



Convergence of the variational iteration method for solving multi-order fractional differential equations

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ABSTRACT

In this paper, the variational iteration method (VIM) is applied to obtain approximate solutions of multi-order fractional differential equations (M-FDEs). We can easily obtain the satisfying solution just by using a few simple transformations and applying the VIM. A theorem for convergence and error estimates of the VIM for solving M-FDEs is given. Moreover, numerical results show that our theoretical analysis are accurate and the VIM is a powerful method for solving M-FDEs.

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1. Introduction

Many phenomena in engineering physics, chemistry, and other sciences can be described very successfully by models that use mathematical tools of fractional calculus, i.e. the theory of derivatives and integrals of non-integer order [1,2]. For example, they have been successfully used in modeling frequency dependent damping behavior of many viscoelastic materials [3]. There are numerous research which demonstrate the applications of fractional derivatives in the areas of electrochemical processes, dielectric polarization, colored noise, and chaos [4–7].

The numerical solution of differential equations of integer order has been a hot topic in numerical and computational mathematics for a long time. The solution of fractional differential equations has been recently studied by numerous authors [2,8–16]. However, the state of the art is far less advanced for general fractional order differential equations. Moreover, to the best of the authors' knowledge, very few algorithms for the numerical solution of M-FDEs have been suggested [17–19], particularly algorithms for analytical solutions and approximate solutions of M-FDEs [20,21].

The VIM that was first proposed by He [22–24] as a modification of the general Lagrange multiplier method [25] has been successfully applied to many situations [22–24,26,27]. Recently, Darvishi et al. [28] have applied the VIM to solve systems of linear and nonlinear stiff ordinary differential equations. Very recently, convergence of the VIM for solving linear systems of ordinary differential equations with constant coefficients has been given [29]. In 1998, the VIM was first proposed to solve fractional differential equations with great success. Following the above idea, some authors [30–39] have applied the VIM to more complex fractional differential equations, and they have showed the effectiveness and the accuracy of the method. But as far as we know, very few research has been done on convergence of the VIM for solving fractional differential equations. In [21], Sweilam et al. have used the VIM to solve the fractional differential system which was generated by an M-FDE. They

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have presented an algorithm to convert the M-FDE into a fractional differential system, then found the analytical solution by using the VIM to solve each equation of the system. If the order of the M-FDE is a very high, it is difficult to find the analytical solution because the fractional differential system has many equations. In particular, convergence of the VIM for solving the M-FDE has not yet been discussed. In this work, we can easily obtain approximate analytical solutions of M-FDEs by means of the VIM and a few simple transformations which are based on the properties of the fractional calculus. Moreover, the convergence of the proposed method is also studied.

This paper is organized as follows. In Section 2, we review the basic definitions and the properties of the fractional calculus. In Section 3, a brief description of the VIM is given. In Section 4, we give the outline of the solution of M-FDEs by using the VIM. The convergence of the proposed method is also given. In Section 5, some illustrative examples are given. Some concluding remarks are given in Section 6.

2. Fractional calculus

In this section, we first review the basic definitions and the operational properties of fractional integral and derivative for the purpose of acquainting with sufficient fractional calculus theory. Many definitions and studies of fractional calculus have been proposed in the past two centuries. These definitions include Riemann–Liouville, Reize, Caputo and Grünwald–Letnikov fractional operators (see [2]). The two most commonly used definitions are the Riemann–Liouville operator and the Caputo operator. We give some definitions and properties of the fractional calculus.

Definition 2.1 (See [2]). The Riemann–Liouville fractional integral operator of order α ($\alpha \geq 0$) is defined as

$$J_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} y(\tau) d\tau, \quad t > a, \alpha > 0,$$

$$J_a^0 y(t) = y(t).$$

Definition 2.2 (See [2]). The Riemann–Liouville fractional derivatives of order α and the Caputo fractional derivatives of order α are defined as

$${}^R D_t^\alpha y(t) = D^n J_a^{n-\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - \tau)^{n-\alpha-1} y(\tau) d\tau, \quad t > a, n - 1 < \alpha \leq n,$$

$${}^C D_t^\alpha y(t) = J_a^{n-\alpha} D^n y(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} y(\tau) d\tau \quad t > a, n - 1 < \alpha \leq n,$$
(2.1)

where D^n is the classical differential operator of order n .

