_2 \tilde{\Phi}_3 \tilde{\Phi}_4 \Phi_1$$

$$= -[\tilde{\Phi}_3 \tilde{\Phi}_4 \Phi_2, \Phi_1] - [\tilde{\Phi}_3 \tilde{\Phi}_4 \Phi_1, \Phi_2]$$

from this relation, we get easily $[\langle \Phi_1, \Phi_2 \rangle \Phi_3, \Phi_4] + [\langle \Phi_1, \Phi_2 \rangle \Phi_4, \Phi_3] = -[\langle \Phi_3, \Phi_4 \rangle \Phi_1, \Phi_2] - [\langle \Phi_3, \Phi_4 \rangle \Phi_2, \Phi_1]$, that is

$$f(\Phi_1, \Phi_2; \Phi_3, \Phi_4) + f(\Phi_3, \Phi_4; \Phi_1, \Phi_2) = 0.$$

Assume that $f=0$, then it is proved that the two points $A=\langle \Phi_1, \Phi_2\rangle$ and $B=\langle \Phi_3, \Phi_4\rangle$ are hinged. From the assumption, we have $[\Phi_1, \tilde{\Phi}_3 A\Phi_4 + + \tilde{\Phi}_4 A\Phi_3]=0$, which can be written as

$$2\tilde{\Phi}_1\tilde{\Phi}_3\tilde{\Phi}_1\tilde{\Phi}_2\Phi_4+2\tilde{\Phi}_1\tilde{\Phi}_4\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3+(\Phi_1, \Phi_4)\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3+$$
$$+(\Phi_1, \Phi_3)\tilde{\Phi}_1\tilde{\Phi}_2\Phi_4+(\Phi_2, \Phi_4)(\Phi_1, \Phi_3)\Phi_1+(\Phi_2, \Phi_3)(\Phi_1, \Phi_4)\Phi_1=0,$$

hence, using [2, § 40], we have

$$2(\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3, \Phi_4)-(\Phi_1, \Phi_3)(\Phi_2, \Phi_4)-(\Phi_1, \Phi_4)(\Phi_2, \Phi_3)=0$$

since $\Phi_1\neq 0$. This relation shows that $(A, B)=0$, i.e. A and B are hinged because of $(A, B)=(A\Phi_3, \Phi_4)$ [2, (56.1)].

Conversely, assume that the points A and B are hinged, then, from [2, p. 480, 65.7′], there are symplecta Φ_2 in $\{A\}$ and Φ_4 in $\{B\}$ such that Φ_2 and Φ_4 are joined (verbunden). So we may suppose that $A=\langle\Phi_1, \Phi_2\rangle$ and $B=\langle\Phi_3, \Phi_4\rangle$, in which $\langle\Phi_2, \Phi_4\rangle=0$. Then it will be shown that $\tilde{\Phi}_3 A\Phi_4+\tilde{\Phi}_4 A\Phi_3=0$. From [2, § 43], we have $(\Phi_1, \Phi_4)=0$ and $(\Phi_2, \Phi_3)=0$, hence $A\Phi_4=0$ and $\tilde{\Phi}_4 A\Phi_3=\tilde{\Phi}_4\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3$. To show that $\tilde{\Phi}_4\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3=0$, we may assume that $\Phi_4=\ulcorner\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\urcorner$. Then $\Phi_i=\ulcorner\begin{pmatrix}\Theta_i+\gamma_i & \underline{\delta}_i\\ \bar{\delta}_i & \Theta_i-\gamma_i\end{pmatrix}, \begin{pmatrix}P_i\\ P_i'\end{pmatrix}\urcorner$ $(i=1, 2, 3)$ get simpler because of $\langle\Phi_2, \Phi_4\rangle=0$, $[\Phi_3, \Phi_4]=0$, $(\Phi_1, \Phi_4)=0$, and $[\Phi_1, \Phi_2]=0$ implies that $P_2\times P_1'=0$. A calculation shows that

$$\tilde{\Phi}_4\tilde{\Phi}_1\tilde{\Phi}_2\Phi_3=\ulcorner\begin{pmatrix}0 & \underline{\delta}\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\urcorner,$$

where $\underline{\delta}=\frac{1}{4}\{\Theta_3P_2, P_1'\}$, then $\underline{\delta}=-\dfrac{1}{4\varepsilon}\ \mathrm{sp}\ \Theta_3(P_2\times P_1')=0$ [2, (32.4)] and $\tilde{\Phi}_3 A\Phi_4+\tilde{\Phi}_4 A\Phi_3=0$ follows.

