Topology and its Applications 156 (2009) 525–532

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Chaos via Furstenberg family couple ☆

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A R T I C L E I N F O

Article history:
Received 13 November 2007
Received in revised form 8 August 2008
Accepted 8 August 2008

MSC:
primary 37B40, 37B05, 37B10, 54H20

Keywords:
Furstenberg family
Distributional chaos
$(\mathcal{F}_1, \mathcal{F}_2)$-chaos

A B S T R A C T

In this paper we define $(\mathcal{F}_1, \mathcal{F}_2)$-chaos via Furstenberg family couple $\mathcal{F}_1$ and $\mathcal{F}_2$. It turns out that the Li–Yorke chaos and distributional chaos can be treated as chaos in Furstenberg families sense. Some sufficient conditions such that a system is the $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic (Theorems 4.2 and 4.4) are given. In addition, we construct an example as an application. It is showed that the second type of distributional chaos cannot imply the first type of distributional chaos even though the scrambled set is uncountable.

1. Introduction

By a *topological dynamical system* (briefly, dynamical system or system), we mean a pair (X, f), where X is a complete metric space dense in itself with a metric d, and $f : X \to X$ is a continuous map from X to itself. For any nonempty subset $A \subset X$, $\bar{A}$ is the closure of A and we set $[A]_\delta = \{x \in X \mid d(x, A) < \delta\}$ for any $\delta > 0$. We also use $[A]_0$ to denote the closure of A. Let the diagonal $\Delta = \{(x, x) \mid x \in X\}$ and d^2 be the product metric on the product space $X \times X$, i.e.,

$$d^2\big((x_1, x_2), (y_1, y_2)\big) = \max\big\{d(x_1, y_1), d(x_2, y_2)\big\}$$

for any $(x_1, x_2), (y_1, y_2) \in X \times X$.

Li and Yorke [2] first used the word chaos to describe the complexity of the orbits of points in a system determined by iterations of a map. According to [2] and [3], some authors defined the chaos in the Li–Yorke sense as follows.

A system (X, f) is *chaotic in the Li–Yorke sense* (or Li–Yorke chaotic), if there exists an uncountable set $C \subset X$ such that for any distinct points $x, y \in X$,

$$\limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0 \quad \text{and} \quad \liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0.$$

Many authors have discussions on the Li–Yorke chaotic system (see [2] and [3] etc.). The relationship between the topological entropy and the Li–Yorke chaos was discussed in [4] and [6]. It was showed that every transitive system with a periodic point is Li–Yorke chaotic in [7].

☆ The authors are supported by National Science Foundation of China, No. 10771079.

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.08.006

Schweizer and Smítal [5] gave another definition for chaos, which is called *the chaos in the distribution sense* (or distributional chaos).

Let (X, f) be a dynamical system and $x, y \in X$. For a positive integer n and real parameter t, put

$$\Phi^n_{xy}(t) = \frac{1}{n}\#\{0 \leqslant i \leqslant n-1:\ d\big(f^i(x), f^i(y)\big) < t\},$$
$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi^n_{xy}(t), \qquad \Phi^\star_{xy}(t) = \limsup_{n\to\infty} \Phi^n_{xy}(t),$$

where $\#(\cdot)$ denotes the cardinality. Call Φ_{xy} and $\Phi^\star_{xy}$ the *upper distribution function* and the *lower distribution function* of x and y, respectively.

Obviously, $\Phi_{xy}(t) \leqslant \Phi^\star_{xy}(t)$ for any $t \geqslant 0$. If there exists an interval I such that $\Phi_{xy}(t) < \Phi^\star_{xy}(t)$ for all $t \in I$, we simply write $\Phi_{xy} < \Phi^\star_{xy}$. The chaos in the distribution sense appears when $\Phi_{xy} < \Phi^\star_{xy}$ for any distinct points x, y in an uncountable set. It turns out that three mutually nonequivalent versions of distributional chaos, DC1, DC2 and DC3, can be considered. (see [8].) Namely, if there is an uncountable set $C \subset X$ such that for any distinct points $x, y \in C$,

(1) $\Phi^\star_{xy} \equiv 1$ and $\Phi_{xy}(t) = 0$ for some $t > 0$, then the system is of the first type of distributional chaos (briefly, DC1).
(2) $\Phi^\star_{xy} \equiv 1$ and $\Phi_{xy} < \Phi^\star_{xy}$, then the system is of the second type of distributional chaos (briefly, DC2).
(3) $\Phi_{xy} < \Phi^\star_{xy}$, then the system is of the third type of the distributional chaos (briefly, DC3).

Moreover, the relationship between the topological entropy and the distributional chaos was discussed in [9] and [10].

Recently, the notion of chaos via $\mathcal{F}$ was introduced (see [1] for more details) for a given Furstenberg family $\mathcal{F}$. It turned out that the Li–Yorke chaos and some version of distributional chaos can be described as chaos in Furstenberg families sense.

The aim of this paper is to introduce the definition of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos for two given Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$. Moreover, we give some sufficient conditions for a system to be $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic. As an application, we construct a system and show that Furstenberg family couple is necessary for us to describe the chaotic phenomena of the system.

This paper is organized as follows. In Section 2 we introduce some results of Furstenberg families. Section 3 is devoted to give the definitions of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos for Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$. In the following section, we give some sufficient conditions such that a system is $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic (Theorems 4.2, 4.4). Finally, as an application of the results mentioned above, we construct a subsystem of shift systems on symbolic spaces, showing that DC2 cannot imply DC1 even though the scrambled set is uncountable.

