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Cellular structures on Hecke algebras of type B

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Abstract

The aim of this paper is to gather and (try to) unify several approaches for the modular representation theory of Hecke algebras of type B. We attempt to explain the connections between Geck's cellular structures (coming from Kazhdan–Lusztig theory with unequal parameters) and Ariki's Theorem on the canonical basis of the Fock spaces.

Keywords: Representation theory; Hecke algebras; Kazhdan–Lusztig theory; Cellular algebras

Contents

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Introduction

The modular representation theory of Hecke algebras of type B was first studied by Dipper, James and Murphy [10]: one of their essential tools was to construct a family of modules (called *Specht modules*) playing the same role as Specht modules in type A. Each of these new Specht modules have a canonical quotient which is zero or simple: one of the main problem raised by this construction is to determine which ones are non-zero. Later, Graham and Lehrer [20] developed the theory of *cellular algebras*, which contains, as a particular case, the construction of Dipper–James–Murphy. The problem of parametrizing the simple modules and computing the decomposition matrix of Specht modules were then solved by Ariki [1] using the canonical basis of Fock spaces of higher level. In fact, Ariki's Theorem provides different parametrizations of the simple modules of the Hecke algebra: only one of them (*asymptotic case*) has an interpretation in the framework of Dipper–James–Murphy and Graham–Lehrer. Recently, Geck showed that the Kazhdan–Lusztig theory with unequal parameters should provide a cell datum for each choice of a *weight function* on the Weyl group (if Lusztig's Conjectures (P1)–(P15) hold [30, Conjecture 14.2]).

Our main aim in this paper is to present an overview of all these results, focusing particularly on conjectural connections between Uglov's point of view on the Fock space theory and Geck cellular structures. This should (if Lusztig's Conjectures (P1)–(P15) hold in type B) lead to a unified approach for a better understanding of the representation theory of Hecke algebras. As a by-product, we should get an interpretation of *all* Ariki's parametrizations of simple modules.

More precisely, if Q and q are two indeterminates, if $\mathcal{H}_n$ denotes the Hecke A-algebra with parameters Q and q (here, $A = \mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$), if ξ is a positive irrational number (!) and if r denotes the unique natural number such that $r \leqslant \xi < r + 1$, then Kazhdan–Lusztig theory *should* provide a cell datum $\mathcal{C}^\xi = ((\mathrm{Bip}(n), \trianglelefteq_r), \mathcal{SBT}, C^\xi, *)$ where

- $\mathrm{Bip}(n)$ is the set of bipartitions of n and $\trianglelefteq_r$ is a partial order on $\mathrm{Bip}(n)$ depending on r (see Section 3.2).
- If $\lambda \in \mathrm{Bip}(n)$, $\mathcal{SBT}(\lambda)$ denotes the set of standard bitableaux of (bi-)shape λ (filled with $1, \ldots, n$).

- If S and T are two standard bitableaux of size n and of the same shape, $C^{\xi}_{S,T}$ is an element of $\mathcal{H}_n$ coming from a Kazhdan–Lusztig basis of $\mathcal{H}_n$ (it heavily depends on ξ).
- $*: \mathcal{H}_n \to \mathcal{H}_n$ is the A-linear anti-involution of $\mathcal{H}_n$ sending the element T_w of the standard basis to $T_{w^{-1}}$

(see [6, Conjecture C]). If this conjecture holds then, by the general theory of cellular algebras, we can associate to each bipartition λ of n a *Specht module* S^{ξ}_{λ} endowed with a bilinear form ϕ^{ξ}_{λ}. If K is the field of fractions of A, then $KS^{\xi}_{\lambda} = K \otimes_A S^{\xi}_{\lambda}$ is the simple $K\mathcal{H}_n$-module associated to λ [19, Theorem 10.1.5]. Now, if Q_0 and q_0 are two elements of $\mathbb{C}^{\times}$ then, through the specialization $Q \mapsto Q_0$, $q \mapsto q_0$, we can construct the $\mathbb{C}\mathcal{H}_n$-module

$$D^{\xi}_{\lambda} = \mathbb{C}S^{\xi}_{\lambda}/\operatorname{Rad}\bigl(\mathbb{C}\phi^{\xi}_{\lambda}\bigr).$$

By the general theory of cellular algebras, it is known that the non-zero D^{ξ}_{λ} give a set of representatives of simple $\mathbb{C}\mathcal{H}_n$-modules. At this stage, it must be noticed that, even if $KS^{\xi}_{\lambda} \simeq KS^{\xi'}_{\lambda}$ (where ξ' is another positive irrational number), it might happen that $S^{\xi}_{\lambda} \not\simeq S^{\xi'}_{\lambda}$ and $D^{\xi}_{\lambda} \not\simeq D^{\xi'}_{\lambda}$ (it is probable that $(S^{\xi}_{\lambda}, \phi^{\xi}_{\lambda}) \simeq (S^{\xi'}_{\lambda}, \phi^{\xi'}_{\lambda})$ for all $\lambda \in \operatorname{Bip}(n)$ if and only if we have also $r \leqslant \xi' < r+1$).

On the other hand, if we assume further that q_0^2 is a primitive eth root of unity, if $Q_0^2 = -q_0^{2d}$ for some $d \in \mathbb{Z}$ (which is only well-defined modulo e), and if $s = (s_0, s_1) \in \mathbb{Z}^2$ is such that $s_0 - s_1 \equiv d \bmod e$, then Ariki's Theorem provides a bijection between the set of Uglov's bipartitions $\operatorname{Bip}^s_e(n)$ and the set of simple $\mathbb{C}\mathcal{H}_n$-modules. Moreover, the decomposition matrix is given by $(d^s_{\lambda\mu}(1))_{\lambda \in \operatorname{Bip}(n), \mu \in \operatorname{Bip}^s_e(n)}$, where $(d^s_{\lambda\mu}(q))_{\lambda,\mu \in \operatorname{Bip}(n)}$ is the transition matrix between the standard basis and Uglov–Kashiwara–Lusztig's canonical basis of the Fock space (see Section 2.2). The first result of this paper insures that [6, Conjecture C] is "compatible" with Ariki's Theorem in the following sense:

Theorem. *Assume that* [6, Conjecture C] *holds and assume that* $s_0 - s_1 \equiv d \bmod e$ *and* $s_0 - s_1 \leqslant r < s_0 - s_1 + e$*. Then* $D^{\xi}_{\lambda} \neq 0$ *if and only* $\lambda \in \operatorname{Bip}^s_e(n)$ *and, if* $\lambda \in \operatorname{Bip}(n)$ *and* $\mu \in \operatorname{Bip}^s_e(n)$*, then*

$$\bigl[\mathbb{C}S^{\xi}_{\lambda} : D^{\xi}_{\mu}\bigr] = d^s_{\lambda\mu}(1).$$

In particular, we have

$$\bigl[\mathbb{C}S^{\xi}_{\lambda} : D^{\xi}_{\mu}\bigr] \neq 0 \quad \Rightarrow \quad \lambda \trianglelefteq_r \mu$$

Note that, in the *asymptotic case* (in other words, if $\xi > n-1$), then [6, Conjecture C] holds (see [16]) and the cellular datum $\mathcal{C}^{\xi}$ is more or less equivalent to the one constructed by Dipper, James and Mathas (see the work of Geck, Iancu and Pallikaros [18]).

One of the problems raised by the previous theorem (in fact, essentially by Ariki's Theorem) is the following: if $s' = (s'_0, s'_1) \in \mathbb{Z}^2$ is such that $s'_0 - s'_1 \equiv s_0 - s_1 \bmod e$, then the sets $\operatorname{Bip}^s_e(n)$ and $\operatorname{Bip}^{s'}_e(n)$ are in bijection. Our second result (see Theorem 5.3) is to construct this bijection by means of an isomorphism between the crystals associated to the simple sub-modules $\mathcal{M}[s]$ and $\mathcal{M}[s']$ of the Fock spaces (see Section 2.1 for the definition of these modules). This shows that the "abstract crystal" of an irreducible highest weight module is canonically associated to the representation theory of Hecke algebras of type B. By contrast, each realization of this crystal is associated with a natural parametrization of the simple modules.