3. FREUDENTHAL proved that a point A and a symplecton Φ are incident if and only if $\tilde{\Phi}A=0$ (or $A\tilde{\Phi}=0$) [2, §§ 54 and 59]. The following proposition is an analogy for half incidence.

Proposition 3.1: Let A be a point and Φ be a symplecton, then the following conditions are equivalent:

(a) The point A is half incident with the symplecton Φ.
(b) $\tilde{\Phi}A\tilde{\Phi}=0$.
(c) $\langle A\tilde{\Phi}\Phi_3, A\tilde{\Phi}\Phi_4\rangle=0$ for all Φ_3, Φ_4 in $\mathfrak{R}_4$.

Proof: (a) → (b). Put $A=\langle\Phi_1, \Phi_2\rangle$, $[\Phi_1, \Phi_2]=0$. Then, $\tilde{\Phi}A\tilde{\Phi}\Phi^*= =A\tilde{\Phi}^2\Phi^*+(\tilde{\Phi}A-A\tilde{\Phi})\tilde{\Phi}\Phi^*=(\Phi, \Phi^*)A\Phi+[\tilde{\Phi}, A]\tilde{\Phi}\Phi^*=0$, using [2, § 69]. Hence (b) follows.

(b) → (a). Using the transitivity of metasymplectic group, we may assume that $\Phi=\ulcorner\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\urcorner$.

Put $A=\langle\Phi_1, \Phi_2\rangle$, $\Phi_i=\ulcorner\begin{pmatrix}\Theta_i+\gamma_i & \underline{\delta}_i \\ \bar{\delta}_i & \Theta_i-\gamma_i\end{pmatrix}, \begin{pmatrix}P_i \\ P_i'\end{pmatrix}\urcorner$, $i=1, 2$, and assume

(b)′ $\tilde{\Phi}A\tilde{\Phi}\Phi^*=0$ for all Φ^* in $\mathfrak{R}_4$.

In the condition (b)′, let Φ^* is one of the form $\ulcorner\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 \\ 0\end{pmatrix}\urcorner$.

After a straight forward calculation we have $\tilde{\Phi}A\tilde{\Phi}\Phi^*=\ulcorner\begin{pmatrix}\gamma & \underline{\delta} \\ 0 & -\gamma\end{pmatrix}, \begin{pmatrix}P \\ 0\end{pmatrix}\urcorner$, where,

$$\gamma=2\bar{\delta}_1\gamma_2-2\bar{\delta}_2\gamma_1-\tfrac{1}{4}\{P_1', P_2'\}, \tag{3.1}$$

$$\underline{\delta}=-4\underline{\delta}_1\bar{\delta}_2-4\underline{\delta}_2\bar{\delta}_1+\tfrac{1}{4}\{P_1, P_2'\}+\tfrac{1}{4}\{P_2, P_1'\}-8\gamma_1\gamma_2, \tag{3.2}$$

$$P=(\Theta_1+\gamma_1)P_2'+2\gamma_2P_1'-2\bar{\delta}_2P_1-\bar{\delta}_1P_2. \tag{3.3}$$

From (b)′ it follows $\gamma=\underline{\delta}=0$ and $P=0$. Since $[\Phi_1, \Phi_2]=0$ we have $\tilde{\Phi}_1\tilde{\Phi}_2\Phi=$ $=\tilde{\Phi}_2\tilde{\Phi}_1\Phi$, by comparing the 1–2-element of the left matrix of $\tilde{\Phi}_1\tilde{\Phi}_2\Phi$ and of $\tilde{\Phi}_2\tilde{\Phi}_1\Phi$, we obtain the relation

$$2\underline{\delta}_1\bar{\delta}_2-\tfrac{1}{4}\{P_1, P_2'\}=2\underline{\delta}_2\bar{\delta}_1-\tfrac{1}{4}\{P_2, P_1'\}.$$

Using this relation, the above expression (3.2) can be rewritten as

$$\underline{\delta}=-6\underline{\delta}_1\bar{\delta}_2-2\underline{\delta}_2\bar{\delta}_1+\tfrac{1}{2}\{P_1, P_2'\}-8\gamma_1\gamma_2. \tag{3.2$'$}$$

Let us next assume that $\Phi^*=\ulcorner\begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 \\ P^{*\prime}\end{pmatrix}\urcorner$, then

$$\tilde{\Phi}A\tilde{\Phi}\Phi^*=\ulcorner\begin{pmatrix}0 & * \\ 0 & 0\end{pmatrix}, \begin{pmatrix}Q \\ 0\end{pmatrix}\urcorner,$$

where,

$$\left\{\begin{aligned} Q=&\bar{\delta}_1(\Theta_2+\gamma_2)P^{*\prime}+\bar{\delta}_2(\Theta_1-\gamma_1)P^{*\prime} \\ &+\tfrac{1}{4}\{P_1', P^{*\prime}\}P_2'+\tfrac{1}{8}\{P_2', P^{*\prime}\}P_1'+(P^{*\prime}\times P_2')P_1'. \end{aligned}\right. \tag{3.4}$$