2. Preliminary on Furstenberg families

In this section we recall the basic facts related to the Furstenberg families (see [11] and [1] for the details). Let $\mathbb{Z}_+$ be the set of positive integers and $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}_+$.

A collection $\mathcal{F} \subset \mathcal{P}$ is called a *Furstenberg family* if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

Let $\mathcal{B}$ be the collection of all infinite subsets of $\mathbb{Z}_+$ and $\kappa\mathcal{B}$ be the collection of the subsets of $\mathbb{Z}_+$ with finite complement. Obviously, both $\mathcal{B}$ and $\kappa\mathcal{B}$ are Furstenberg families.

Furstenberg family $\mathcal{F}$ is said to be *proper* if it is a nonempty proper set of the family $\mathcal{P}$. It is clear that a Furstenberg family $\mathcal{F}$ is proper if and only if $\mathbb{Z}_+ \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Furstenberg families. Then

$$\mathcal{F}_1 \cdot \mathcal{F}_2 = \{F_1 \cap F_2:\ F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}.$$

By recalling the definition of upper density of a subset of $\mathbb{Z}_+$, we define a class of Furstenberg families via upper (lower) density.

Let $J = \{n_1 < n_2 < \cdots\} \subset \mathbb{Z}_+$ be an infinite set. Define

$$\mu^*(J) = \limsup_{n\to\infty} \frac{\#(J \cap \{1, \ldots, n\})}{n}, \qquad \mu_*(J) = \liminf_{n\to\infty} \frac{\#(J \cap \{1, \ldots, n\})}{n}.$$

Then $\mu^*(J)$ and $\mu_*(J)$ are the *upper density* and the *lower density* of $J \subset \mathbb{Z}_+$, respectively.

It is easy to check that

$$\mu^*(J) = \limsup_{i\to\infty} \frac{i}{n_i}.$$

For every $s \in [0, 1]$, define

$$\overline{\mathcal{M}}(s) = \{F \in \mathcal{B}:\ \mu^*(F) \geqslant s\}, \qquad \underline{\mathcal{M}}(s) = \{F \in \mathcal{B}:\ \mu_*(F) \geqslant s\}.$$

Obviously, $\overline{\mathcal{M}}(0) = \mathcal{B}$.

Let F be a subset of $\mathbb{Z}_+$. We say F is *thick* if for any $m \in \mathbb{N}$, there exists $n \in F$ such that $n, n+1, \ldots, n+m \in F$. That is, F contains arbitrary long blocks of consecutive positive integers.

According to the definition, the following theorems (Theorems 2.1 and 2.2) are immediate.

Theorem 2.1. *The subset with upper density* 1 *is thick.*

Theorem 2.2. *Let* a, b *be real numbers with* $1 \geqslant a, b > 0$ *and* $a + b > 1$. *If* F, H *are subsets of* $\mathbb{Z}_+$ *such that* $\mu_*(F) \geqslant a$, $\mu^*(H) \geqslant b$, *then* $\mu^*(F \cap H) \geqslant a + b - 1$.

3. $(\mathcal{F}_1, \mathcal{F}_2)$-chaos

In this section we introduce the definition of $(\mathcal{F}_1, \mathcal{F}_2)$-chaos and characterize the Li–Yorke chaos and the distributional chaos via Furstenberg family couple.

Let A be a subset of X. For any $x \in X$, write

$$N(x, A, f) = \{n:\ f^n(x) \in A\}.$$

Let $\mathcal{F}$ be a Furstenberg family and A be a nonempty subset. Then $x \in X$ is *the $\mathcal{F}$-attaching point of A* if $N(x, A, f) \in \mathcal{F}$. We call

$$\mathcal{F}(A, f) = \bigcup_{F \in \mathcal{F}} \bigcap_{n \in F} f^{-n}(A)$$

the $\mathcal{F}$-attaching set of A. Clearly, $\mathcal{F}(A, f)$ is the set of all $\mathcal{F}$-attaching points of A.

Let (X, f) be a system and $\mathcal{F}$ be a Furstenberg family. The family $\mathcal{F}$ is *compatible with the system* (X, f), if the set $\mathcal{F}(U, f)$ is a G_δ set for every open set $U \subset X$.

Many Furstenberg families familiar to us are compatible with any system (X, f). According to [1], we have the following theorem.

Theorem 3.1. *(See [1].) For any* $t \in [0, 1]$, *Furstenberg family* $\overline{\mathcal{M}}(t)$ *is compatible with any system* (X, f).

Let $A \subset X$ be a nonempty set and $\mathcal{F}$ be a proper Furstenberg family. We say the point $x \in X$ is an *$\mathcal{F}$-adherent point* of A, if $N(x, [A]_\delta, f) \in \mathcal{F}$ for any $\delta > 0$. Put

$$\alpha_{\mathcal{F}}(A, f) = \{x \mid N(x, [A]_\delta, f) \in \mathcal{F} \text{ for any } \delta > 0\}.$$

Let (X, f) be a system and $\mathcal{F}_1, \mathcal{F}_2$ be Furstenberg families. Let $A \subset X \times X$. The system (X, f) is *$(\mathcal{F}_1, \mathcal{F}_2)$-chaotic with respect to A*, if there exists an uncountable set K such that for any distinct points $x, y \in K$, (x, y) is both $\mathcal{F}_1$-adherent point of A and $\mathcal{F}_2$-attaching point of $X \times X \setminus \overline{[A]_\delta}$ for some $\delta > 0$.

A system (X, f) is *$(\mathcal{F}_1, \mathcal{F}_2)$-chaotic*, if the system (X, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic with respect to the diagonal Δ.