Properties of the operators $J_a^\alpha, {}^R D_t^\alpha$ and ${}^C D_t^\alpha$ can be found in [2], we mention the following.

For $y(t) \in C^m[a, b], \alpha, \beta \geq 0, n - 1 < \alpha \leq n, \alpha + \beta \leq m, a \geq 0$ and $\gamma \geq -1$,

$$(J_a^\alpha J_a^\beta y)(t) = (J_a^\beta J_a^\alpha y)(t) = (J_a^{\alpha+\beta} y)(t),$$
(2.2)

$$J_a^\alpha (t - a)^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} (t - a)^{\gamma+\alpha},$$
(2.3)

$$(J_a^\alpha {}^C D_t^\alpha y)(t) = J^n D^n y(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(a^+) \frac{(x - a)^k}{k!}.$$
(2.4)

Furthermore, if $y^{(i)}(a) = 0, i = 0, 1, \dots, m - 1$, then

$${}^R D_t^\alpha y(t) = {}^C D_t^\alpha y(t),$$
(2.5)

$${}^R D_t^\alpha (t - a)^\gamma = {}^C D_t^\alpha (t - a)^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \alpha + 1)} (t - a)^{\gamma-\alpha},$$
(2.6)

$${}^C D_t^\alpha {}^C D_t^\beta y(t) = {}^C D_t^\beta {}^C D_t^\alpha y(t) = {}^C D_t^{\alpha+\beta} y(t),$$
(2.7)

$${}^C D_t^\alpha J_a^\beta y(t) = J_a^\beta {}^C D_t^\alpha y(t) = {}^C D_t^{\alpha-\beta} y(t) = J_a^{\beta-\alpha} y(t).$$
(2.8)

Lemma 2.1. If $n - 1 < \alpha \leq n$, then

$${}^C D_t^\alpha (t^i) = 0, \quad \forall i = 0, 1, \dots, n - 1.$$
(2.9)

Proof. Note the definition of Caputo derivatives, obviously, it is true. □

3. A brief description of the VIM

To illustrate the basic concept of the VIM, we consider the following general nonlinear equation

$$Lu(t) + Nu(t) = g(t), \tag{3.1}$$

where L is the linear operator, N is the nonlinear operator, and $g(t)$ is the inhomogeneous term. In [22–24] the VIM was proposed by He, where a correction functional for Eq. (3.1) can be constructed as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))d\xi. \tag{3.2}$$

Obviously, successive approximations $u_j, j \geq 0$ can be established by determining the Lagrange multiplier λ . The function $\delta\tilde{u}_n(t)$ is a restricted variation, which means that $\delta\tilde{u}_n(t) = 0$. Therefore, we first determine the Lagrange multiplier λ , which can be identified optimally via the variational theory. The successive approximations $u_n(t), n \geq 1$ of the solution $u(t)$ will be readily obtained upon using Lagrange's multiplier obtained by using any selective initial function $u_0(t)$.

4. The VIM for solving M-FDEs

In this section, we shall apply the VIM to solve M-FDEs. Convergence results of the VIM for solving M-FDEs will be given. We define the norm $\|u(t)\|_\infty = \max_{0 \leq t \leq T} |u(t)|, \forall u(t) \in \mathbb{C}[0, T]$.

