Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
48 (2008) 667–669
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

Foreword

Special Issue on Choquet integration in honor of Gustave Choquet (1915–2006) ☆

There are many functionals on function spaces, which are called integrals. We will confine this issue to the non-linear Choquet integral for monotone measures. In this foreword, we point out some of its basic ideas and where the articles of this issue are positioned in the building of Choquet integration. But first let us say some words about its naming.

Whereas Lebesgue's classical integration theory is based on σ-additive measures, there have been made, during the second half of the 20th century and often independently, many attempts to depart from additivity of the measure, guided by requirements of different applications. One of the earliest and also deepest and most comprehensive of these approaches had been worked out by Gustave Choquet in his 1953/54 paper [2]. He died November 14 last year in Lyon at the age of 91, so this is a good time and place to appreciate those parts of his scientific work related to this issue[1] and to see why, nowadays and with full justification, the integral w.r.t. a monotone non-additive measure is called Choquet integral.

In fact, Choquet's famous paper [2], comprising 165 pages, is a monograph rather than an ordinary article. It contains the essentials of non-additive measure theory, especially the theory of ∞-alternating set functions and their dual, totally monotone ones, which later have been called belief functions as well. In this context he investigated, in its dual version, the Möbius transform of a non-additive measure under topological assumptions.[2] These parts of [2] are written in a very general setting, so reading the paper costs some effort (the first paper in this issue is the outcome of such an effort).

Choquet's motivation for introducing non-additive measures and their integrals had been potential theory. Electrostatic capacities of bodies in 3-dimensional space being non-additive measures, the latter are often named *capacities* as Choquet did (under topological requirements due to the context of his research). We prefer the name *monotone set function* or *non-additive measure*. The name *fuzzy measure* is also in use.

In classical integration theory (or likewise probability theory) the integral (or expectation) of a real function (or random variable) $f \geqslant 0$ can be characterized as the Riemann integral of the respective decreasing distribution function. Choquet's core idea of generalizing classical integration on p. 265 of [2] is to adopt this view and to omit those properties of the set function which are not necessary to define distribution functions. Therefore, the Choquet integral $\int f, \mathrm{d}\mu$ of $f \geqslant 0$ is defined as the Riemann integral $\int_0^\infty \mu(f \geqslant x)\mathrm{d}x$ of the decreasing distribution function w.r.t. μ, a monotone but not necessarily σ-additive or additive set function. Monotonicity alone guarantees existence of the integral. The Choquet integral is fully characterized by monotonicity and

☆ The authors thank A. Chateauneuf, J.-Y. Jaffray and U. Krause for their support concerning the appreciation of Choquet.

[1] For other aspects of his work, see [5] and the papers announced there.

[2] The relation with the number theoretic Möbius function was pointed out only in 1976 by G. Shafer based on Rota's 1964 generalization of the Möbius function in combinatorics and Dempster's 1967 work on upper and lower probabilities. It is also interesting that Shapley at the same time as [2] developed the theory of cooperative games and a cooperative game can be perceived as a non-additive measure as well. The Möbius transform appears here as the coefficients of the linear combination of the game by unanimity games. But integrals w.r.t. games are playing no role there.

0888-613X/$ - see front matter
doi:10.1016/j.ijar.2007.05.008

comonotonic additivity.[3] For further properties of the integral respective assumptions have to be made about the set function μ. For example, to get a subadditive integral μ has to be submodular or strongly subadditive as Choquet calls it in [2]. Similarly, convergence theorems need σ-continuity of μ.

In economic or psychological decision theory one needs non-linear functionals on random variables (lotteries) and, in this context and about 30 years after Choquet, the Choquet integral had been invented anew and independently by Quiggin, Schmeidler and Yaari to overcome the shortcomings of expected utility as a decision functional. In the last decades the increasing research in this context and related ones like finance and insurance contributed much to popularize the Choquet integral.