In particular, we say the system (X, f) is *$(\mathcal{F}_1, \mathcal{F}_2)$-$\delta$-chaotic* $(\delta > 0)$, if there exists an uncountable set $K \subset X$, for any distinct points $x, y \in K$, (x, y) is both $\mathcal{F}_1$-adherent point of the diagonal Δ and $\mathcal{F}_2$-attaching point of $X \times X \setminus \overline{[\Delta]_\delta}$.

Now, we state the strong-distributional chaos as follows.

Let (X, d) be a metric space and $f : X \to X$ be a continuous map. If there exists an uncountable subset $C \subset X$,

(1) for some $1 > b \geqslant 0$ and any distinct points $x, y \in C$,

$$\Phi^\star_{xy} \equiv 1 \quad \text{and} \quad \Phi_{xy}(t) \leqslant b < 1 \quad \text{for some } t > 0,$$

then we say that the system (X, f) exhibits the second type of strong-distributional chaos (briefly, strong DC2).

(2) for some $a > b \geqslant 0$ and any distinct points $x, y \in C$,

$$\Phi^\star_{xy}(t) \geqslant a > 0 \quad \text{for any } t > 0 \quad \text{and} \quad \Phi_{xy}(t_0) \leqslant b \quad \text{for some } t_0 > 0,$$

then we say that the system (X, f) exhibits the third type of strong-distributional chaos (briefly, strong DC3).

Now we characterize the Li–Yorke chaos and the distributional chaos via Furstenberg family couple.

Theorem 3.2. *Let* (X, f) *be a dynamical system.*

(1) *Then system* (X, f) *is Li–Yorke chaotic if and only if it is* $(\mathcal{B}, \mathcal{B})$*-chaotic.*
(2) *Let* a, b *be real numbers with* $1 \geqslant a, b > 0$ *and* $a + b > 1$. *Then*
  (a) *the system* (X, f) *is DC1 if and only if it is* $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$*-chaotic*;
  (b) *the system* (X, f) *is strong DC2 if and only if it is* $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(b))$*-chaotic*;
  (c) *the system* (X, f) *is strong DC3 if and only if it is* $(\overline{\mathcal{M}}(a), \overline{\mathcal{M}}(b))$*-chaotic with respect to the set* $[\Delta]_{t_0}$ *for some* $t_0 \geqslant 0$.

4. Criteria for $(\mathcal{F}_1, \mathcal{F}_2)$-$\delta$-chaos

In this section some sufficient conditions such that a system is $(\mathcal{F}_1, \mathcal{F}_2)$-chaotic are showed. Now, let us begin with Mycielski theorem. For more details see [12].

Theorem 4.1 *(Mycielski theorem). Let X be a complete metric space dense in itself. Suppose for every $N \in \mathbb{N}$, R_N is a set of first category in the product space X^{r_N} and G_j, $j = 1, 2, \ldots$, is a sequence of nonempty open subsets in X. Then there exists a nonempty perfect compact $C_j \subset G_j$ such that for every $N \in \mathbb{N}$ and any r_N distinct points $x_1, x_2, \ldots, x_{r_N} \in \bigcup_{j=1}^{\infty} C_j$, we have $(x_1, x_2, \ldots, x_{r_N}) \notin R_N$.*

Now, we give the criteria for $(\mathcal{F}_1, \mathcal{F}_2)$-$\delta$-chaotic system.

Theorem 4.2. *Let (X, f) be a dynamical system and $\mathcal{F}_1$, $\mathcal{F}_2$ be proper Furstenberg families which are compatible with the product system $(X \times X, f \times f)$. Suppose there exists a nonempty subset $W \subset X \times X \setminus \overline{[\Delta]_\lambda}$ for some $\lambda > 0$ such that both $\mathcal{F}_2(W, f \times f)$ and $\alpha_{\mathcal{F}_1}(\Delta, f \times f)$ are dense in $X \times X$. Then the system (X, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-$\lambda$-chaotic.*

Proof. Since $W \subset X \times X \setminus \overline{[\Delta]_\lambda}$ for some $\lambda > 0$ and $\overline{\mathcal{F}_2(W, f \times f)} = X \times X$, then $\mathcal{F}_2(X \times X \setminus \overline{[\Delta]_\lambda}, f \times f)$ is dense in $X \times X$. Moreover, the Furstenberg families $\mathcal{F}_1$, $\mathcal{F}_2$ are compatible with the product system $(X \times X, f \times f)$, and we have that both $\mathcal{F}_2(X \times X \setminus \overline{[\Delta]_\lambda}, f \times f)$ and $\alpha_{\mathcal{F}_1}(\Delta, f \times f)$ are dense G_δ subsets in $X \times X$. Therefore,

$$\mathcal{F}_2\big(X \times X \setminus \overline{[\Delta]_\lambda}, f \times f\big) \bigcap \alpha_{\mathcal{F}_1}(\Delta, f \times f)$$

is a dense G_δ set. By Theorem 4.1 and the definition of $(\mathcal{F}_1, \mathcal{F}_2)$-$\lambda$-chaos, the proof of this theorem is completed. □

Lemma 4.3. *Let A, B be two subsets of X and $\mathcal{F}$, $\tilde{\mathcal{F}}$ be Furstenberg families. Then*

$$\alpha_{\tilde{\mathcal{F}}}(A, f) \times \alpha_{\mathcal{F}}(B, f) \subset \alpha_{\mathcal{F}.\tilde{\mathcal{F}}}(A \times B, f \times f).$$