Consider the following initial value problem

$$\begin{cases} {}_0^C D_t^{\alpha_n} x(t) = F(t, x(t), {}_0^C D_t^{\alpha_1} x(t), \dots, {}_0^C D_t^{\alpha_{n-1}} x(t)), & t \in [0, T], \\ x^{(i)}(0) = x_0^i, & i = 0, 1, \dots, n-1, \end{cases} \tag{4.1}$$

where $n-1 \leq \alpha_n \leq n, 0 \leq \alpha_i < \alpha_n \leq n, i = 1, 2, \dots, n-1, F : D = [0, T] \times \mathbb{R} \times \dots \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous mapping, and the function $F(t, u_0, u_1, \dots, u_{n-1})$ exists with continuous and bounded partial derivatives $\frac{\partial F}{\partial u_i}$,

and $\beta_i = \sup_{0 \leq t \leq T} \left| \frac{\partial F}{\partial u_i} \right|, i = 0, 1, \dots, n-1$.

If we take $\bar{x}(t) = x(t) - \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i$, Eq. (4.1) can be written as

$$\begin{cases} {}_0^C D_t^{\alpha_n} \bar{x}(t) = F\left(t, \bar{x}(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, {}_0^C D_t^{\alpha_1} \bar{x}(t) + {}_0^C D_t^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, \dots, {}_0^C D_t^{\alpha_{n-1}} \bar{x}(t) + {}_0^C D_t^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i\right), \\ \bar{x}^{(i)}(0) = 0, & i = 0, 1, \dots, n-1. \end{cases} \tag{4.2}$$

According to properties (2.1) and (2.5), we have ${}_0^C D_t^{\alpha_n} \bar{x}(t) = D^n J_0^{n-\alpha_n} \bar{x}(t)$. Let $J_0^{n-\alpha_n} \bar{x}(t) = y(t)$, then problem (4.2) has the following form

$$\begin{aligned} y^{(n)}(t) = & F\left(t, {}_0^C D_t^{n-\alpha_n} y(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, {}_0^C D_t^{n-\alpha_n+\alpha_1} y(t) + {}_0^C D_t^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, \dots, {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} y(t) \right. \\ & \left. + {}_0^C D_t^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i\right). \end{aligned} \tag{4.3}$$

In view of the VIM, we construct the correction functional in the following form

$$\begin{aligned} y_{m+1}(t) = & y_m(t) + \int_0^t \lambda(\xi) \left[y_m^{(n)}(\xi) - F\left(\xi, {}_0^C D_\xi^{n-\alpha_n} \tilde{y}_m(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} \tilde{y}_m(\xi) \right. \right. \\ & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} \tilde{y}_m(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i\right) \right] d\xi. \end{aligned} \tag{4.4}$$

Making the above correction functional stationary, and noting that $\delta\tilde{y}_m(t) = 0$,

$$\delta y_{m+1}(t) = \delta y_m(t) + \delta \int_0^t \lambda(\xi) y_m^{(n)}(\xi) d\xi.$$

This yields the stationary conditions

$$\lambda(\xi)|_{\xi=t} = 0, \quad \lambda'(\xi)|_{\xi=t} = 0, \dots, 1 + (-1)^{n-1} \lambda^{(n-1)}(\xi)|_{\xi=t} = 0, \quad \lambda^{(n)}(\xi) = 0,$$

which gives $\lambda(\xi) = -\frac{(t-\xi)^{n-1}}{(n-1)!}$. For example, if we take $n = 1$, then $\lambda(\xi) = -1$, and if $n = 2$, then $\lambda(\xi) = -(t - \xi)$.

As a result, we get the following iteration formula

$$\begin{aligned}
 y_{m+1}(t) = & y_m(t) - \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left[y_m^{(n)}(\xi) - F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y_m(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y_m(\xi) \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y_m(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right] d\xi. \tag{4.5}
 \end{aligned}$$

Beginning with an initial approximation $y_0(t)$, we obtain the successive approximations from which the exact solution can be obtained by

$$y(t) = \lim_{m \rightarrow \infty} y_m(t).$$

So problem (4.1) has the following exact solution

$$x(t) = {}_0^C D_t^{n-\alpha_n} y(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i.$$

Now, we show that the sequence $\{y_m(t)\}_{m=1}^\infty$ defined by Eq. (4.5) with $y_0(t) = y_0$ converges to the solution of (4.3). To do this we state and prove the following theorem.