Once we accept [6, Conjecture C], it is natural to ask whether the matrix $(d^s_{\lambda\mu}(q))$ can be interpreted as a q-decomposition matrix using Jantzen's filtration (see our Conjecture C in Section 4.5). It would also be interesting to see if it should be possible to construct different *Schur algebras* of type B directly from Kazhdan–Lusztig's theory (for the asymptotic case, this construction should give rise to an algebra which is Morita equivalent to the two Schur algebras of type B constructed by Du and Scott [11] and Dipper, James and Mathas [9]). If so, it is then natural to ask for a generalization of Varagnolo–Vasserot Theorem for Hecke algebras of type A as well as a generalization of Yvonne's Conjecture [34, Conjecture 2.13]. Note that a construction of *Schur algebras* of type B is provided by the theory of Cherednik algebras [13], but this does not provide a *generic* Schur algebra.

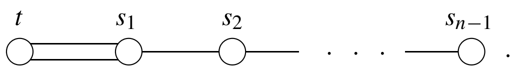
If the reader wants more arguments for [6, Conjecture C], he or she is encouraged to read the original source of this conjecture [6]. Note also that recent works by Gordon and Martino (see [21,22]) on Cherednik algebras show other compatibilities between this conjecture and the geometry of the Calogero–Moser spaces, as well as with Baby Verma modules of Cherednik algebras *at* $t = 0$.

Finally, it is natural to ask whether there exist different cellular structures for Ariki–Koike algebras associated to the complex reflection groups $G(d, 1, n)$, these cellular structures being indexed by d-cores (or by d-uples of elements of $\mathbb{Z}$): however, in this case, no Kazhdan–Lusztig's theory is available at that time (will there be one in the future?) so we have no candidate for the different cellular bases.

This paper is organized as follows. In the first section, we present the setting of our problem. Then, we recall Ariki's Theorem which allows to compute the decomposition matrices for Hecke algebras of type B_n using objects coming from quantum group theory. In the third section, we introduce the combinatorial notions that we need to describe the Kazhdan–Lusztig theory in type B_n. The last two sections are devoted to the main results of our paper. Under some conjectures, we show the existence of several deep connections between Kazhdan–Lusztig theory and the canonical basis theory for $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$.

1. Notation

Let (W_n, S_n) be a Weyl group of type B_n and assume that the elements of S_n are denoted by $t, s_1, \dots, s_{n-1}$ in such a way that the Dynkin diagram is given by

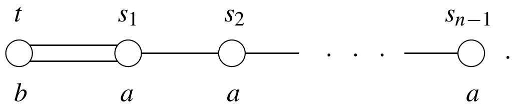


The length function with respect to S_n will be denoted by $\ell : W_n \to \mathbb{N} = \{0, 1, 2, \dots\}$.

Let Γ be a torsion-free finitely generated abelian group and let A denote the group ring $\mathbb{Z}[\Gamma]$. The group law on Γ will be denoted additively and we shall use an exponential notation for A: more precisely,

$$A = \bigoplus_{\gamma \in \Gamma} \mathbb{Z}e^{\gamma},$$

where $e^{\gamma} e^{\gamma'} = e^{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$. We also fix two elements a and b in Γ and we denote by $\mathcal{H}_n$ the Hecke algebra of W_n over A associated to the choice of parameters $t \mapsto b$ and $s_i \mapsto a$ for $1 \leqslant i \leqslant n-1$ symbolized by the following diagram:



More precisely, there exists a basis $(T_w)_{w\in W_n}$ of the A-module $\mathcal{H}_n$ such that the multiplication on $\mathcal{H}_n$ is A-bilinear and completely determined by the following properties:

$$\begin{cases} T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ \big(T_t - e^b\big)\big(T_t + e^{-b}\big) = 0, & \\ \big(T_{s_i} - e^a\big)\big(T_{s_i} + e^{-a}\big) = 0, & \text{if } 1 \leqslant i \leqslant n-1. \end{cases}$$

All along this paper, we also fix a field k of characteristic 0 and a morphism of groups $\theta : \Gamma \to k^\times$: the morphism θ extends uniquely to a morphism of rings $\mathbb{Z}[\Gamma] \to k$ that we still denote by θ. We then set

$$k\mathcal{H}_n = k \otimes_A \mathcal{H}_n$$

and we still denote by θ the specialization morphism $\mathcal{H}_n \to k\mathcal{H}_n$. Moreover, we assume that the following holds: there exist natural numbers d and e such that $e \geqslant 1$ and

$$\begin{cases} \theta(a)^2 \text{ is a primitive } e\text{th root of unity;} \\ \theta(b)^2 = -\theta(a)^{2d}. \end{cases}$$

If M is an $\mathcal{H}_n$-module, we denote by kM the $k\mathcal{H}_n$-module $k \otimes_A M$. The Grothendieck group of $k\mathcal{H}_n$ will be denoted by $\mathcal{R}(k\mathcal{H}_n)$ and, if L is a $k\mathcal{H}_n$-module, its class in $\mathcal{R}(k\mathcal{H}_n)$ will be denoted by $[L]$. Note that the algebra $k\mathcal{H}_n$ is split.

Since Γ is torsion-free, the ring $\mathbb{Z}[\Gamma]$ is integral and we denote by K its field of fractions. Then the algebra $K\mathcal{H}_n = K \otimes_A \mathcal{H}_n$ is split semisimple. Its simple modules are parametrized by the set $\mathrm{Bip}(n)$ of bipartitions of n: we shall denote this bijection by

$$\begin{array}{ccc} \mathrm{Bip}(n) & \longrightarrow & \mathrm{Irr}\, K\mathcal{H}_n, \\ \lambda & \longmapsto & V_\lambda. \end{array}$$

This bijection is chosen as in [19, 10.1.2]. We denote by Bip the set of all bipartitions (i.e. $\mathrm{Bip} = \coprod_{n\geqslant 0} \mathrm{Bip}(n)$), the empty partition will be denoted by $\emptyset$ and the empty bipartition $(\emptyset,\emptyset)$ (which is the unique bipartition in $\mathrm{Bip}(0)$) will be denoted by $\boldsymbol{\emptyset}$. Since A is integrally closed (it is a Laurent polynomial ring in several algebraically independent indeterminates), there is a well-defined decomposition map [19, Theorem 7.4.3]

$$\mathbf{d}_n : \mathcal{R}(K\mathcal{H}_n) \longrightarrow \mathcal{R}(k\mathcal{H}_n).$$

2. Fock space, canonical basis and Ariki's Theorem

The aim of this section is to recall Ariki's Theorem relating the canonical basis of the Fock space and the decomposition matrix $\mathbf{d}_n$. The main references are [2,14] and [35].

2.1. The Fock space

Let v be an indeterminate. Let $\mathfrak{h}$ be a free $\mathbb{Z}$-module with basis $(h_0, \ldots, h_{e-1}, \mathfrak{d})$ and let $(\Lambda_0, \ldots, \Lambda_{e-1}, \delta)$ be its dual basis in $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}, \mathbb{Z})$. The quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ is defined as the unital associative $\mathbb{C}(v)$-algebra generated by elements $\{e_i, f_i \mid 0 \leqslant i \leqslant e-1\}$ and $\{k_h \mid h \in \mathfrak{h}\}$ subject to the relations given for example in [31, Chapter 6]. We denote by $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})'$ the subalgebra of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ generated by $e_i, f_i, k_{h_i}, k_{h_i}^{-1}$ for $i \in \{0, 1, \ldots, e-1\}$.

We fix a pair $s = (s_0, s_1) \in \mathbb{Z}^2$. To s is associated a *Fock space* (*of level* 2) $\mathfrak{F}^s$: this is an $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module defined as follows. As a $\mathbb{C}(v)$-vector space, it has a basis given by the symbols $|\lambda, s\rangle$ where λ runs over the set of bipartitions:

$$\mathfrak{F}^s = \bigoplus_{\lambda \in \mathrm{Bip}} \mathbb{C}(v)|\lambda, s\rangle.$$

The action of the generators $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ is given for instance in [33]. By [26], $\mathfrak{F}^s$ is an *integrable* $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module.

If $m \in \mathbb{Z}$, we denote by $\bar{m}$ the unique element of $\{0, 1, \ldots, e-1\}$ such that $m \equiv \bar{m} \bmod e$. We then set

$$\Delta(s) = \frac{1}{2}\sum_{j=0}^{1} \frac{s_j - \bar{s}_j}{e}(s_j + \bar{s}_j - e).$$

Since $|\emptyset, s\rangle$ is a highest weight vector of $\mathfrak{F}^s$ and as $\mathfrak{F}^s$ is an integrable module, it follows that the submodule $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})|\emptyset, s\rangle$ generated by $|\emptyset, s\rangle$ is an irreducible module. We denote it by $\mathcal{M}[s]$ and it is isomorphic to the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with highest weight $-\Delta(s)\delta + \Lambda_{\bar{s}_0} + \Lambda_{\bar{s}_1}$. In addition, the submodule $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})'|\emptyset, s\rangle$ generated by $|\emptyset, s\rangle$ is also an irreducible highest weight module with weight $-\Delta(s)\delta + \Lambda_{\bar{s}_0} + \Lambda_{\bar{s}_1}$. We denote it by $\mathcal{M}[s]'$.