Using [2, (32.5)] with $P=P_2'$, $P_1=P^{*\prime}$, $P_2=P_1'$, and a relation $\tfrac{1}{8}\{P_1', P_2'\}=$ $=\bar{\delta}_1\gamma_2-\bar{\delta}_2\gamma_1$, which follows from (3.1), we obtain

$$\begin{aligned}(P^{*\prime}\times P_2')P_1'=&(P_1'\times P_2')P^{*\prime}+(\gamma_1\bar{\delta}_2-\gamma_2\bar{\delta}_1)P^{*\prime}+ \\ &+\tfrac{1}{4}\{P^{*\prime}, P_1'\}P_2'+\tfrac{1}{8}\{P^{*\prime}, P_2'\}P_1'.\end{aligned}$$

Hence, (3.4) can be rewritten as:

$$Q=(\bar{\delta}_1\Theta_2+\bar{\delta}_2\Theta_1)P^{*\prime}+(P_1'\times P_2')P^{*\prime}. \tag{3.4$'$}$$

Now, $A\Phi$ has the form

$$A\Phi=\ulcorner\begin{pmatrix}\Theta-\tfrac{1}{2}\gamma & -\tfrac{1}{2}\underline{\delta} \\ 0 & \Theta+\tfrac{1}{2}\gamma\end{pmatrix}, \begin{pmatrix}-P \\ 0\end{pmatrix}\urcorner,$$

where $\Theta=\bar{\delta}_1\Theta_2+\bar{\delta}_2\Theta_1+P_1'\times P_2'$.

Above relations (3.1), (3.2)′, (3.3) and (3.4)′ show that the assumption (b)′ implies $\gamma=\underline{\delta}=0$, $P=0$, $\Theta=0$, and it follows $A\Phi=0$.

(b) $\leftrightarrow$ (c). From [2, § 52],

$$\langle A\tilde{\Phi}\Phi_3, A\tilde{\Phi}\Phi_4\rangle = -\tfrac{1}{2}(A\tilde{\Phi}\Phi_3, \tilde{\Phi}\Phi_4)\, A = \tfrac{1}{2}(\tilde{\Phi}A\tilde{\Phi}\Phi_3, \Phi_4)\, A.$$

Hence it follows that (b) is equivalent to (c).

Remark: (a) $\rightarrow$ (b) in Proposition 3.1 is also proved, without using [2, p. 481 69.1 $\rightarrow$ 69.2] explicitly, as follows. From [2, § 72, p. 482 (*)], $A\Phi=0$ implies $\tilde{\Phi}_1\tilde{\Phi}_2\Phi=0$. Using Jacobi-associativity

$$[[\Phi, \Phi_1], [\Phi_2, [\Phi, \Phi^*]]] + [\Phi_2, [[\Phi, \Phi^*], [\Phi, \Phi_1]]] + [[\Phi, \Phi^*], [\Phi_2, [\Phi_1, \Phi]]] = 0,$$

or $\widetilde{[\Phi, \Phi_1]}\,\tilde{\Phi}_2\tilde{\Phi}\Phi^* + \tilde{\Phi}_2\widetilde{[\Phi, \Phi^*]}\,\tilde{\Phi}\Phi_1 = 0$. Therefore,

$$\begin{aligned}\tilde{\Phi}A\tilde{\Phi}\Phi^* &= A\tilde{\Phi}^2\Phi^* + [\tilde{\Phi}, A]\,\tilde{\Phi}\Phi^* \\ &= \langle\tilde{\Phi}\Phi_1, \Phi_2\rangle\,\tilde{\Phi}\Phi^* + \langle\Phi_1, \tilde{\Phi}\Phi_2\rangle\,\tilde{\Phi}\Phi^* \quad [2,\ (35.4),\ \S\ 37] \\ &= \tilde{\Phi}_2\widetilde{[\Phi, \Phi_1]}\,\tilde{\Phi}\Phi^* + \tilde{\Phi}_1\widetilde{[\Phi, \Phi_2]}\,\tilde{\Phi}\Phi^* \\ &\qquad - \tfrac{1}{2}(\Phi_2, \tilde{\Phi}\Phi^*)\,\tilde{\Phi}\Phi_1 - \tfrac{1}{2}(\Phi_1, \tilde{\Phi}\Phi^*)\,\tilde{\Phi}\Phi_2 \\ &= 0,\end{aligned}$$

because of

$$\widetilde{[\Phi, \Phi_i]}\,\tilde{\Phi}\Phi^* = -[\tilde{\Phi}^2\Phi_i, \Phi^*] - \tilde{\Phi}\tilde{\Phi}^*\tilde{\Phi}\Phi_i = -\tfrac{1}{2}(\tilde{\Phi}\Phi^*, \Phi_i)\,\Phi + \tfrac{1}{2}(\Phi, \Phi^*)\,\tilde{\Phi}\Phi_i$$

