On Monochromatic Paths in Edge-Coloured Digraphs*

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Let $G$ be a directed graph whose edges are coloured with two colours. Call a set $S$ of vertices of $G$ independent if no two vertices of $S$ are connected by a monochromatic directed path. We prove that if $G$ contains no monochromatic infinite outward path, then there is an independent set $S$ of vertices of $G$ such that, for every vertex $x$ not in $S$, there is a monochromatic directed path from $x$ to a vertex of $S$. In the event that $G$ is infinite, the proof uses Zorn's lemma. The last part of the paper is concerned with the case when $G$ is a tournament.

In this paper $G$ will denote a directed graph, possibly infinite, possibly with multiple edges. A directed path, or simply a path, is a (finite or infinite) sequence $x_1, x_2, x_3, \ldots$ of distinct vertices of $G$ such that there is a directed edge from $x_i$ to $x_{i+1}$ for each $i$. If the sequence $x_1, x_2, x_3, \ldots$ is infinite we call the path an infinite outward path. Let the edges of $G$ be coloured with two colours. A set $S$ of vertices of $G$ is called independent if no two vertices of $S$ are connected by a monochromatic directed path.

The following is our main result. It arose from consideration of an unpublished problem of I. Rival and the first author.

**Theorem 1.** Let $G$ be a directed graph whose edges are coloured with two colours, such that $G$ contains no monochromatic infinite outward path. Then there is an independent set $S$ of vertices of $G$ such that, for every vertex $x$ not in $S$, there is a monochromatic path from $x$ to a vertex of $S$.

**Proof.** Let the two colours be red and blue. We first introduce some notation. For distinct vertices $x$, $y$ of $G$, $x \rightarrow \text{red} y$ will mean that there is a

directed path from $x$ to $y$, all of whose edges are coloured red. If $S$ is a set of vertices of $G$ and $x$ is a vertex of $G$, $x \rightarrow \text{red} S$ ($S \rightarrow \text{red} x$) will mean that $x \rightarrow \text{red} s$ ($s \rightarrow \text{red} x$) for some $s \in S$. The negation of, for instance, $x \rightarrow \text{red} s$ will be denoted $x \not\rightarrow \text{red} s$. Similarly we define, e.g., $x \rightarrow \text{blue} y$ and $x \rightarrow \text{mono} y$, the latter meaning that there is a monochromatic path from $x$ to $y$.

For sets $S, T$ of vertices of $G$, put $S \leq T$ if for all $s \in S$ there is a $t \in T$ such that either $s = t$, or $s \rightarrow \text{blue} t$ and $t \not\rightarrow \text{blue} s$. In particular, $S \leq T$ implies $S \subseteq T$. It is easy to see that the collection of all independent sets of vertices of $G$ is partially ordered by $\leq$.

Let $\mathcal{S}$ be the family of all nonempty independent sets $S$ of vertices of $G$ such that $S \rightarrow \text{red} y$ implies $y \rightarrow \text{mono} S$ for all vertices $y$ of $G$. Note that $\mathcal{S}$ is nonempty; there is a vertex $v$ such that $v \rightarrow \text{red} y$ implies $y \rightarrow \text{red} v$ for all $y$ (since otherwise we can construct an infinite outward red path), whence $\{v\} \in \mathcal{S}$.

We claim that $(\mathcal{S}, \leq)$ has maximal elements. To see this, let $\mathcal{C}$ be a chain in $(\mathcal{S}, \leq)$, and define

$$S^\infty = \left\{ s \in \bigcup \mathcal{C} : \exists S \in \mathcal{C} \text{ such that } s \in T \text{ whenever } T \in \mathcal{C} \text{ and } T \geq S \right\}.$$  