Remark 2.1. If $\mathcal{M}$ and $\mathcal{N}$ are simple $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})'$-modules with highest weight Λ and Λ' respectively, then $\mathcal{M} \simeq \mathcal{N}$ if and only if $\Lambda \equiv \Lambda' \bmod \mathbb{Z}\delta$. Therefore, if $s' = (s'_0, s'_1) \in \mathbb{Z}^2$, then $\mathcal{M}[s]' \simeq \mathcal{M}[s']'$ if and only if $(s_0, s_1) \equiv (s'_0, s'_1) \bmod e\mathbb{Z}^2$ or $(s_0, s_1) \equiv (s'_1, s'_0) \bmod e\mathbb{Z}^2$.

2.2. Uglov's canonical basis

We shall recall here Uglov's construction of a canonical basis of the Fock space, which contains as a particular case the Kashiwara–Lusztig canonical basis of the simple $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module $\mathcal{M}[s]$ (which is the same as the one of $\mathcal{M}[s]'$). The reader may refer to Uglov's original paper [33] for further details.

First, recall that there is a unique $\mathbb{C}$-linear involutive automorphism of algebra $\bar{\ }: \mathcal{U}_v(\widehat{\mathfrak{sl}_e}) \longrightarrow \mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ such that, for all $i \in \{0, 1, \ldots, e-1\}$ and $h \in \mathfrak{h}$, we have

$$\bar{v} = v^{-1}, \qquad \overline{k_h} = k_{-h}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i.$$

It is $\mathbb{C}(v)$-antilinear (with respect to the restriction of $\bar{?}$ to $\mathbb{C}(v)$).

One of the main ingredients in Uglov's construction is a $\mathbb{C}(v)$-antilinear involution

$$\bar{\ }: \mathfrak{F}^{\mathbf{s}} \longrightarrow \mathfrak{F}^{\mathbf{s}}$$

which is defined using the *wedge realization* of the Fock space. This involution is $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-antilinear, that is, for all $u \in \mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ and $f \in \mathfrak{F}^s$, we have

$$\overline{u \cdot f} = \bar{u} \cdot \overline{f}. \tag{2.2}$$

Moreover, recall that

$$\overline{|\emptyset, s\rangle} = |\emptyset, s\rangle. \tag{2.3}$$

Now, if $\lambda \in \mathrm{Bip}$, there exists a unique $G(\lambda, s) \in \mathfrak{F}^s$ satisfying

$$\begin{cases} \overline{G(\lambda, s)} = G(\lambda, s), \\ G(\lambda, s) \equiv |\lambda, s\rangle \mod v\mathbb{C}[v]. \end{cases}$$

The proof of this result is constructive and gives an algorithm to compute this canonical basis. This algorithm has been improved by Yvonne [36]. Let us write, for $\mu \in \mathrm{Bip}$,

$$G(\mu, s) = \sum_{\lambda \in \mathrm{Bip}} d^s_{\lambda,\mu}(v) |\lambda, s\rangle,$$

where $d^s_{\mu,\mu}(v) = 1$ and $d^s_{\lambda,\mu}(v) \in v\mathbb{C}[v]$ if $\lambda \neq \mu$.

Finally, recall that there exists a unique subset Bip^s_e of Bip such that

$$\big(G(\lambda, s)\big)_{\lambda \in \mathrm{Bip}^s_e} \text{ coincides with Kashiwara–Lusztig canonical basis of } \mathcal{M}[s]. \tag{2.4}$$

We set

$$\mathrm{Bip}^s_e(n) = \mathrm{Bip}^s_e \cap \mathrm{Bip}(n).$$

2.3. Ariki's Theorem

We shall recall here the statement of Ariki's Theorem, using the later construction of Uglov:

Ariki's Theorem. *Assume that $d \equiv s_0 - s_1 \mod e$. Then there exists a unique bijection*

$$\begin{array}{ccc} \mathrm{Bip}^s_e(n) & \longrightarrow & \mathrm{Irr}\, k\mathcal{H}_n, \\ \mu & \longmapsto & D^s_\mu \end{array}$$

such that

$$\mathbf{d}_n[V_\lambda] = \sum_{\mu \in \mathrm{Bip}^\sigma_e(n)} d^s_{\lambda,\mu}(1)\, \big[D^s_\mu\big]$$

for all $\lambda \in \mathrm{Bip}(n)$.

This theorem, together with the algorithm for computing the Kashiwara–Lusztig canonical bases [23], provides an efficient tool for computing the decomposition map $\mathbf{d}_n$.

3. Domino insertion

We shall review here some combinatorial results about the *domino insertion algorithm*: the reader may refer to [27,28] and [32] for further details.

3.1. Domino tableaux, bitableaux

Let $r \in \mathbb{N}$. We denote by δ_r the 2-core $(r, r-1, \dots, 1)$, with the convention that $\delta_0 = \emptyset$. Let $\mathcal{P}$ denote the set of all partitions and let $\mathcal{P}_r$ denote the set of partitions with 2-core δ_r. We denote by $\mathcal{P}_r(n)$ the set of partitions of 2-weight n and 2-core δ_r, so that

$$\mathcal{P}_r = \coprod_{n \geqslant 0} \mathcal{P}_r(n)$$

and

$$\mathcal{P} = \coprod_{r \geqslant 0} \mathcal{P}_r.$$

Note that partitions in $\mathcal{P}_r(n)$ are partitions of $2n + \frac{r(r-1)}{2}$. Let $\mathbf{q} : \mathcal{P} \to \mathrm{Bip}$ the map sending a partition to its 2-quotient. By composition with the bijection

$$\begin{array}{ccc} \mathrm{Bip} & \longrightarrow & \mathrm{Bip} \\ (\lambda^0, \lambda^1) & \longmapsto & \begin{cases} (\lambda^0, \lambda^1) & \text{if } r \text{ is even} \\ (\lambda^1, \lambda^0) & \text{if } r \text{ is odd} \end{cases} \end{array}$$

the restriction of $\mathbf{q}$ to $\mathcal{P}_r$ induces a bijection

$$\mathbf{q}_r : \mathcal{P}_r \xrightarrow{\sim} \mathrm{Bip}.$$

The restriction of $\mathbf{q}_r$ to $\mathcal{P}_r(n)$ induces a bijection

$$\mathcal{P}_r(n) \xrightarrow{\sim} \mathrm{Bip}(n)$$

that we still denote by $\mathbf{q}_r$ if no confusion may arise.

Let $\mathcal{SDT}_r(n)$ denote the set of standard domino tableaux whose shape lies in $\mathcal{P}_r(n)$ (and filled with dominoes with entries $1, 2, \ldots, n$) and let $\mathcal{SBT}(n)$ denote the set of standard bitableaux of total size n (filled again with boxes with entries $1, 2, \ldots, n$). We denote by $\tilde{\mathbf{q}}_r : \mathcal{SDT}_r(n) \to \mathcal{SBT}(n)$ the bijection obtained as a particular case of [8, Theorem 7.3]. If $\boldsymbol{\lambda} : \mathcal{SBT}(n) \to \mathrm{Bip}(n)$ (respectively $\boldsymbol{\Delta}$) sends a bitableau (respectively a domino tableau) to its shape, then the diagram

$$\begin{tikzcd} \mathcal{SDT}_r(n) \arrow[r, "\tilde{\mathbf{q}}_r", "\sim"'] \arrow[d, "\boldsymbol{\Delta}"'] & \mathcal{SBT}(n) \arrow[d, "\boldsymbol{\lambda}"] \\ \mathcal{P}_r(n) \arrow[r, "\mathbf{q}_r", "\sim"'] & \mathrm{Bip}(n) \end{tikzcd} \tag{3.1}$$

is commutative.