Proof. Assume $(x, y) \in \alpha_{\tilde{\mathcal{F}}}(A, f) \times \alpha_{\mathcal{F}}(B, f)$. For any $\delta > 0$, choose $\delta_1 > 0$ such that $[A]_{\delta_1} \times [B]_{\delta_1} \subset [A \times B]_\delta$. Since $x \in \alpha_{\tilde{\mathcal{F}}}(A, f)$, $y \in \alpha_{\mathcal{F}}(B, f)$, we have

$$N\big((x, y), [A \times B]_\delta, f \times f\big) \supset N\big((x, y), [A]_{\delta_1} \times [B]_{\delta_1}, f \times f\big) \supset N\big(x, [A]_{\delta_1}, f\big) \cdot N\big(y, [B]_{\delta_1}, f\big) \in \mathcal{F} \cdot \tilde{\mathcal{F}}.$$

Thus $N_{f \times f}((x, y), [A \times B]_\delta) \in \mathcal{F} \cdot \tilde{\mathcal{F}}$. It follows that $(x, y) \in \alpha_{\mathcal{F}.\tilde{\mathcal{F}}}(A \times B, f \times f)$. □

Let (X, f) be a dynamical system. For any point $a \in X$ and nonempty set $B \subset X$, write $d(a, B) = \inf\{d(a, b) \mid b \in B\}$. We say that a, B are $(\mathcal{F}, \lambda)$-*distal* ($\lambda > 0$) if

$$\big\{n \bigm| \big\{f^n(a)\big\} \times f^n(B) \subset X \times X \setminus \overline{[\Delta]_\lambda}\big\} \in \mathcal{F}.$$

In particular, if $B = \{b\}$, $b \in X$ then we say that a, b are $(\mathcal{F}, \lambda)$-distal.

Theorem 4.4. *Let (X, f) be a dynamical system and $\mathcal{F}$, $\tilde{\mathcal{F}}$ and $\mathcal{G}$ be proper Furstenberg families. Suppose that $\mathcal{F} \cdot \mathcal{F}$ and $\mathcal{F} \cdot \tilde{\mathcal{F}} \cdot \mathcal{G}$ are also proper Furstenberg families. Let $a \in X$ and nonempty subset $K \subset X$ be $(\mathcal{G}, \lambda)$-distal for some $\lambda > 0$. Set*

$$A = \big\{x \bigm| \text{for any } \epsilon > 0, \text{ there is } F \in \mathcal{F} \text{ s.t. } d\big(f^n(x), f^n(a)\big) < \epsilon \text{ for any } n \in F\big\},$$

$$B = \big\{y \bigm| \text{for any } \epsilon > 0, \text{ there is } F \in \tilde{\mathcal{F}} \text{ s.t. } d\big(f^n(y), f^n(K)\big) < \epsilon \text{ for any } n \in F'\big\}.$$

If $\bar{A} = \bar{B} = X$, then for any proper Furstenberg families $\mathcal{F}_1$ and $\mathcal{F}_2$ which are compabitiable with the product space $(X \times X, f \times f)$ and $\mathcal{F}_1 \supseteq \mathcal{F} \cdot \mathcal{F}$, $\mathcal{F}_2 \supseteq \mathcal{F} \cdot \tilde{\mathcal{F}} \cdot \mathcal{G}$, we have that the system (X, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-$\delta$-chaotic for some $\delta > 0$.

Proof. It remains to show that $\alpha_{\mathcal{F}_1}(\Delta, f \times f)$ and $\mathcal{F}_2(X \times X \setminus \overline{[\Delta]_\delta}, f \times f)$ are dense in $X \times X$ for some $\delta > 0$.

Since $A \times A$ is dense in $X \times X$ and $\mathcal{F}_1 \supseteq \mathcal{F} \cdot \mathcal{F}$, by Lemma 4.3 $\alpha_{\mathcal{F}_1}(\Delta, f \times f)$ is dense in $X \times X$. Choose $\epsilon > 0$ with $\lambda > \sqrt{2}\epsilon$. Assume $x \in A$ and $y \in B$. Then

$$H_1 = \big\{n \bigm| d\big(f^n(x), f^n(a)\big) < \epsilon\big\} \in \mathcal{F}, \qquad H_2 = \big\{n \bigm| d\big(f^n(y), f^n(K)\big) < \epsilon\big\} \in \tilde{\mathcal{F}}.$$

Thus $F_1 = H_1 \cap H_2 \in \mathcal{F} \cdot \tilde{\mathcal{F}}$. By the definition of the product metric, we have

$$F_1 = \big\{n \bigm| d^2\big(\big(f^n(x), f^n(y)\big), \big\{f^n(a)\big\} \times f^n(K)\big) < \epsilon\big\} = \big\{n \bigm| \big(f^n(x), f^n(y)\big) \in \big[\big\{f^n(a)\big\} \times f^n(K)\big]_\epsilon\big\}.$$

Since a and K are $(\mathcal{G}, \lambda)$-distal, we write

$$G = \big\{n \bigm| \big\{f^n(a)\big\} \times f^n(K) \subset X \times X \setminus \overline{[\Delta]_\lambda}\big\}.$$

It follows that $G \in \mathcal{G}$, and

$$G \subset \big\{n \bigm| \big[\big\{f^n(a)\big\} \times f^n(K)\big]_\epsilon \subset X \times X \setminus \overline{[\Delta]}_{\lambda-\epsilon}\big\}.$$