($S^\infty$ consists of all vertices of $G$ that belong to every member of $\mathcal{C}$ from some point on.) Let $S \in \mathcal{C}$ and $s \in S$. If $s \in S^\infty$, there is $S_1 \in \mathcal{C}$ such that $S_1 \geq S$ and $s \in S_1$. Thus there must be $s_1 \in S_1$ such that $s \rightarrow \text{blue} s_1$ and $s_1 \not\rightarrow \text{blue} s$. If $s_1 \notin S^\infty$, then there is $S_2 \in \mathcal{C}$ such that $S_2 \geq S_1$ and $s_1 \notin S_2$. Thus there must be $s_2 \in S_2$ such that $s_1 \rightarrow \text{blue} s_2$ and $s_2 \not\rightarrow \text{blue} s_1$. It follows that $s \rightarrow \text{blue} s_2$ and $s_2 \not\rightarrow \text{blue} s$. Now if $s \notin S^\infty$ we may continue, but, since $G$ has no infinite outward blue path, this procedure must eventually terminate. That is, we obtain some $S_n \in \mathcal{C}$ and $s_n \in S_n$ such that $s \rightarrow \text{blue} s_n$, $s_n \not\rightarrow \text{blue} s$, and $s_n \in S^\infty$. We have proven that $S^\infty$ is nonempty and that $S^\infty \geq S$ for all $S \in \mathcal{C}$. To show $S^\infty$ is independent, let $s, t \in S^\infty$, and suppose without loss of generality that $S, T \in \mathcal{C}$ are such that $s \in S, s \in U$ whenever $U \in \mathcal{C}$ and $U \geq S, t \in T$, and $S \leq T$. Then $s \in T$, and since $T$ is independent $s$ and $t$ cannot be connected by a monochromatic path. Thus $S^\infty$ is independent. Finally, to show $S^\infty \in \mathcal{S}$ let $s \rightarrow \text{red} y$ for $s \in S^\infty$ and $y$ a vertex of $G$. There is $s \in S$ such that $s \in S$, and from the definition of $\mathcal{S}$ there is $t \in S$ such that $y \rightarrow \text{mono} t$. If $t \in S^\infty$, then we are done. Suppose $t \notin S^\infty$, so that in particular $t \neq s$. Then $y \not\rightarrow \text{red} t$, since $S$ is independent and $s \rightarrow \text{red} y$; thus $y \rightarrow \text{blue} t$, and since $S^\infty \geq S$ there is $t^\infty \in S^\infty$ such that $t \rightarrow \text{blue} t^\infty$. Hence $y \rightarrow \text{blue} S^\infty$, showing that $S^\infty \in \mathcal{S}$. We have proven that any chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$, and so, by Zorn's lemma, $(\mathcal{S}, \leq)$ contains maximal elements.

Let $S$ be a maximal element of $(\mathcal{S}, \leq)$; we claim that $S$ is the set we seek. Suppose to the contrary that there is a vertex $x$ not in $S$ such that $x \not\rightarrow \text{mono} S$. 
Since \( G \) has no infinite outward red path, we may choose \( x \) so that \( y \rightarrow^\text{red} x \) for each vertex \( y \) not in \( S \) satisfying \( y \leftrightarrow^\text{mono} S \) and \( x \rightarrow^\text{red} y \). Also note that, by the definition of \( \mathcal{S} \), \( S \not\rightarrow^\text{red} x \).

Let \( T = \{ t \in S \mid t \leftrightarrow^\text{blue} x \} \), so that \((T \not\subset S) S - T \rightarrow^\text{blue} x \). Then \( T \cup \{ x \} \) is independent and \( T \cup \{ x \} \gg S \). By the maximality of \( S \), there is a vertex \( y \) such that \( T \cup \{ x \} \rightarrow^\text{red} y \) and \( y \leftrightarrow^\text{mono} T \cup \{ x \} \). Clearly, \( y \notin S \). If \( T \rightarrow^\text{red} y \), then \( S \rightarrow^\text{red} y \), and it follows from the definition of \( \mathcal{S} \) that \( y \rightarrow^\text{mono} S - T \). But this is impossible; \( y \not\rightarrow^\text{red} S - T \) because \( T \rightarrow^\text{red} y \) and \( S \) is independent, and \( y \not\leftrightarrow^\text{blue} S - T \) because \( S - T \rightarrow^\text{blue} x \) and \( y \not\leftrightarrow^\text{blue} x \). Hence we must have \( x \rightarrow^\text{red} y \). Now \( y \rightarrow^\text{red} S \) because \( x \not\rightarrow^\text{red} S \), and \( y \not\leftrightarrow^\text{blue} S \) because \( S - T \not\leftrightarrow^\text{blue} x \) and \( y \not\leftrightarrow^\text{blue} T \cup \{ x \} \). Also \( y \not\rightarrow^\text{red} x \), and this contradicts the choice of \( x \). Thus \( S \) satisfies the conditions of the theorem.

We point out a special case.

**Corollary 2.** Let \( T \) be a finite tournament whose edges are coloured with two colours. Then there is a vertex \( v \) of \( T \) such that for every other vertex \( x \) of \( T \) there is a monochromatic path from \( x \) to \( v \).

If the edges of the tournament are not coloured (or all coloured the same), then this result is fairly well known. In fact, every finite tournament \( T \) contains a vertex \( v \) such that for every other vertex \( x \) of \( T \) there is a directed path of length at most two from \( x \) to \( v \) (e.g., [2]; see [1] for an extension to directed graphs in general). On the other hand, no such bounding of path lengths is possible in Corollary 2. For example, consider the tournament with vertices \( \{ t_1, t_2, \ldots, t_n \} \) and coloured directed edges as follows:

\[
\begin{align*}
(t_i, t_{i+1}) & \quad \text{coloured red for } 1 \leq i \leq n - 1 \\
(t_j, t_i) & \quad \text{coloured blue for } j > i + 1.
\end{align*}
\]

Then \( t_n \) is the only vertex satisfying the conclusion of Corollary 2; moreover, the only monochromatic directed path from \( t_1 \) to \( t_n \) passes through all the other vertices.