3.2. Orders between bipartitions

The bijection $\mathbf{q}_r$ allows us to define several partial orders on the set of bipartitions. First, let $\trianglelefteq$ denote the dominance order on $\mathcal{P}$. We then define the partial order $\trianglelefteq_r$ on Bip as follows: if λ, $\mu \in \mathrm{Bip}$, then

$$\lambda \trianglelefteq_r \mu \quad \Longleftrightarrow \quad \mathbf{q}_r^{-1}(\lambda) \trianglelefteq \mathbf{q}_r^{-1}(\mu).$$

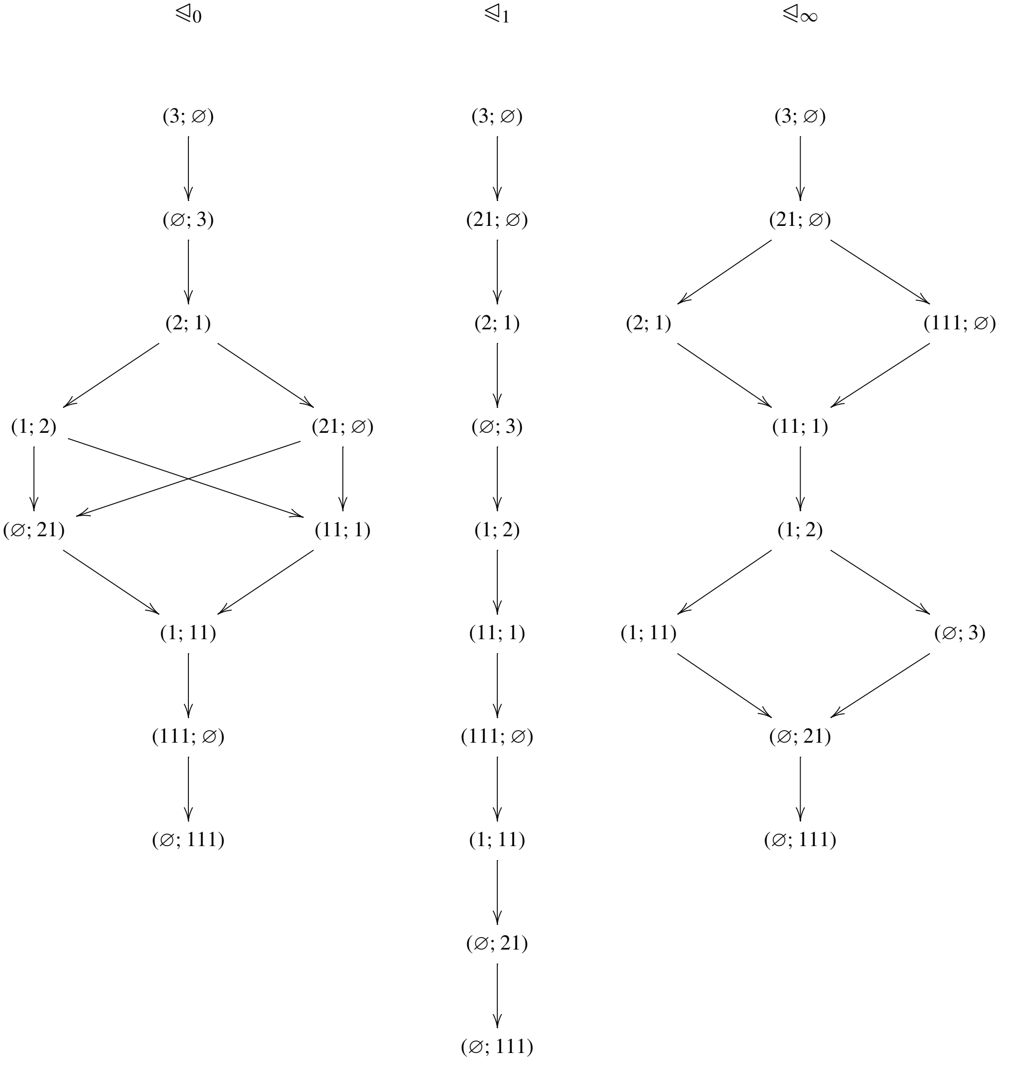
Remark 3.2. If $r \geqslant n-1$ and $s \geqslant n-1$, then the partial orders $\trianglelefteq_r$ and $\trianglelefteq_s$ coincide on $\mathrm{Bip}(n)$. In fact, they coincide with the classical dominance order on bipartitions that we shall denote by $\trianglelefteq_\infty$.

Example 3.3. Here are the orders $\trianglelefteq_0$ and $\trianglelefteq_1 = \trianglelefteq_2 = \cdots = \trianglelefteq_\infty$ on Bip(2):

$$(\emptyset; 11) \trianglelefteq_0 (11; \emptyset) \trianglelefteq_0 (1; 1) \trianglelefteq_0 (\emptyset; 2) \trianglelefteq_0 (2; \emptyset),$$

$$(\emptyset; 11) \trianglelefteq_\infty (\emptyset; 2) \trianglelefteq_\infty (1; 1) \trianglelefteq_\infty (11; \emptyset) \trianglelefteq_\infty (2; \emptyset)$$

Example 3.4. Here are the orders $\trianglelefteq_0$, $\trianglelefteq_1$ and $\trianglelefteq_2 = \trianglelefteq_3 = \cdots = \trianglelefteq_\infty$ on Bip(3). In the diagram, we put an arrow between two bipartitions $\lambda \longrightarrow \mu$ if $\mu \trianglelefteq_i \lambda$ and if there is no other bipartition ν such that $\mu \trianglelefteq_i \nu \trianglelefteq_i \lambda$.



3.3. Domino insertion algorithm

Recall that $r \in \mathbb{N}$ is fixed. If $w \in W_n$, then the *domino insertion algorithm*, as described for instance in [27], associates to w a standard domino tableau $P_r(w) \in \mathcal{SDT}_r(n)$. We set

$$S_r(w) = \tilde{\mathbf{q}}_r\bigl(P_r(w)\bigr).$$

Then $S_r(w)$ is a standard bitableau of total size n. We also set

$$Q_r(w) = P_r\bigl(w^{-1}\bigr) \quad \text{and} \quad T_r(w) = S_r\bigl(w^{-1}\bigr) = \tilde{\mathbf{q}}_r\bigl(Q_r(w)\bigr).$$

It turns out that $P_r(w)$ and $Q_r(w)$ have the same shape, as well as $S_r(w)$ and $T_r(w)$. Also, if we set

$$\mathcal{SBT}^{(2)}(n) = \{(S,T) \in \mathcal{SBT}(n) \times \mathcal{SBT}(n) \mid \boldsymbol{\lambda}(S) = \boldsymbol{\lambda}(T)\},$$

then the map

$$\begin{array}{rcl} W_n & \longrightarrow & \mathcal{SBT}^{(2)}(n), \\ w & \longmapsto & \big(S_r(w), T_r(w)\big) \end{array} \tag{3.5}$$

is bijective. If $w \in W_n$, then we set

$$\boldsymbol{\lambda}_r(w) = \boldsymbol{\lambda}\big(S_r(w)\big) \in \mathrm{Bip}(n).$$

Note that

$$\boldsymbol{\lambda}_r(w) = \boldsymbol{\lambda}\big(T_r(w)\big) = \boldsymbol{\lambda}_r\big(w^{-1}\big). \tag{3.6}$$

Remark 3.7. If r, $r' \geqslant n-1$, then the bijection $W_n \to \mathcal{SBT}^{(2)}(n)$, $w \mapsto (S_{r'}(w), T_{r'}(w))$ coincides with the bijection 3.5. We shall denote it by $W_n \to \mathcal{SBT}^{(2)}(n)$, $w \mapsto (S_\infty(w), T_\infty(w))$. Similarly, we shall set $\boldsymbol{\lambda}_\infty(w) = \boldsymbol{\lambda}(S_\infty(w))$.

4. Kazhdan–Lusztig theory

The aim of this section is to show that Kazhdan–Lusztig theory *with unequal parameters* should provide cellular data on $\mathcal{H}_n$ which are compatible with Ariki's Theorem. The reader may refer to [30] for the foundations of this theory.

We denote by $\mathcal{H}_n \to \mathcal{H}_n$, $h \mapsto \overline{h}$ the unique automorphism of ring such that $\overline{e^\gamma} = e^{-\gamma}$ and $\overline{T}_w = T_{w^{-1}}^{-1}$ for all $\gamma \in \Gamma$ and $w \in W_n$. It is an A-antilinear involutive automorphism of the ring $\mathcal{H}_n$.