Occasionally, the Choquet integral had been blamed for not being defined uniquely for functions assuming positive *and* negative values. Perhaps for this reason Choquet itself defined it in [2] only for functions $f \geqslant 0$ as explained above. There exist two extensions of Choquet's original integral to arbitrary functions, an asymmetric one, which is only positively homogenous but comonotonic additive, and a symmetric extension, which is fully homogenous but not fully comonotonic additive in general. Forthermore, for infinite μ only the latter one makes sense. In decision theory, the symmetric extension to arbitrary functions seems to be suited to model human behavior under risk or uncertainty if gains *and* losses are involved.

As Choquet emphasized in [2] totally monotone normalized set functions (belief functions), are an important class of monotone set functions. By means of the combinatorial Möbius operator and its inverse, the zeta operator, they can be transformed into a measure on a larger set, thus transforming the Choquet integral in an additive integral of the naturally extended integrands (but, of course, the extension of the integrand is a non-linear operator). **Brüning and Denneberg** derive these well known facts from Choquet's Integral Representation Theorem with a concise proof extracted from the dual setting in [2]. Thus, the other most prominent result of Choquet's research, his generalizations of the Krein-Milman Theorem,[4] are useful also for non-additive integration, that is for Choquet integration theory.

For non-continuous totally monotone μ the problem arises if it can be decomposed into the sum of a σ-continuous and a purely non-continuous monotone set function. This analogue of the famous Yoshida–Hewitt Theorem for finitely additive measures is an open theoretical problem. In this issue **Rébillé** starts a further approach with a suitable topological structure of the basic set and a weaker continuity condition than in Yosida–Hewitt's Theorem.

Interchangeability of limit processes play important roles in many parts of mathematics. A fundamental theorem of this type is Fubini's Theorem, which – in case of monotone (non-additive) measures on algebras without continuity conditions – had been adapted by Ghirardato. Slice-comonotonicity of a function f on a cartesian product $\Omega_1 \times \Omega_2$ is a sufficient and necessary condition for Fubini's Theorem to hold for f w.r.t. all independent products of monotone measures (which are not unique in the non-additive case). **Chateauneuf and Lefort** extend this result – under suitable assumptions – to functions f measurable w.r.t. the product σ-algebra, i.e., to a larger class of functions than Ghirardato. The example of continuous belief functions on $\mathbb{N}$ is treated in all details with a uniqueness result for the independent product on $\mathbb{N} \times \mathbb{N}$.

Modelling risk and uncertainty had been the main incentive for the increasing research on Choquet integration. The risk case, where a basic probability measure P is given, occurs in finance and insurance. The risk functionals (called *risk measure* or *risk value*) considered in this context comprise as special case the Choquet integrals w.r.t. a transformed probability $\mu = \gamma \circ P$ with a convex or concave distribution function γ on the unit interval. The corresponding integrals generate variability parameters $|\int f \mathrm{d}\mu - \int f dP|$. Examples are average absolute deviation and Gini index. **Denneberg and Leufer** elaborate this context and propose order based correlation parameters analogous to covariance. A popular tool for studying dependence of random variables is the copula of the common distribution. It is invariant under isotonic transforms of the margins, so it is related to order like the comonotonic additive Choquet integral, comonotonicity being an ordinal concept as

[3] We don't know, if Choquet – in the early days of non-additive integration – was aware of this fact. But the first paper, in 1971, having shown the importance of comonotonic additivity for non-additive integration is that of Dellacherie, a student of Choquet. Comonotonicity was called there 'même tableau de variations'.

[4] There are several versions of Choquet's Theorem, one being included in Bourbaki's *Éléments de Mathémathique* [1] Chap IV, § 7. See [4] or Choquet's own detailed exposition in [3].

well. Denneberg and Leufer apply their new parameters to the copula, thus finding interesting new concordance parameters analogous to Spearman's rho.

Choquet integration deserves to be integrated in basic academic teaching and offers open problems for further research, theoretical ones and in applications. We hope that the present issue will contribute to this end.

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Available online 4 July 2007