Theorem 4.1. *Let $y(t), y_i(t) \in C^n[0, T], i = 0, 1, \dots$. The sequence defined by Eq. (4.5) with $y_0(t) = y_0$ converges to $y(t)$, i.e. the exact solution of Eq. (4.3). Furthermore, we have the following error estimate*

$$\|E_{m+1}(t)\|_\infty \leq \|E_0(t)\|_\infty \frac{(\beta n T^{\alpha_n})^{m+1}}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}) + 1)},$$

where $\beta = \max_{0 \leq j \leq n-1} \beta_j, E_j(t) = y_j(t) - y(t), j = 1, 2, \dots$

Proof. Obviously from Eq. (4.3) we have

$$\begin{aligned}
 y(t) = & y(t) - \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left[y^{(n)}(\xi) - F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y(\xi) \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right] d\xi. \tag{4.6}
 \end{aligned}$$

Now from Eqs. (4.5) and (4.6), we get

$$\begin{aligned}
 E_{m+1}(t) = & E_m(t) - \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left\{ E_m^{(n)}(\xi) - \left[F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y_m(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y_m(\xi) \right. \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y_m(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right. \\
 & \left. - F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y(\xi) \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right\} d\xi, \tag{4.7}
 \end{aligned}$$

where $E_j(t) = y_j(t) - y(t), j = 1, 2, \dots$

By using the fact that $E_m^{(i)}(0) = 0, m = 0, 1, \dots, i = 0, 1, \dots, n-1$, and integration by parts we obtain

$$\begin{aligned}
 E_{m+1}(t) = & \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left[F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y_m(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y_m(\xi) \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y_m(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right. \\
 & \left. - F \left(\xi, {}_0^C D_\xi^{n-\alpha_n} y(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, {}_0^C D_\xi^{n-\alpha_n+\alpha_1} y(\xi) \right. \right. \\
 & \left. \left. + {}_0^C D_\xi^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i, \dots, {}_0^C D_\xi^{n-\alpha_n+\alpha_{n-1}} y(\xi) + {}_0^C D_\xi^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) \right] d\xi. \tag{4.8}
 \end{aligned}$$

Noting that $F(t, u_0, u_1, \dots, u_{n-1})$ exists with continuous and bounded partial derivatives $\frac{\partial F}{\partial u_i}, \forall i = 0, 1, \dots, n - 1$, and using Lagrange’s theorem and the definition of the Riemann–Liouville fractional integral operator, we have