4.1. Kazhdan–Lusztig basis

From now on, we fix a total order $\leqslant$ on Γ which endows Γ with a structure of ordered group:

$$A_{\leqslant 0} = \bigoplus_{\gamma \leqslant 0} \mathbb{Z} e^\gamma \quad \text{and} \quad A_{<0} = \bigoplus_{\gamma < 0} \mathbb{Z} e^\gamma.$$

Then $A_{\leqslant 0}$ is a subring of A and $A_{<0}$ is an ideal of $A_{\leqslant 0}$. We also set

$$\mathcal{H}_n^{<0} = \bigoplus_{w \in W_n} A_{<0} T_w.$$

It is a sub-$A_{\leqslant 0}$-module of $\mathcal{H}_n$.

If $w \in W_n$, then [30, Theorem 5.2] there exists a unique element $C_w^{\leqslant} \in \mathcal{H}_n$ such that

$$\begin{cases} \overline{C_w^{\leqslant}} = C_w^{\leqslant}; \\ C_w^{\leqslant} \equiv T_w \mod \mathcal{H}_n^{<0}. \end{cases}$$

Also, $(C_w^{\leqslant})_{w \in W_n}$ is an A-basis of $\mathcal{H}_n$ and this basis depends on the choice of the order $\leqslant$ on Γ. We define the preorders $\leqslant_{\mathcal{L},\leqslant}$, $\leqslant_{\mathcal{R},\leqslant}$ and $\leqslant_{\mathcal{LR},\leqslant}$ on W_n as in [30, Section 8.1] and we denote by $\sim_{\mathcal{L},\leqslant}$, $\sim_{\mathcal{R},\leqslant}$ and $\sim_{\mathcal{LR},\leqslant}$ the equivalence relations respectively associated to these preorders. Here, the exponents or the indices $\leqslant$ stand for emphasizing the fact that the objects really depends on the order $\leqslant$ on Γ.

4.2. Bonnafé–Geck–Iancu–Lam conjectures

From now on, we shall assume that the parameters a, b in Γ are strictly positive and that $b \notin \{a, 2a, \dots, (n-1)a\}$.

If $b < (n-1)a$, then we denote by r the unique natural number such that $ra < b < (r+1)a$. If $b > (n-1)a$, then we may choose for r any value in $\{n-1, n, n+1, \dots\} \cup \{\infty\}$.

Note that the flexibility on the choice of r whenever $b > (n-1)a$ will not change the statements below (see Remarks 3.2 and 3.7). Note also that, once a and b are fixed elements of Γ, then the number r depends only on the choice of the order $\leqslant$ on Γ (again with some flexibility if $b > (n-1)a$).

Conjecture A. *With the above definition of r, we have, for all w, $w' \in W_n$:*

(a) $w \sim_{\mathcal{L},\leqslant} w'$ *if and only if* $T_r(w) = T_r(w')$.
(b) $w \sim_{\mathcal{R},\leqslant} w'$ *if and only if* $S_r(w) = S_r(w')$.
(c) $w \sim_{\mathcal{LR},\leqslant} w'$ *if and only if* $\boldsymbol{\lambda}_r(w) = \boldsymbol{\lambda}_r(w')$.
(c^+) $w \leqslant_{\mathcal{LR},\leqslant} w'$ *if and only if* $\boldsymbol{\lambda}_r(w) \trianglelefteq_r \boldsymbol{\lambda}_r(w')$.

Remark 4.1. First, note that the statements (a) and (b) in Conjecture A are equivalent. Note also that, if Lusztig's Conjectures (P1)–(P15) in [30, Chapter 14] hold, then (a) and (b) imply (c).

Note also that the statements (a), (b) and (c) have been proved whenever $b > (n-1)a$ (*asymptotic case*: see [7, Theorem 7.7] and [5, 3.9]): the statement (c$^+$) has been proved only if w and w' have the same *t-length* (i.e. if the numbers of occurrences of t in a reduced expression of w and w' are equal), see [5, 3.8].

Statements (a), (b) and (c$^+$) have also been proved whenever $a = 2b$ or $3a = 2b$ (see [6, Theorem 3.14]).

Finally, we must add that Conjecture A is "highly probable": it has been checked for $n \leqslant 6$ and is compatible with many other properties of Kazhdan–Lusztig cells and other conjectures of Lusztig. For a detailed discussion about this, the reader may refer to [6].

From now on, and until the end of this paper, we assume that Conjecture A holds. We shall now define a basis of $\mathcal{H}_n$ which depends on a, b and $\leqslant$ as follows. If $(S, T) \in \mathcal{SBT}^{(2)}(n)$, let w

be the unique element of W_n such that $S = S_r(w)$ and $T = T_r(w)$: we then set

$$C^{\leqslant}_{S,T} = \left(C^{\leqslant}_w\right)^{\dagger},$$

where $\dagger : \mathcal{H}_n \to \mathcal{H}_n$ is the unique A-algebra involution such that

$$T_s^{\dagger} = -T_s^{-1}$$

for all $s \in S_n$. Now, if $\lambda \in \mathrm{Bip}(n)$, we denote by $\mathcal{SBT}(\lambda)$ the set of standard bitableaux of shape λ (i.e. $\mathcal{SBT}(\lambda) = \boldsymbol{\lambda}^{-1}(\lambda)$). Finally, we denote by $\mathcal{H}_n \to \mathcal{H}_n$, $h \mapsto h^*$ the unique A-linear map such that

$$T_w^* = T_{w^{-1}}$$

for all $w \in W_n$. It is an involutive anti-automorphism of the A-algebra $\mathcal{H}_n$. Then we have constructed a quadruple $((\mathrm{Bip}(n), \trianglelefteq_r), \mathcal{SBT}, C^{\leqslant}, *)$ where

- $(\mathrm{Bip}(n), \trianglelefteq_r)$ is a poset;
- For each $\lambda \in \mathrm{Bip}(n)$, $\mathcal{SBT}(\lambda)$ is a finite set;
- Let $\mathcal{SBT}^{(2)}(n) = \coprod_{\lambda \in \mathrm{Bip}(n)} \mathcal{SBT}(\lambda) \times \mathcal{SBT}(\lambda)$. Then the map

$$\begin{array}{rccc} C^{\leqslant} : & \mathcal{SBT}^{(2)}(n) & \longrightarrow & \mathcal{H}_n, \\ & (S,T) & \longmapsto & C^{\leqslant}_{S,T} \end{array}$$

  is injective and its image is an A-basis $(C^{\leqslant}_{S,T})_{(S,T)\in\mathcal{SBT}^{(2)}(n)}$ of $\mathcal{H}_n$;
- The map $* : \mathcal{H}_n \to \mathcal{H}_n$ is an A-linear involutive anti-automorphism of the ring $\mathcal{H}_n$ such that $(C^{\leqslant}_{S,T})^* = C^{\leqslant}_{T,S}$ for all $(S,T) \in \mathcal{SBT}^{(2)}(n)$.

The next conjecture is taken from [6, Conjecture C].

Conjecture B. *The quadruple* $((\mathrm{Bip}(n), \trianglelefteq_r), \mathcal{SBT}, C^{\leqslant}, *)$ *is a cell datum on* $\mathcal{H}_n$*, in the sense of Graham and Lehrer* [20].

Remark 4.2. For a discussion of some evidences for this conjecture, see again [6]. Note that if Lusztig's Conjectures (P1)–(P15) in [6, Chapter 14] hold and if Conjecture A above holds, then Conjecture B holds by a recent work of Geck [15].

In the rest of this paper, we shall give some more evidences for Conjectures A and B, towards their compatibilities with Ariki's Theorem.

Hypothesis. *From now on, and until the end of this paper, we assume that Conjectures A and B hold. We shall denote by* $\mathcal{C}^{\leqslant}$ *the cell datum* $((\mathrm{Bip}(n), \trianglelefteq_r), \mathcal{SBT}, C^{\leqslant}, *)$.