$(i=1, 2)$ [2, § 40]. Thus $\tilde{\Phi}A\tilde{\Phi}=0$ follows.

Proposition 3.2: Let A be a point and Φ be a symplecton, then it holds

$$A\tilde{\Phi}A\Phi^* = 0 \text{ for all } \Phi^* \text{ in } \mathfrak{R}_4.$$

Proof: Put $A=\langle\Phi_1, \Phi_2\rangle$, $[\Phi_1, \Phi_2]=0$. Since the metasymplectic group operates transitively on $\mathfrak{R}_4$, we may assume that $\Phi_2 = \left\ulcorner\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\right\urcorner$. Then Φ_1 is one of the form

$$\left\ulcorner\begin{pmatrix}\Theta_1 & \underline{\delta}_1\\ 0 & \Theta_1\end{pmatrix}, \begin{pmatrix}P_1\\ 0\end{pmatrix}\right\urcorner, \text{ and } P_1\times P_1 = 2\underline{\delta}_1\,\Theta_1,\ \Theta_1 P_1 = 0,$$

$$P_1\times\Theta_1 P^* = \tfrac{1}{4}\{P_1, P^*\}\,\Theta_1,\ \Theta_1\,\Theta^* P_1 = \frac{1}{4\varepsilon}(\mathrm{sp}\ \Theta_1\,\Theta^*)\,P_1,\ \Theta_1{}^2 = 0,$$

$\Theta_1\,\Theta^*\,\Theta_1 = \dfrac{1}{4\varepsilon}(\mathrm{sp}\ \Theta_1\,\Theta^*)\,\Theta_1$ [2, p. 462–463].

For any $\Phi^* = \left\ulcorner\begin{pmatrix}\Theta^*+\gamma^* & \underline{\delta}^*\\ \delta^* & \Theta^*-\gamma^*\end{pmatrix}, \begin{pmatrix}P^*\\ P^{*\prime}\end{pmatrix}\right\urcorner$ in $\mathfrak{R}_4$, we have

$$A\Phi^* = \left\ulcorner\begin{pmatrix}\delta^*\,\Theta_1 & \frac{1}{4\varepsilon}\mathrm{sp}\ \Theta_1\,\Theta^*\\ 0 & \delta^*\,\Theta_1\end{pmatrix}, \begin{pmatrix}\Theta_1 P^{*\prime}\\ 0\end{pmatrix}\right\urcorner,$$

and using $\Theta_1{}^2=0$, $A\tilde{\Phi}A\Phi^*$ has the expression:

$$\left\ulcorner\begin{pmatrix} 0 & \frac{1}{4\varepsilon}(\bar{\delta}^* \operatorname{sp} \Theta_1[\Theta, \Theta_1] - \operatorname{sp} \Theta_1(\Theta_1 P^{*\prime} \times P')) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\urcorner$$

in which the 1–2-element of left matrix vanishes, because in the first term

$$\operatorname{sp} \Theta_1[\Theta, \Theta_1] = \operatorname{sp} \Theta_1 \Theta \Theta_1 - \operatorname{sp} \Theta_1{}^2 \Theta = \frac{1}{4\varepsilon} (\operatorname{sp} \Theta_1 \Theta) \operatorname{sp} \Theta_1 = 0$$

since $\Theta_1 \in \operatorname{Inv}(\mathfrak{M})$ [1, p. 229], and the second term is also equal 0 since it can be written as $\frac{1}{4}\{\Theta_1{}^2 P^{*\prime}, P'\}$ [2, p. 458 (31.3.1)], Θ being an element of Φ. Therefore it follows $A\tilde{\Phi}A\Phi^* = 0$.

Corollary: For a point A and a symplecton Φ, it holds $A[\tilde{\Phi}, A] = 0$ and $[\tilde{\Phi}, A]A = 0$.

Remark: In [2, § 69], the conditions for a point and a symplecton to be half incident are stated in dual form, but it does not hold the duality for the condition $\tilde{\Phi}A\tilde{\Phi} = 0$ in Proposition 3.1.

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