$$\begin{aligned}
 |E_{m+1}(t)| &= \left| J_0^n F \left(t, {}_0^C D_t^{n-\alpha_n} y_m(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, {}_0^C D_t^{n-\alpha_n+\alpha_1} y_m(t) \right. \right. \\
 &\quad \left. \left. + {}_0^C D_t^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, \dots, {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} y_m(t) + {}_0^C D_t^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i \right) \right. \\
 &\quad \left. - J_0^n F \left(t, {}_0^C D_t^{n-\alpha_n} y(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, {}_0^C D_t^{n-\alpha_n+\alpha_1} y(t) \right. \right. \\
 &\quad \left. \left. + {}_0^C D_t^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i, \dots, {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} y(t) + {}_0^C D_t^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i \right) \right| \\
 &= |J_0^n [F'_2(\eta(t)) {}_0^C D_t^{n-\alpha_n} E_m(t) + \dots + F'_{n+1}(\eta(t)) {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} E_m(t)]| \\
 &\leq \beta_0 |J_0^n {}_0^C D_t^{n-\alpha_n} E_m(t)| + \beta_1 |J_0^n {}_0^C D_t^{n-\alpha_n+\alpha_1} E_m(t)| + \dots + \beta_{n-1} |J_0^n {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} E_m(t)| \\
 &\leq \beta_0 J_0^{\alpha_n} |E_m(t)| + \beta_1 J_0^{\alpha_n-\alpha_1} |E_m(t)| + \dots + \beta_{n-1} J_0^{\alpha_n-\alpha_{n-1}} |E_m(t)| \\
 &= (\beta_0 J_0^{\alpha_n} + \beta_1 J_0^{\alpha_n-\alpha_1} + \dots + \beta_{n-1} J_0^{\alpha_n-\alpha_{n-1}}) |E_m(t)| \\
 &\quad \vdots \\
 &\leq (\beta_0 J_0^{\alpha_n} + \beta_1 J_0^{\alpha_n-\alpha_1} + \dots + \beta_{n-1} J_0^{\alpha_n-\alpha_{n-1}})^{m+1} |E_0(t)| \\
 &\leq (\beta_0 J_0^{\alpha_n} + \beta_1 J_0^{\alpha_n-\alpha_1} + \dots + \beta_{n-1} J_0^{\alpha_n-\alpha_{n-1}})^{m+1} \max_{0 \leq s \leq T} |E_0(s)| \\
 &\leq \beta^{m+1} (J_0^{\alpha_n} + J_0^{\alpha_n-\alpha_1} + \dots + J_0^{\alpha_n-\alpha_{n-1}})^{m+1} \max_{0 \leq s \leq T} |E_0(s)| \\
 &\leq \beta^{m+1} \max_{0 \leq s \leq T} |E_0(s)| n^{m+1} \frac{1}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}))} \int_0^t (t - \tau)^{(m+1)\alpha_n-1} d\tau \\
 &= \beta^{m+1} \max_{0 \leq s \leq T} |E_0(s)| n^{m+1} \frac{t^{(m+1)\alpha_n}}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}))(m+1)\alpha_n}, \tag{4.9}
 \end{aligned}$$

where $\beta = \max_{0 \leq i \leq n-1} \beta_i$, F'_i is the partial derivative of function F for the i th variable,

$$\begin{aligned}
 \eta(t) &= \left(t, {}_0^C D_t^{n-\alpha_n} y(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i + \theta ({}_0^C D_t^{n-\alpha_n} E_m(t)), {}_0^C D_t^{n-\alpha_n+\alpha_1} y(t) + {}_0^C D_t^{\alpha_1} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i \right. \\
 &\quad \left. + \theta ({}_0^C D_t^{n-\alpha_n+\alpha_1} E_m(t)), \dots, {}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} y(t) + {}_0^C D_t^{\alpha_{n-1}} \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i + \theta ({}_0^C D_t^{n-\alpha_n+\alpha_{n-1}} E_m(t)) \right), \quad 0 \leq \theta \leq 1.
 \end{aligned}$$

Noting that $T, \beta, \alpha_n, \|E_0(t)\|_\infty, n$ are constants, $0 < \alpha_n - \alpha_{n-1}$,

$$\begin{aligned}
 \frac{1}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}))(m+1)\alpha_n} &\leq \frac{1}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}))(m+1)(\alpha_n - \alpha_{n-1})} \\
 &= \frac{1}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}) + 1)},
 \end{aligned}$$

and based on the convergence of Mittag-Leffler functions [2], we have

$$\|E_{m+1}(t)\|_\infty \leq \|E_0(t)\|_\infty \frac{(\beta n T^{\alpha_n})^{m+1}}{\Gamma((m+1)(\alpha_n - \alpha_{n-1}) + 1)} \rightarrow 0$$

as $m \rightarrow \infty$. This completes the proof. \square

For example, we consider the initial value of the linear fractional differential equation

$$\begin{cases}
 {}_0^C D_t^{\alpha_n} x(t) = Ax(t) + f(t), & t \in [0, T], \\
 x^{(i)}(0) = x_0^i, & i = 0, 1, \dots, n - 1,
 \end{cases} \tag{4.10}$$

where $n - 1 \leq \alpha_n \leq n, A$ is a constant.