4.3. Interpretation of Ariki's Theorem

Since we assume that Conjectures A and B hold, the general theory of cellular algebras [20] shows that the cell datum $\mathcal{C}^{\leqslant}$ provides us with a family of Specht modules $S^{\leqslant}_{\lambda}$ associated to each bipartition λ of n. Moreover, $S_\lambda^{\leqslant}$ is endowed with a bilinear form $\phi_\lambda^{\leqslant} : S_\lambda^{\leqslant} \times S_\lambda^{\leqslant} \to A$ such that, for all $h \in \mathcal{H}_n$ and $x, y \in S_\lambda^{\leqslant}$, we have

$$\phi_\lambda^{\leqslant}(h \cdot x, y) = \phi_\lambda^{\leqslant}(x, h^* \cdot y). \tag{4.3}$$

We denote by $k\phi_\lambda^{\leqslant}$ the specialization of $\phi_\lambda^{\leqslant}$ to $kS_\lambda^{\leqslant}$ (through $\theta : \Gamma \to k^\times$). We then set

$$D_\lambda^{\leqslant} = kS_\lambda^{\leqslant} / \mathrm{rad}\big(k\phi_\lambda^{\leqslant}\big).$$

By 4.3, $\mathrm{rad}(k\phi_\lambda^{\leqslant})$ is a $k\mathcal{H}_n$-submodule of $S_\lambda^{\leqslant}$, so that $D_\lambda^{\leqslant}$ is a $k\mathcal{H}_n$-module. We set

$$\mathrm{Bip}_{(k)}^{\leqslant}(n) = \big\{\lambda \in \mathrm{Bip}(n) \bigm| D_\lambda^{\leqslant} \neq 0\big\}.$$

Then, by the theory of cellular algebras [20, Theorem. 3.4], $(D_\lambda^{\leqslant})_{\lambda \in \mathrm{Bip}_{(k)}^{\leqslant}(n)}$ is a family of representatives of isomorphism classes of irreducible $k\mathcal{H}_n$-modules.

Remark 4.4. Note that the $k\mathcal{H}_n$-modules $D_\lambda^{\leqslant}$, as well as the $\mathcal{H}_n$-modules $S_\lambda^{\leqslant}$, depend heavily on the choice of the order $\leqslant$ on Γ (they depend on r): several choices for $\leqslant$ will lead to non-isomorphic $k\mathcal{H}_n$-modules. In particular, the set $\mathrm{Bip}_{(k)}^{\leqslant}(n)$ does depend on $\leqslant$ (see Section 5).

Once we accept Conjectures A and B, it is highly probable that the Specht modules satisfy the following property (which has been proved if $b > (n-1)a$ in [7, Proposition 7.11]).

Conjecture B$^+$. *For all $\lambda \in \mathrm{Bip}(n)$, we have $KS_\lambda^{\leqslant} \simeq V_\lambda$.*

Recall that θ is the specialization morphism $\mathcal{H}_n \to k\mathcal{H}_n$ such that there exist natural numbers d and e such that $e \geqslant 1$ and

$$\begin{cases} \theta(a)^2 \text{ is a primitive } e\text{th root of unity;} \\ \theta(b)^2 = -\theta(a)^{2d}. \end{cases}$$

The following result is a generalization of [1] and [17] (see also [15]).

Theorem 4.5. *Assume that Conjectures A, B and B$^+$ hold. Let $\lambda \in \mathrm{Bip}(n)$. Let p be the unique integer such that $d + pe \leqslant r < d + (p+1)e$. Put $s = (d + pe, 0)$. Then $\mathrm{Bip}_{(k)}^{\leqslant}(n) = \mathrm{Bip}_e^s(n)$ and*

$$\big[kS_\lambda^{\leqslant} : D_\mu^{\leqslant}\big] = d_{\lambda,\mu}^s(1)$$

for all $\mu \in \mathrm{Bip}_{(k)}^{\leqslant}(n)$. In other words, $D_\mu^{\leqslant} \simeq D_\mu^s$ and

$$\mathbf{d}_n[V_\lambda] = \sum_{\mu\,\mathrm{Bip}_{(k)}^{\leqslant}(n)} d_{\lambda,\mu}^s(1)\big[D_\mu^{\leqslant}\big].$$

Proof. Assume that p is the unique integer such that

$$d + pe \leqslant r < d + (p+1)e$$

and put

$$s = (d + pe, 0).$$

Let $\mu \in \mathrm{Bip}_e^s(n)$. Then, because of the above condition on p, we can use the same strategy as in the proof of [17, Theorem 5.4] to show that:

$$G(\mu, s) = |\mu, s\rangle + \sum_{\lambda \trianglelefteq_r \mu,\, \lambda \neq \mu} d^s_{\lambda,\mu}(v)|\lambda, s\rangle.$$

Now, assume that $\nu \in \mathrm{Bip}_{(k)}^{\leqslant}(n)$. Then by Ariki's Theorem there exists $\mu \in \mathrm{Bip}_e^s(n)$ such that for all $\lambda \in \mathrm{Bip}(n)$

$$\mathbf{d}_{\lambda,\nu} = d^s_{\lambda,\mu}(1),$$

where $\mathbf{d}_{\lambda,\nu} = [kS_\lambda^{\leqslant} : D_\nu^{\leqslant}]$. We have:

$$\mathbf{d}_{\mu,\nu} = 1 \quad \text{and} \quad \mathbf{d}_{\lambda,\nu} = 0 \quad \text{if } \mu \trianglelefteq_r \lambda \text{ and } \mu \neq \lambda.$$

By the property of cellular algebras [20, Proposition 3.6], this implies that $\nu = \mu$. As we have a bijection between $\mathrm{Bip}_{(k)}^{\leqslant}(n)$ and $\mathrm{Bip}_e^s(n)$, this implies that these two sets are equal. The rest of the theorem is now obvious. $\square$

Remark 4.6. The above theorem gives a conjectural interpretation of Ariki's parametrization of simple $k\mathcal{H}_n$-modules in terms of a new cellular structure on $\mathcal{H}_n$ (coming from Kazhdan–Lusztig's theory).

4.4. Uglov bipartitions

The bipartitions in $\mathrm{Bip}_e^s(n)$ are known as Uglov bipartitions. They are constructed by using the crystal graph of the associated Fock space representation (see for example [2] for details on crystal graphs). In general, we only have a recursive definition for them. However, there exist non-recursive characterizations of such bipartitions in particular cases:

- in the case where $-e \leqslant d + pe < 0$, see [12] and [24, Proposition 3.1] (the Uglov bipartitions are then called the FLOTW bipartitions);
- in the case where $d + pe > n - 1 - e$, see [3,4] (the Uglov bipartitions are then called the Kleshchev bipartitions). Note that this includes the "asymptotic case," that is, the case where $b > (n-1)a$.

4.5. Jantzen filtration

Once we believe in Conjectures A and B, it is natural to try to develop the theory as in the type A case. For instance, one can define a Jantzen filtration on $kS_\lambda^\leqslant$ as follows. First, there exists a discrete valuation ring $\mathcal{O} \subset K$ containing A such that, if we denote by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, then $\mathfrak{p} \cap A = \operatorname{Ker}\theta$. Since K is the field of fractions of A, the map $\theta : A \to k$ extends to a map $\tilde{\theta} : \mathcal{O} \to k$ with kernel $\mathfrak{p}$ and we have

$$k\mathcal{H}_n = k \otimes_{\mathcal{O}} \mathcal{O}\mathcal{H}_n$$

where $\mathcal{O}\mathcal{H}_n = \mathcal{O} \otimes_A \mathcal{H}_n$. Similarly, if $\lambda \in \operatorname{Bip}(n)$, then $\mathcal{O}S_\lambda^\leqslant$ is an $\mathcal{O}\mathcal{H}_n$-module and the extension of scalars defines a bilinear form $\mathcal{O}\phi_\lambda^\leqslant$ on $\mathcal{O}S_\lambda^\leqslant$. We then set, for all $i \geqslant 0$,

$$\mathcal{O}S_\lambda^\leqslant(i) = \bigl\{x \in \mathcal{O}S_\lambda^\leqslant \bigm| \forall y \in \mathcal{O}S_\lambda^\leqslant,\ \mathcal{O}\phi_\lambda^\leqslant(x, y) \in \mathfrak{p}^m\bigr\}.$$

We then set

$$kS_\lambda^\leqslant(i) = \bigl(\mathcal{O}S_\lambda^\leqslant(i) + \mathfrak{p}S_\lambda^\leqslant\bigr)/\mathfrak{p}S_\lambda^\leqslant.$$

Then there exists m_0 such that $kS_\lambda^\leqslant(m_0) = 0$ and the $S_\lambda^\leqslant(i)$'s are $k\mathcal{H}_n$-submodules of $kS_\lambda^\leqslant$. Moreover, $kS_\lambda^\leqslant(0) = kS_\lambda^\leqslant$ and $kS_\lambda^\leqslant(1) = \operatorname{rad} k\phi_\lambda^\leqslant$. The *Jantzen filtration* of $kS_\lambda^\leqslant$ is the filtration

$$0 = kS_\lambda^\leqslant(m_0) \subseteq kS_\lambda^\leqslant(m_0 - 1) \subseteq \cdots \subseteq kS_\lambda^\leqslant(1) \subseteq kS_\lambda^\leqslant(0) = kS_\lambda^\leqslant.$$

The next conjecture proposes an interpretation of the polynomials $d_{\lambda,\mu}^s$ as a *v-decomposition matrix*. This is a generalization of a conjecture by Lascoux, Leclerc and Thibon [29, Section 9].