Let $F_2 = F_1 \cap G$. Then $F_2 \in \mathcal{F} \cdot \tilde{\mathcal{F}} \cdot \mathcal{G}$ and

$$\big(f^n(x), f^n(y)\big) \in \big[\big\{f^n(a)\big\} \times f^n(K)\big]_\epsilon \subset X \times X \setminus \overline{[\Delta]}_{\lambda-\epsilon}$$

for any $n \in F_2$. Take $\delta = \lambda - \epsilon$. Then $A \times B \subset \mathcal{F}_2(X \times X \setminus \overline{[\Delta]}_\delta, f \times f)$. Thus $\mathcal{F}_2(X \times X \setminus \overline{[\Delta]}_\delta, f \times f)$ is dense in $X \times X$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are compatible with the product space $(X \times X, f \times f)$, by Theorem 4.1, the system (X, f) is $(\mathcal{F}_1, \mathcal{F}_2)$-$\delta$-chaotic. □

Let (X, f) be a dynamical system and $\mathcal{F}$ be a Furstenberg family. For any $x, y \in X$ and infinite set $F \in \mathcal{F}$, $F = \{n_1 < n_2 < \cdots\}$,

$$d\big(f^F(x), f^F(y)\big) \to 0$$

means

$$\lim_{i\to\infty} d\big(f^{n_i}(x), f^{n_i}(y)\big) = 0.$$

By Theorems 4.4, 2.2 and the properties of the Furstenberg families defined via the density, it is not hard to prove the following corollaries.

Corollary 4.5. *Let (X, f) be a dynamical system and $1 \geqslant s > \frac{1}{2}$, $s + t > 1$, $\lambda > 0$. Suppose $a, b \in X$ are $(\underline{\mathcal{M}}(1), \lambda)$-distal. Set*

$$A = \big\{x \bigm| \text{there is an } F \in \underline{\mathcal{M}}(s) \text{ s.t. } d\big(f^F(x), f^F(a)\big) \to 0\big\},$$

$$B = \big\{y \bigm| \text{there is an } F' \in \overline{\mathcal{M}}(t) \text{ s.t. } d\big(f^{F'}(y), f^{F'}(b)\big) \to 0\big\}.$$

If $\bar{A} = \bar{B} \neq \emptyset$, then the system (X, f) is $(\overline{\mathcal{M}}(2s-1), \overline{\mathcal{M}}(s+t-1))$-$\delta$-chaotic for some $\delta > 0$.

Corollary 4.6. *Let (X, f) be a dynamical system and $a, b \in X$ be $(\overline{\mathcal{M}}(t), \lambda)$-distal for some $1 \geqslant t > 0$, $\lambda > 0$. Set*

$$A = \big\{x \bigm| \text{there exists } F \in \underline{\mathcal{M}}(1) \text{ we have } d\big(f^F(x), f^F(a)\big) \to 0\big\},$$

$$B = \big\{y \bigm| \text{there exists } F' \in \underline{\mathcal{M}}(1) \text{ we have } d\big(f^{F'}(y), f^{F'}(b)\big) \to 0\big\}.$$

If $\bar{A} = \bar{B} \neq \emptyset$, then the system (X, f) is $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(t))$-$\delta$-chaotic for some $\delta > 0$.

5. Applications

In this section we construct a system which is not $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$-chaotic but $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(\frac{1}{\sqrt{e}}))$-$\delta$-chaotic. It follows that the DC2 does not imply the DC1 even though the scrambled set is uncountable.

Let $\Lambda = \{0, 1, \ldots, n-1\}$ $(n \geqslant 2)$ with discrete topology, and $\Sigma_n = \Lambda^{\mathbb{N}}$ with the product topology. The shift $\sigma : \Sigma_n \to \Sigma_n$ is defined by $\sigma(a_1 a_2 \cdots) = a_2 a_3 \cdots$ for any $a_1 a_2 \cdots \in \Sigma_n$. The system (Σ_n, σ) is called the shift system on the symbolic space Σ_n.

Let $A = a_1 \cdots a_n \in \Lambda^n$, $B = b_1 \cdots b_m \in \Lambda^m$ for some $n, m \in \mathbb{N}$. We denote by $|A|$ the length of the segment $A = a_1 \cdots a_n$, i.e., $|A| = n$. Write $AB = a_1 \cdots a_n b_1 \cdots b_m$. The segment $A = a_1 \cdots a_n$ is called a subsegment of $B = b_1 \cdots b_m$ $(m \geqslant n)$ if $b_i \cdots b_{i+n-1} = a_1 \cdots a_n$ for some $1 \leqslant i \leqslant m - n + 1$, denoted by $A = B|_{[i,n]}$.

Now we define a minimal system. Take $A_1 = 0$. Let $\bar{A}_1 = 1$ and $\mathcal{P}_1 = \{A_1, \bar{A}_1\}$. Take $A_2 = 01100$, $\bar{A}_2 = 11100$. For any $n = 3, 4, \ldots,$ set

$$\mathcal{P}_{n-1} = \big\{B_1 \cdots B_{n-1} \bigm| B_i \in \{A_i, \bar{A}_i\},\ 1 \leqslant i \leqslant n-1\big\}.$$

For any $P \in \mathcal{P}_{n-1}$, $P = B_1 B_2 \cdots B_{n-1}$, we write $\bar{P} = \bar{B}_1 \bar{B}_2 \cdots \bar{B}_{n-1}$. Define A_n and $\bar{A}_n$ by

$$A_n = P_1 P_1 P_2 \cdots P_{2^{n-1}}, \qquad \bar{A}_n = P_1 \bar{P}_1 \bar{P}_2 \cdots \bar{P}_{2^{n-1}},$$

where $P_i \in \mathcal{P}_{n-1}$ and note that $i \neq j$ implies $P_i \neq P_j$ for any $1 \leqslant i, j \leqslant 2^{n-1}$.