The same idea when applied to problem (4.10), we have

$$y^{(n)}(t) = A \left({}^C_0D_t^{n-\alpha_n} y(t) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i \right) + f(t) \tag{4.11}$$

where $y(t) = J_0^{n-\alpha_n} \left(x(t) - \sum_{i=0}^{n-1} \frac{x_0^i}{i!} t^i \right)$.

Therefore, from the above analysis the following iteration formula for computing $y_m(t)$ can be obtained

$$y_{m+1}(t) = y_m(t) - \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!} \left[y_m^{(n)}(\xi) - A \left({}^C_0D_\xi^{n-\alpha_n} y_m(\xi) + \sum_{i=0}^{n-1} \frac{x_0^i}{i!} \xi^i \right) - f(\xi) \right] d\xi, \tag{4.12}$$

where $m = 0, 1, \dots$. According to Theorem 4.1, we give the following corollary.

Corollary 4.1. Let $y(t), y_i(t) \in C^n[0, T], i = 0, 1, \dots$. The sequence $\{y_m(t)\}_{m=1}^\infty$ defined by (4.12) with $y_0(t) = y_0$ converges to $y(t)$, which is the exact solution of (4.11).

Proof. Let $F(t, x(t)) = Ax(t) + f(t)$. Corollary 4.1 is easily proved by means of Theorem 4.1. \square

5. Illustrative examples

In this section, some illustrative examples are given to show the efficiency of the VIM for solving M-FDEs. All of the computations have been done by using the MAPLE software.

Example 5.1. Consider the following initial value problem

$$\begin{cases} {}^C_0D_t^\alpha x(t) + x(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + t^3, & \alpha \in (1, 2), \\ x(0) = 0, & x'(0) = 0. \end{cases} \tag{5.1}$$

If we take $\alpha = 1.9$ and $y(t) = J_0^{2-\alpha}(x(t))$, Eq. (5.1) can be written as

$$y''(t) + {}^C_0D_t^{0.1} y(t) = \frac{2}{\Gamma(1.1)} t^{0.1} + t^3. \tag{5.2}$$

To solve Eq. (5.2) by means of the VIM, we have the following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\xi - t) \left[y_n''(\xi) + {}^C_0D_\xi^{2-\alpha} y_n(\xi) - \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - t^3 \right] d\xi. \tag{5.3}$$

To get the components of the solution, we start with an initial approximation

$$y_0(t) = 0. \tag{5.4}$$

By using the above iteration formula (5.3), we can obtain the other components by using the MAPLE package as follows

$$\begin{aligned} y_1(t) &= 0.083333333333t^4 + 0.9100753299t^{2.1}, \\ y_2(t) &= 0.9100753299t^{2.1} - 0.003347313231t^{5.9}, \\ y_3(t) &= 0.9100753299t^{2.1} + 0.0007593018360t^{7.8}, \\ y_4(t) &= 0.9100753299t^{2.1} - 0.000001111207366t^{9.7}, \\ y_5(t) &= 0.9100753299t^{2.1} - \text{small terms}, \\ &\vdots \\ y_n(t) &= 0.9100753299t^{2.1} - \text{small terms}, \end{aligned}$$

where the small terms are the terms whose coefficients can converge to 0 as $n \rightarrow \infty$. As we observed, the sequence is convergent to $y(t) = 0.9100753299t^{2.1}$, i.e.

$$y(t) = \lim_{n \rightarrow \infty} y_n(t) = 0.9100753299t^{2.1},$$

and the exact solution of problem (5.1) is

$$x(t) = {}^C_0D_t^{2-\alpha} y(t) = t^2.$$

It can be verified that $x(t) = t^2$ is indeed the exact solution of problem (5.1).