Conjecture C. *Let* $\lambda \in \operatorname{Bip}(n)$ *and* $\mu \in \operatorname{Bip}_{(k)}^\leqslant(n)$. *Let p be the unique integer such that*

$$(d + pe)a < b < \bigl(d + (p+1)e\bigr)a.$$

Put

$$s = (d + pe, 0),$$

then

$$d_{\lambda,\mu}^s(v) = \sum_{i \geqslant 0} \bigl[kS_\lambda^\leqslant(i)/kS_\lambda^\leqslant(i+1) : D_\mu^\leqslant\bigr] v^i.$$

It would also be very interesting to find an analogue of *Jantzen's sum formula*. This formula could be obtained using the matrix described by Yvonne in [35, Section 7.4].

4.6. An example: the asymptotic case

Assume here, and only in this subsection, that $b > (n-1)a$ (*in other words, if* $r \geqslant n-1$). Then Theorem 4.5 holds without assuming that Conjectures A and B are true in this case. However, note that, at the time this paper is written, the Conjecture A is not fully proved. As it is explained in Remark 4.1, only (a), (b) and (c) are proved and part of (c^+). However, we can define an order $\trianglelefteq'_\infty$ on $\mathrm{Bip}(n)$ as follows: if $\lambda, \mu \in \mathrm{Bip}(n)$, we write $\lambda \trianglelefteq'_\infty \mu$ if $w \leqslant_{\mathcal{LR},\leqslant} w'$ for some (or all) w and w' in W_n such that $\boldsymbol{\lambda}_\infty(w) = \lambda$ and $\boldsymbol{\lambda}_\infty(w') = \mu$. Then, by the work of Geck and Iancu [16] and Geck [15], we have that $((\mathrm{Bip}(n), \trianglelefteq'_\infty), \mathcal{SBT}, C^{\leqslant}, *)$ is a cell datum on $\mathcal{H}_n$ and it is easily checked that the proof of Theorem 4.5 remains valid if we replace everywhere $\trianglelefteq_r$ by $\trianglelefteq'_\infty$.

Note also that the cell datum $((\mathrm{Bip}(n), \trianglelefteq'_\infty), \mathcal{SBT}, C^{\leqslant}, *)$ is roughly speaking equivalent to the cell data constructed by Graham and Lehrer [20, Theorem 5.5] or Dipper, James and Mathas [9, Theorem 3.26].

5. Generic Hecke algebra

In the last section, we have shown that, if we assume that Conjectures A and B hold, the choice of a pair $(a, b) \in \Gamma^2$ and the choice of a total order on Γ lead to an "appropriate" representation theory for the associated Hecke algebra of type B_n. In this section, we show that these results can be applied to find several "Specht modules" theories for the *same* algebra: we shall work with the group $\Gamma = \mathbb{Z}^2$, with $a = (1, 0)$ and $b = (0, 1)$ and the main theme of this section will be to make the order $\leqslant$ vary.

Hypotheses and notation. *From now on, we assume that* $\Gamma = \mathbb{Z}^2$, $a = (1, 0)$ *and* $b = (0, 1)$. *We set* $Q = e^b$ *and* $q = e^a$ *so that* $A = \mathbb{Z}[Q, Q^{-1}, q, q^{-1}]$.

In this case, $\mathcal{H}_n$ is called the *generic Hecke algebra* of type B_n. Recall that

$$(T_t - Q)\big(T_t + Q^{-1}\big) = (T_{s_i} - q)\big(T_{s_i} + q^{-1}\big) = 0$$

for all $i \in \{1, 2, \ldots, n-1\}$. Now, any choice of elements Q_0 and q_0 in $k^\times$ defines a unique morphism $\theta : \Gamma \to k^\times$, $a \mapsto v_0$, $b \mapsto V_0$.

5.1. Orders on $\mathbb{Z}^2$

Our main theme in this section is to use the possibility of endowing $\Gamma = \mathbb{Z}^2$ with several orders. For instance, if $\xi \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ (a positive irrational number), then we can define

$$(m, n) \leqslant_\xi (m', n') \quad \text{if and only if} \quad m + \xi n \leqslant m' + \xi n'.$$

This defines a total order on $\mathbb{Z}^2$. Moreover, $0 <_\xi a$ and $0 <_\xi b$, so that all the results of this paper can be applied. If r denotes the entire part $[\xi]$ of ξ, then

$$ra < b < (r+1)a.$$

For simplification, we shall denote by S_λ^ξ the $\mathcal{H}_n$-module $S_\lambda^{\leqslant_\xi}$ and D_λ^ξ the $k\mathcal{H}_n$-module $D_\lambda^{\leqslant_\xi}$. We also set $\mathrm{Bip}_{(k)}^\xi(n) = \mathrm{Bip}_{(k)}^{\leqslant_\xi}(n)$. We also assume that the following holds: there exist natural numbers d and e such that $e \geqslant 1$ and

$$\begin{cases} \theta(a)^2 \text{ is a primitive } e\text{th root of unity;} \\ \theta(b)^2 = -\theta(a)^{2d}. \end{cases}$$

We can now restate the Theorem 4.5 in this case:

Theorem 5.1. *Assume that Conjectures A, B and B^+ hold. Let $\lambda \in \mathrm{Bip}(n)$. Let p be the unique integer such that $d + pe < \xi < d + (p+1)e$. Put $s = (d + pe, 0)$. Then $\mathrm{Bip}_{(k)}^\xi(n) = \mathrm{Bip}_e^s(n)$ and*

$$\left[kS_\lambda^\xi : D_\mu^\xi\right] = d_{\lambda,\mu}^s(1)$$

for all $\mu \in \mathrm{Bip}_{(k)}^\xi(n)$. In other words, $D_\mu^\xi \simeq D_\mu^s$ and

$$\mathbf{d}_n[V_\lambda] = \sum_{\mu \in \mathrm{Bip}_{(k)}^\xi(n)} d_{\lambda,\mu}^s(1)\left[D_\mu^\xi\right].$$

This theorem shows that, if Conjectures A and B hold, then the different choices of ξ (or of other orders on $\mathbb{Z}^2$) lead to:

- Different cellular structures on $\mathcal{H}_n$;
- Different families of Specht modules;
- Different parametrizations of the simple $k\mathcal{H}_n$-modules,
- Different "v-decomposition matrices" as defined in 4.5 (despite the fact that the decomposition matrices must be equal up to permutations of the rows and the columns).

For instance, if ξ and ξ' are two positive irrational numbers, then the $K\mathcal{H}_n$-modules KS_λ^ξ and $KS_\lambda^{\xi'}$ are isomorphic, but the $\mathcal{H}_n$-modules S_λ^ξ and $S_\lambda^{\xi'}$ might be non-isomorphic.

Remark 5.2. If $\xi > n - 1$, then Geck, Iancu and Pallikaros [18] have shown that the Specht modules S_λ^ξ are isomorphic to the ones constructed by Dipper, James and Murphy [10].

5.2. Crystal isomorphisms

Let ξ_1 and ξ_2 be irrational numbers. Let r_1 and r_2 be the natural numbers such that

$$r_1 < \xi_1 < (r_1 + 1) \quad \text{and} \quad r_2 < \xi_2 < (r_2 + 1).$$

Then if we use the order $\leqslant_{\xi_1}$, we obtain a complete set of non-isomorphic simple modules for the specialized algebra $k\mathcal{H}_n$:

$$\left\{D_\lambda^{\xi_1} \,\middle|\, \lambda \in \mathrm{Bip}_{(k)}^{\xi_1}\right\}.$$

On the other hand, if we use the order $\leqslant_{\xi_2}$, we obtain a complete set of non-isomorphic simple modules for the same algebra:

$$\{D_\lambda^{\xi_2} \mid \lambda \in \mathrm{Bip}_{(k)}^{\xi_2}\}.$$

By Theorem 5.1, there exist $s^1 \in \mathbb{Z}^2$ and $s^2 \in \mathbb{Z}^2$ such that:

$$\mathrm{Bip}_{(k)}^{\xi_1}(n) = \mathrm{Bip}_e^{s^1}(n) \quad \text{and} \quad \mathrm{Bip}_{(k)}^{\xi_2}(n) = \mathrm{Bip}_e^{s^2}(n)$$

and we have a bijection

$$c : \mathrm{Bip}_e^{s^1}(n) \to \mathrm{Bip}_e^{s^2}(n)$$

which is uniquely determined by the condition that:

$$\text{For all } \mu \in \mathrm{Bip}_e^{s^1}(n), \quad D_\mu^{\xi_1} \simeq D_{c(\mu)}^{\xi_2}.$$

In this section, we want to explicitly determine the bijection c. To do this, we first note that this map induces a bijection between the vertices of the crystal graphs of $\mathcal{M}[s^1]$ and $\mathcal{M}[s^2]$.