Let

$$a = A_1 A_2 \cdots.$$

According to the definition mentioned above, the following claim is immediate.

Claim 1.

(1) *For any $n \geqslant 1$, we have $a = P_1 P_2 P_3 \cdots$, where $P_i \in \mathcal{P}_n$, $i = 1, 2, \ldots.$*
(2) *For any $n \geqslant 2$,*

$$|A_{n+1}| = |\bar{A}_{n+1}| = \big(2^n + 1\big)|A_1 A_2 \cdots A_n|.$$

Lemma 5.1. *Let $a = A_1 A_2 \cdots$. Then a is an almost periodic point in the system (Σ_n, σ). Put $Y = \omega(a, \sigma)$. We have (Y, σ) is a minimal system.*

Proof. Assume that $a = a_1a_2\cdots$, $a_i \in \{0,1\}$, $i = 1,2,\ldots.$ We can prove that for any $n \geqslant 1$, there exists N such that $a_1a_2\cdots a_n$ is a subsegment of $a_{i+1}\cdots a_{i+N}$ for any $i \in \mathbb{N}$. Note that for any $n \geqslant 2$, $a_1\cdots a_n$ is a subsegment of $A_1A_2\cdots A_n$. Thus $a_1\cdots a_n$ is a subsegment of both A_{n+1} and $\bar{A}_{n+1}$. By Claim 1(1), $a = A_1A_2\cdots$ can be generated by the elements of

$$\mathcal{P}_{n+1} = \{B_1\cdots B_{n+1} \mid B_i \in \{A_i, \bar{A}_i\},\ 1 \leqslant i \leqslant n+1\}.$$

Take $N = 3|A_1\cdots A_{n+1}|$. Then A_{n+1} or $\bar{A}_{n+1}$ is a subsegment of $a_i\cdots a_{i+N}$ for any $i \in \mathbb{N}$. Thus, $a_1\cdots a_n$ is a subsegment of $a_i\cdots a_{i+N}$. It finishes the proof. □

Lemma 5.2. *Let $a = A_1A_2\cdots$, $b = \bar{A}_1\bar{A}_2\cdots$. Then a, b are $(\overline{\mathcal{M}}(\frac{1}{\sqrt{e}}), \delta)$-distal for some $\delta > 0$.*

Proof. Assume $a = A_1A_2\cdots$, $b = \bar{A}_1\bar{A}_2\cdots$. Then

$$\frac{\#\{i \mid d(\sigma^i(a), \sigma^i(b)) \geqslant \frac{1}{2^{12}},\ 1 \leqslant i \leqslant |A_1A_2A_3A_4|\}}{|A_1A_2A_3A_4|} \geqslant \frac{\frac{2^3}{2^3+1}|A_4|}{|A_1A_2A_3A_4|} = \frac{2^3}{2^3+2}.$$

Therefore, for $n = 5$ we have

$$\frac{\#\{i \mid d(\sigma^i(a), \sigma^i(b)) \geqslant \frac{1}{2^{12}},\ 1 \leqslant i \leqslant |A_1A_2A_3A_4A_5|\}}{|A_1A_2A_3A_4A_5|} \geqslant \frac{2^4\frac{2^3}{2^3+1}|A_4|}{|A_1A_2A_3A_4A_5|} = \frac{2^3}{2^3+1}\cdot\frac{2^2}{2^2+1}.$$

In the similar way, for any $n \geqslant 5$, we have

$$\begin{aligned}\frac{\#\{i \mid d(\sigma^i(a), \sigma^i(b)) \geqslant \frac{1}{2^{12}},\ 1 \leqslant i \leqslant |A_1A_2\cdots A_n|\}}{|A_1A_2\cdots A_n|} &\geqslant \frac{2^{n-1}2^{n-2}\cdots 2^5 2^4\frac{2^3}{2^3+1}|A_4|}{(2^{n-1}+2)(2^{n-2}+2)\cdots(2^5+2)|A_1A_2A_3A_4A_5|} \\ &= \frac{2^{n-2}}{2^{n-2}+1}\cdot\cdots\cdot\frac{2^3}{2^3+1}\frac{2^2}{2^2+1} \geqslant \frac{1}{\sqrt{e}}.\end{aligned}$$

By the definition of the product metric, $d(f^n(a), f^n(b)) \geqslant \lambda$ implies $(f^n(a), f^n(b)) \in X\times X\setminus[\Delta]_{\frac{\lambda}{2}} \subset X\times X\setminus\overline{[\Delta]}_{\frac{\lambda}{4}}$. Thus, take $\delta = 2^{\frac{1}{14}}$,

$$\limsup_{n\to\infty}\frac{\#\{i \mid 1 \leqslant i \leqslant n,\ (\sigma^i(a), \sigma^i(b)) \in X\times X\setminus\overline{[\Delta]}_{\frac{1}{2^{14}}}\}}{n} \geqslant \limsup_{n\to\infty}\frac{\#\{i \mid 1 \leqslant i \leqslant n,\ d(\sigma^i(a), \sigma^i(b)) \geqslant \frac{1}{2^{14}}\}}{n} \geqslant \frac{1}{\sqrt{e}}. \qquad \square$$

Theorem 5.3. *Let $Y = \omega(a, \sigma)$. Then (Y, σ) is $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(\frac{1}{\sqrt{e}}))$-$\delta$-chaotic for some $\delta > 0$.*