Example 5.2. Consider the following initial value problem

$$\begin{cases} {}_0^C D_t^\alpha x(t) + t \cdot {}_0^C D_t^\beta x(t) + x(t) = \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{6}{\Gamma(4-\beta)} t^{4-\beta} + t^3 + t, & \alpha \in (2, 3), \beta \in (1, 2), \\ x(0) = 0, & x'(0) = 1, & x''(0) = 0. \end{cases} \quad (5.5)$$

Let $\bar{x}(t) = x(t) - t$, Eq. (5.5) can be written as

$$\begin{cases} {}_0^C D_t^\alpha \bar{x}(t) + t \cdot {}_0^C D_t^\beta \bar{x}(t) + \bar{x}(t) = \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{6}{\Gamma(4-\beta)} t^{4-\beta} + t^3, & \alpha \in (2, 3), \beta \in (1, 2), \\ \bar{x}(0) = 0, & \bar{x}'(0) = 0, & \bar{x}''(0) = 0. \end{cases} \quad (5.6)$$

If we take $\alpha = 2.5$, $\beta = 1.9$, and $\bar{x}(t) = {}_0^C D_t^{0.5} y(t)$ then we can obtain

$$y^{(3)}(t) + t \cdot {}_0^C D_t^{2.4} y(t) + {}_0^C D_t^{0.5} y(t) = \frac{6}{\Gamma(1.5)} t^{0.5} + \frac{6}{\Gamma(2.1)} t^{2.1} + t^3. \quad (5.7)$$

To solve Eq. (5.7) by means of the VIM, we have the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t \frac{(t-\xi)^2}{2} \left[y_n^{(3)}(\xi) + \xi \cdot {}_0^C D_\xi^{2.4} y_n(\xi) + {}_0^C D_\xi^{0.5} y_n(\xi) - \frac{6}{\Gamma(1.5)} \xi^{0.5} - \frac{6}{\Gamma(2.1)} \xi^{2.1} - \xi^3 \right] d\xi. \quad (5.8)$$

To get the components of the solution, we start with an initial approximation

$$y_0(t) = 0. \quad (5.9)$$

By using the above iteration formula (5.8), we can obtain the other components by using the MAPLE package as follows

$$\begin{aligned} y_1(t) &= 0.5158304763t^{3.5} + 0.008333333333t^6 + 0.08845088132t^{5.1}, \\ y_2(t) &= 0.5158304763t^{3.5} - 0.00005029655604t^{8.5} - 0.01683135569t^{6.7} - 0.002325009960t^{7.6}, \\ y_3(t) &= 0.5158304763t^{3.5} + 0.00001802710497t^{10.1} + 0.0000001503126503t^{11} \\ &\quad + 0.0005185986048t^{9.2} + 0.003207427930t^{8.3} - \text{small terms}, \\ y_4(t) &= 0.5158304763t^{3.5} - 0.0000000002598266671t^{13.5} - 0.00000006348363523t^{12.6} \\ &\quad - 0.0001041376657t^{10.8} - 0.000004510069285t^{11.7} - 0.0005941036288t^{9.9} - \text{small terms}, \\ &\vdots \\ y_n(t) &= 0.5158304763t^{3.5} - \text{small terms}, \end{aligned}$$

where the small terms are the terms whose coefficients can converge to 0 as $n \rightarrow \infty$. As we observed, the sequence is convergent to $y(t) = 0.5158304763t^{3.5}$, i.e.

$$y(t) = \lim_{n \rightarrow \infty} y_n(t) = 0.5158304763t^{3.5},$$

and the exact solution problem (5.5) is

$$x(t) = {}_0^C D_t^{0.5} y(t) + t = t^3 + t.$$

It can be verified that $x(t) = t^3 + t$ is indeed the exact solution of problem (5.5).