The modules $\mathcal{M}[s^1]'$ and $\mathcal{M}[s^2]'$ are isomorphic as $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})'$ modules. As a consequence, the crystal graphs of $\mathcal{M}[s^1]$ and $\mathcal{M}[s^2]$ are also isomorphic. This bijection may be obtained by following a sequence of arrows back to the empty bipartition in the crystal graph of $\mathcal{M}[s^1]$ and then applying the reversed sequence from the empty bipartition of $\mathcal{M}[s^2]$. One can also define it as follows. Let

$$\mathcal{B}_{s^1} := \{G(\mu, s^1) \mid \mu \in \mathrm{Bip}_e^{s^1}\}$$

be the elements of the canonical basis of $\mathcal{M}[s^1]$ and let

$$\mathcal{B}_{s^2} := \{G(\mu, s^2) \mid \mu \in \mathrm{Bip}_e^{s^2}\}$$

be the elements of the canonical basis of $\mathcal{M}[s^1]$. Then if we specialize the element of $\mathcal{B}_{s^1}$ or $\mathcal{B}_{s^2}$ to $v = 1$, we obtain a basis of the same irreducible highest weight $\mathcal{U}(\widehat{\mathfrak{sl}_e})$-modules. Hence, for all $\mu \in \mathrm{Bip}_e^{s^1}(n)$, there exists $\gamma(\mu) \in \mathrm{Bip}_e^{s^2}(n)$ such that

$$\text{for all } \lambda \in \mathrm{Bip}(n), \quad d_{\lambda,\mu}^{s^1}(1) = d_{\lambda,\gamma(\mu)}^{s^2}(1).$$

Then we have a bijection

$$\gamma : \mathrm{Bip}_e^{s^1}(n) \to \mathrm{Bip}_e^{s^2}(n).$$

This bijection has been explicitly described, in a non-recursive way in [24] (see also the generalization in [25]).

Theorem 5.3. *Assume that Conjectures A and B hold. Then for all $\mu \in \mathrm{Bip}_e^{s^1}(n)$, we have*

$$c(\mu) = \gamma(\mu).$$

Proof. Let r_1 (respectively r_2) denote the entire part of ξ_1 (respectively ξ_2). Let $\mu \in \mathrm{Bip}_e^{s^1}(n)$. By Ariki's Theorem and the theory of cellular algebras, $c(\mu)$ is the maximal element with respect to $\trianglelefteq_{r_2}$ such that $|c(\mu), s^1\rangle$ appears with non-zero coefficient in $G(\mu, s^1)$.

We have $[\mathbb{C}S_\lambda^\xi : D_\mu^{\xi_1}] = [\mathbb{C}S_\lambda^\xi : D_{c(\mu)}^{\xi_2}]$ and $[\mathbb{C}S_\lambda^\xi : D_\mu^{\xi_1}] = d_{\lambda,\mu}^{s^1}(1) = d_{\lambda,\gamma(\mu)}^{s^2}(1) = [\mathbb{C}S_\lambda^\xi : D_{c(\mu)}^{\xi_2}]$. Hence $\gamma(\mu)$ is the maximal element with respect to $\trianglelefteq_{r_2}$ such that

$$d_{\gamma(\mu),\mu}^{s^1}(1) \neq 0.$$

This implies that

$$c(\mu) = \gamma(\mu).$$

This concludes the proof. □

Remark 5.4. Note that using the non-recursive parametrization of $\mathrm{Bip}_e^s(n)$ known in particular cases (see Section 4.4), it is possible to obtain nice characterizations of $\mathrm{Bip}_e^s(n)$ in all cases by applying the above bijection.

Example 5.5. Assume that $n = 4$ and that $\theta(a)^2 = \theta(b)^2 = -1$. Then $\theta(a)$ is a primitive 2-root of unity and we have $\theta(b)^2 = -\theta(a)^0$. If $0 < \xi_1 < 2 < \xi_2 < 4$ (i.e. $r_1 \in \{0, 1\}$ and $r_2 \in \{2, 3\}$), then we have

$$\mathrm{Irr}(\mathbb{C}\mathcal{H}_4) = \{D_\mu^{\xi_1} \mid \mu \in \mathrm{Bip}_2^{(0,0)}(4)\},$$
$$\mathrm{Irr}(\mathbb{C}\mathcal{H}_4) = \{D_\mu^{\xi_2} \mid \mu \in \mathrm{Bip}_2^{(2,0)}(4)\}.$$

To compute $\mathrm{Bip}_2^{(0,0)}(4)$ (respectively $\mathrm{Bip}_2^{(2,0)}(4)$), we compute the crystal graph associated to the action of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}}_2)$ on the Fock space $\mathcal{F}^{(0,0)}$ (respectively $\mathcal{F}^{(2,0)}$). The submodule generated by the empty bipartition gives an irreducible highest weight module $\mathcal{M}[0, 0]$ (respectively $\mathcal{M}[2, 0]$) with highest weight $2\Lambda_0$. We only give the part of both crystals containing the bipartitions up to rank 4. They are computed as explained in [24, 2.1, 2.2]:

Crystal graph of $\mathcal{M}[0, 0]$

$$\begin{tikzcd}
 & (\emptyset,\emptyset) \arrow[d, "0"'] & & & \\
 & (1,\emptyset) \arrow[dl, "0"] \arrow[dr, "1"'] & & & \\
(1,1) \arrow[d, "1"] & & (2,\emptyset) \arrow[dl, "1"] \arrow[dr, "0"'] & & \\
(2,1) \arrow[d, "0"] & (2.1,\emptyset) \arrow[d, "0"] & & (3,\emptyset) \arrow[dl, "0"'] \arrow[dr, "1"] & \\
(2,2) & (3.1,\emptyset) & (3,1) & & (4,\emptyset)
\end{tikzcd}$$

Crystal graph of $\mathcal{M}[2,0]$

$$
\begin{tikzcd}
 & (\emptyset,\emptyset) \arrow[d, "0"'] & & & \\
 & (1,\emptyset) \arrow[dl, "0"] \arrow[dr, "1"'] & & & \\
(1,1) \arrow[d, "1"] & & (2,\emptyset) \arrow[dl, "1"] \arrow[dr, "0"'] & & \\
(2,1) \arrow[d, "0"] & (2.1,\emptyset) \arrow[d, "0"] & & (3,\emptyset) \arrow[dl, "0"'] \arrow[dr, "1"] & \\
(2.1,1) & (3.1,\emptyset) & (3,1) & & (4,\emptyset)
\end{tikzcd}
$$

By Theorem 4.5, we then have

$$
\begin{aligned}
\operatorname{Irr}(\mathbb{C}\mathcal{H}_4) &= \left\{D^{\xi_1}_{(4,\emptyset)}, D^{\xi_1}_{(31,\emptyset)}, D^{\xi_1}_{(3,1)}, D^{\xi_1}_{(2,2)}\right\} \\
&= \left\{D^{\xi_2}_{(4,\emptyset)}, D^{\xi_2}_{(31,\emptyset)}, D^{\xi_2}_{(3,1)}, D^{\xi_2}_{(2,2)}\right\}
\end{aligned}
$$

and, by Theorem 5.3, we have

$$
\begin{aligned}
D^{\xi_1}_{(4,\emptyset)} &\simeq D^{\xi_2}_{(4,\emptyset)}, \\
D^{\xi_1}_{(31,\emptyset)} &\simeq D^{\xi_2}_{(31,\emptyset)}, \\
D^{\xi_1}_{(3,1)} &\simeq D^{\xi_2}_{(3,1)}, \\
D^{\xi_1}_{(2,2)} &\simeq D^{\xi_2}_{(2.1,1)}.
\end{aligned}
$$

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