Proof. By Lemma 5.2, the point $a = A_1A_2\cdots$ and $b = \bar{A}_1\bar{A}_2\cdots$ are $(\overline{\mathcal{M}}(\frac{1}{\sqrt{e}}), \frac{1}{2^{14}})$-distal in (Y, σ). Choose $\lambda = \frac{1}{2^{14}}$. Define the map $f: \Sigma_2 \to \Sigma_2$ by $f(x) = C_1C_2\cdots$ for any $x = x_1x_2\cdots \in \Sigma_2$, where for any $i = 1,2,\ldots,$

$$C_i = \begin{cases} A_i, & x_i = 1, \\ \bar{A}_i, & x_i = 0. \end{cases}$$

Then f is continuous. Let $A = f(\Sigma_2)$. So $A \subset Y$ is a closed set and $a = f(111\cdots)$, $b = f(000\cdots)$. Since preimages of $111\cdots$ and $000\cdots$ are dense in (Σ_2, σ), we have

$$A' = \{x \mid \text{there is an } F \in \kappa\mathcal{B} \text{ s.t. } d(f^F(x), f^F(a)) \to 0\},$$

$$B' = \{y \mid \text{there is an } F' \in \kappa\mathcal{B} \text{ s.t. } d(f^{F'}(y), f^{F'}(b)) \to 0\}$$

are dense in A. Thus, by Corollary 4.6, the system (Y, σ) is $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(\frac{1}{\sqrt{e}}))$-$\delta$-chaotic. □

In the rest of the section, we shall point out that the system (Y, σ) is not $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$-chaotic.

Proposition 5.4. *For any $P, Q \in \mathcal{P}_n$ $(n \geqslant 2)$, let $A = PQ|_{[i,|A_1A_2\cdots A_n|]}$. Then $A \in \mathcal{P}_n$ if and only if $i = 1$ or $i = |A_1A_2\cdots A_n| + 1$.*

Proof. The sufficiency is evident. Conversely, we prove the necessity by induction.

If $n = 2$ then it is clear. Let us assume that the result holds for $n = m$, $m \geqslant 2$. Since

$$A_{m+1} = P_1P_1P_2\cdots P_{2^m}, \qquad \bar{A}_{m+1} = P_1\bar{P}_1\bar{P}_2\cdots\bar{P}_{2^m},$$

where each $P_i \in \mathcal{P}_m$ $(1 \leqslant i \leqslant 2^m)$ is a given segment, we write, for any $P \in \mathcal{P}_{m+1}$,

$$P = B_1B_2\cdots B_mB_{m+1} = P'P_1P_1P_2\cdots P_{2^m},$$

or

$$P = B_1 B_2 \cdots B_m B_{m+1} = P' P_1 \bar{P}_1 \bar{P}_2 \cdots \bar{P}_{2^m},$$

where $B_i \in \{A_i, \bar{A}_i\}$, $1 \leqslant i \leqslant m+1$ and $P' = B_1 B_2 \cdots B_m \in \mathcal{P}_m$. Thus, for any $A \in \mathcal{P}_{m+1}$, set

$$A = \tilde{P} P_1 P_1 P_2 \cdots P_{2^m}, \quad \text{or} \quad A = \tilde{P} P_1 \bar{P}_1 \bar{P}_2 \cdots \bar{P}_{2^m},$$

where $\tilde{P} \in \mathcal{P}_m$.

We now consider $PQ|_{[i,|A_1A_2\cdots A_{m+1}|]}$ for $i = k|A_1 \cdots A_m| + 1$, $0 \leqslant k \leqslant 2^m + 2$. If $k = 0$ or $k = 2^m + 2$ then $A = P$ or $A = Q$. It is the desired result. Otherwise, note that for any $1 \leqslant i, j \leqslant 2^m$, $i \neq j$ implies $P_i \neq P_j$. It is easy to check that if $1 \leqslant k \leqslant 2^m + 1$ then $A \neq PQ|_{[i,|A_1A_2\cdots A_n|]}$ for any $i = k|A_1 \cdots A_m| + 1$.

Thus, the result does not hold unless $k = 0$ or $k = 2^m + 2$. □

Theorem 5.5. *The system (Y, σ) is not $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$-chaotic.*

Proof. Choose arbitrarily $x, y \in Y$, we show that either $N((x, y), Y \times Y \setminus \overline{[\Delta]}_\delta, f \times f) \notin \overline{\mathcal{M}}(1)$ for any $\delta > 0$ or $(x, y) \notin \alpha_{\overline{\mathcal{M}}(1)}(\Delta, f \times f)$ holds.

By Claim 1, $a = A_1 A_2 \cdots$ can be generated by the elements of $\mathcal{P}_n$ for any $n \geqslant 1$. Thus, by Proposition 5.4, for a given $n \geqslant 2$ and each $x \in Y$, there exists i such that $x = x_1 x_2 \cdots x_i P_1 P_2 \cdots$, where $x_1, x_2, \ldots, x_i \in \{0, 1\}$, $P_j \in \mathcal{P}_n$, $j = 1, 2, \ldots$. Let

$$I(x, n) = \min\bigl\{i \geqslant 0 \bigm| \sigma^i(x) = P_1 P_2 \ldots, P_j \in \mathcal{P}_n,\ j = 1, 2, \ldots\bigr\}.$$

We consider the following cases:

(1) Let $x, y \in Y$ be distinct points. Assume that $I(x, n) = I(y, n)$ for any $n \geqslant 2$.