Example 5.3. Consider the following initial value problem

$$\begin{cases} {}_0^C D_t^\alpha x(t) + {}_0^C D_t^\gamma x(t) \cdot {}_0^C D_t^\beta x(t) + x^2(t) = \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{36}{\Gamma(4-\gamma)\Gamma(4-\beta)} t^{6-\beta-\gamma} + t^6, \\ \alpha \in (2, 3), \beta \in (1, 2), \gamma \in (0, 1), \\ x(0) = 0, & x'(0) = 0, & x''(0) = 0. \end{cases} \quad (5.10)$$

If we take $\alpha = 2.5$, $\beta = 1.9$, $\gamma = 0.5$, and $x(t) = {}_0^C D_t^{0.5} y(t)$ then we can obtain

$$y^{(3)}(t) + y'(t) \cdot {}_0^C D_t^{2.4} y(t) + ({}_0^C D_t^{0.5} y(t))^2 = \frac{6}{\Gamma(1.5)} t^{0.5} + \frac{36}{\Gamma(3.5)\Gamma(2.1)} t^{3.6} + t^6. \quad (5.11)$$

To solve Eq. (5.11) by means of the VIM, we have the following iteration formula

$$y_{n+1}(t) = y_n(t) - \int_0^t \frac{(t-\xi)^2}{2} \left[y_n^{(3)}(\xi) + y_n'(\xi) \cdot {}^C_0D_\xi^{2.4} y_n(\xi) + ({}^C_0D_\xi^{0.5} y_n(\xi))^2 - \frac{6}{\Gamma(1.5)} \xi^{0.5} - \frac{36}{\Gamma(3.5)\Gamma(2.1)} \xi^{3.6} - \xi^6 \right] d\xi. \quad (5.12)$$

To get the components of the solution, we start with an initial approximation

$$y_0(t) = 0. \quad (5.13)$$

By using the above iteration formula (5.12), we can obtain the other components by using the MAPLE package as follows

$$\begin{aligned} y_1(t) &= 0.5158304763 t^{7/2} + 0.06088399462 t^{33/5} + 0.001984126984 t^9, \\ y_2(t) &= 0.5158304763 t^{7/2} - 0.000004933302048 t^{29/2} - 0.00002632928733 t^{46/5} - 0.007437462059 t^{67/10} \\ &\quad - 0.0004966762372 t^{121/10} - 0.000001239970840 t^{88/5} - 0.003545547055 t^{97/10} - 0.0000005769973533 t^{73/5} \\ &\quad - 0.000009473952122 t^{61/5} - 0.0004570164956 t^{91/10} - 0.00004463608173 t^{76/5} - 0.0004417137938 t^{49/5}, \\ y_3(t) &= 0.5158304763 t^{7/2} + 0.000001117333580 t^{73/5} - 0.00001571036794 t^{13} - \text{small terms}, \\ y_4(t) &= 0.5158304763 t^{7/2} - \text{small terms}, \\ &\vdots \\ y_n(t) &= 0.5158304763 t^{7/2} - \text{small terms}, \end{aligned}$$

where the small terms are the terms whose coefficients can converge to 0 as $n \rightarrow \infty$. As we observed, the sequence is convergent to $y(t) = 0.5158304763 t^{3.5}$, i.e.

$$y(t) = \lim_{n \rightarrow \infty} y_n(t) = 0.5158304763 t^{3.5},$$

and the exact solution of problem (5.10) is

$$x(t) = {}^C_0D_t^{0.5} y(t) = t^3.$$

It can be verified that $x(t) = t^3$ is indeed the exact solution of problem (5.10).

So it can be clearly seen that when we begin by taking the initial condition as the initial approximation, there will be very good agreement between the approximate solutions obtained by the VIM and the exact solution.

6. Conclusions

In this research, we have successfully used the VIM to solve fractional order differential equations. A theorem for the convergence of the VIM for solving M-FDEs has been given. Our work shows that the method can be used in a direct way without restrictive assumptions and high computational cost. The results obtained by the proposed method confirm the robustness and the efficiency of it. And we hope that the work in this paper is a step in this direction.

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