For any $m \in \mathbb{N}$, choose $l \in \mathbb{N}$ with $\frac{m+1}{|A_1A_2\cdots A_l|} \leqslant \frac{1}{4}$. Without loss of generality, we can assume that

$$x = P_1 P_2 \ldots, \qquad y = P'_1 P'_2 \ldots,$$

where $P_j, P'_j \in \mathcal{P}_{l+1}$, $j = 1, 2, \ldots$. Take $n \in \mathbb{N}$ with $n > |A_1 A_2 \cdots A_{l+1}|$. We write $k|A_1 A_2 \cdots A_{l+1}| < n \leqslant (1+k)|A_1 A_2 \cdots A_{l+1}|$ for some $k \geqslant 1$. Thus,

$$\begin{aligned}
\frac{\#\{j \mid d(\sigma^j(x), \sigma^j(y)) > \frac{1}{2^m},\ 0 \leqslant j \leqslant n-1\}}{n} &< 1 - \frac{k(|A_1A_2\cdots A_l| - m - 1)}{(k+1)|A_1A_2\cdots A_{l+1}|} \\
&= 1 - \frac{k}{k+1}\bigl(|A_1A_2\cdots A_l| - m - 1\bigr)\frac{1}{(2^l+2)|A_1A_2\cdots A_l|} \\
&= 1 - \frac{k}{k+1}\left[\frac{1}{2^l+2} - \frac{m+1}{(2^l+2)|A_1A_2\cdots A_l|}\right] \\
&\leqslant 1 - \frac{k}{k+1}\left(\frac{1}{2^l+2} - \frac{1}{2^l+2}\cdot\frac{1}{4}\right) \\
&\leqslant 1 - \frac{1}{2^{l+2}+2^3}.
\end{aligned}$$

By the definition of the product metric, $(f^n(x), f^n(y)) \in X \times X \setminus \overline{[\Delta]}_\lambda$ implies $d(f^n(x), f^n(y)) > \lambda$. Thus

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\#\{j \mid (\sigma^j(x), \sigma^j(y)) \in X \times X \setminus \overline{[\Delta]}_{\frac{1}{2^m}},\ 0 \leqslant j \leqslant n-1\}}{n} &\leqslant \limsup_{n\to\infty} \frac{\#\{j \mid d(\sigma^j(x), \sigma^j(y)) > \frac{1}{2^m},\ 0 \leqslant j \leqslant n-1\}}{n} \\
&\leqslant 1 - \frac{1}{2^{l+2}+2^3} < 1.
\end{aligned}$$

It follows that $N((x, y), Y \times Y \setminus \overline{[\Delta]}_\delta, f \times f) \notin \overline{\mathcal{M}}(1)$ for any $\delta > 0$.

(2) Let $x, y \in Y$ be distinct points. Assume that $I(x, N) \neq I(y, N)$ for some $N \geqslant 2$.

Without loss of generality, we can assume that

$$x = x_1 x_2 \cdots = P_1 P_2 \cdots \quad \text{and} \quad y = y_1 y_2 \cdots = y_1 P'_1 P'_2 \cdots, \tag{5.1}$$

where $x_i, y_i \in \{0, 1\}$, $P_i, P'_i \in \mathcal{P}_N$, $i = 1, 2, \ldots$. If there exists a thick set F such that $x_n = y_n$ for any $n \in F$, then for any $m \in \mathbb{N}$ large enough, there exists $i \in F$ such that $x_i x_{i+1} \cdots x_{i+m} = y_i y_{i+1} \cdots y_{i+m} = P_k P_{k+1} \cdots P_s$. By (5.1), we have $y_{i+1} y_{i+2} \cdots y_{i+m} y_{i+m+1} = P'_k P'_{k+1} \cdots P'_s$. According to Proposition 5.4, it is a contradiction. By Theorem 2.1, (x, y) is not $\overline{\mathcal{M}}(1)$-adherent point of the diagonal Δ.

Thus, the system (Y, σ) is not $(\overline{\mathcal{M}}(1), \overline{\mathcal{M}}(1))$-chaotic. □

Acknowledgement

The authors would like to thank the referee for helpful suggestions and corrections.

References

[1] J.C. Xiong, J. Lü, F. Tan, Furstenberg family and chaos, Sci. China Ser. A 50 (2007) 1325–1333.
[2] T.Y. Li, J.A. Yorke, Periodic three implies chaos, Amer. Math. Monthly 82 (1975) 985–992.
[3] J. Smítal, A chaotic function with some extremal properties, Proc. Amer. Math. Soc. 87 (1983) 54–56.
[4] J.C. Xiong, A chaotic map with topological entropy 0, Acta Math. Sci. 6 (1986) 439–443.
[5] B. Schweizer, J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc. 344 (1994) 737–854.
[6] F. Blanchard, E. Glasner, S. Kolyada, A. Maass, On Li–Yorke pairs, J. Reine Angew. Math. 547 (2002) 51–68.
[7] W. Huang, X.D. Ye, Devaney's chaos or 2-scattering implies Li–Yorke's chaos, Topology Appl. 117 (2002) 259–272.
[8] F. Balibrea, J. Smítal, M. Stefankova, The three versions of distributional chaos, Chaos Solitons Fractals 23 (2005) 1581–1583.
[9] G.L. Forti, L. Paganoni, J. Smítal, Dynamics of homeomorphism on minimal sets generated by triangular mappings, Bull. Austral. Math. Soc. 59 (1999) 1–20.
[10] G.F. Liao, Q.J. Fan, Minimal subshifts which display Schweizer–Smítal chaos and have zero topological entropy, Sci. China Ser. A 41 (1998) 33–38.
[11] E. Akin, Recurrence in Topological Dynamics: Furstenberg and Ellis Actions, Plenum Press, New York, 1997.
[12] J. Mycielski, Independent sets in topological algebras, Fund. Math. 55 (1964) 139–147.