Interestingly, if we use more than two colours the corollary is false. A simple counterexample is the three-element tournament \( \{ a, b, c \} \) with directed edges \((a, b), (b, c), \) and \((c, a)\), all coloured differently (for future reference, we call this edge-coloured tournament \( T_3 \)). It is not hard to add vertices to this tournament to construct arbitrarily large finite counterexamples.

Now consider the tournament with vertices

\[ a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \]
and coloured directed edges

\[(a_1, a_2), (b_1, b_2), (c_1, c_2)\] coloured red,
\[(a_2, a_3), (b_2, b_3), (c_2, c_3)\] coloured blue,
\[(a_3, a_1), (b_3, b_1), (c_3, c_1)\] coloured green,
\[(a_i, b_j)\] coloured red for all \(i, j\),
\[(b_i, c_j)\] coloured blue for all \(i, j\),
\[(c_i, a_j)\] coloured green for all \(i, j\).

This is a tournament whose edges are coloured with three colours and for which no pair of vertices satisfies the conclusion of Corollary 2. More precisely, there is no set \(S\) of exactly two vertices such that for every other vertex \(v\) there is a monochromatic path from \(v\) to a vertex of \(S\). For example, \(S = \{a_1, b_3\}\) does not work because there is no monochromatic path from \(b_3\) to either \(a_1\) or \(b_2\). The following question (due also to Erdős) is still open.

**PROBLEM.** For each \(n\), is there a (least) positive integer \(f(n)\) so that every finite tournament whose edges are coloured with \(n\) colours contains a set \(S\) of \(f(n)\) vertices with the property that for every vertex \(u\) not in \(S\) there is a monochromatic path from \(u\) to a vertex of \(S\)? In particular, is \(f(3) = 3\)?

Corollary 2, of course, just says that \(f(2) = 1\).

Under certain circumstances Corollary 2 may still hold for tournaments whose edges are coloured with three colours. For instance, if the tournament is transitive (i.e., acyclic), Corollary 2 will hold no matter how many colours are used. Our final theorem gives a less trivial example.

**THEOREM 3.** Let \(T\) be a tournament whose edges are coloured with three colours, and whose vertices can be partitioned into disjoint blocks such that

(i) two vertices in different blocks are always connected by a red edge;

(ii) two vertices in the same block are always connected by a blue or a green edge.

Then there is a vertex \(v\) of \(T\) such that for every other vertex \(x\) of \(T\) there is a monochromatic path from \(x\) to \(v\).

**Proof.** Let the blocks of \(T\) be \(B_1, B_2, \ldots, B_n\). We first define a partial order on \(T\) consisting simply of a linear order on each block. The order on block \(B_i\) is determined as follows. The edges of \(B_i\) are two-coloured, so by Corollary 2 we may choose a vertex \(v_1\) of \(B_i\) such that \(x \rightarrow_{\text{mono}} v_1\) for all \(x \in B_i - \{v_1\}\). Then \(B_i - \{v_1\}\) is still a tournament whose edges are two-
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coloured, so we may choose a vertex \( v_2 \) of \( B_1 - \{ v_1 \} \) such that \( x \rightarrow^{\text{mono}} v_2 \) for all \( x \in B_1 - \{ v_1, v_2 \} \). Put \( v_2 < v_1 \). By continuing this process we construct a linear order on each \( B_i \) such that if \( v \) and \( w \) are vertices of \( B_i \) with \( v < w \), then there is a monochromatic path in \( B_i \) from \( v \) to \( w \).

For vertices \( x \) and \( u \) of \( T \) let us write \( x \rightarrow^m u \) if

(i) \( x \) and \( u \) are in different blocks of \( T \) and \( x \rightarrow^{\text{red}} u \), or

(ii) \( x \) and \( u \) are in the same block of \( T \) and \( x < u \), or

(iii) \( x \) and \( u \) are in the same block of \( T \), \( v < x \), and \( x \rightarrow^{\text{red}} u \).

Note that \( x \rightarrow^m u \) implies \( x \rightarrow^{\text{mono}} u \), but the converse need not hold. For each \( u \), let \( N(u) \) denote the number of vertices \( x \) for which \( x \rightarrow^m u \). Choose a vertex \( u \) for which \( N(u) \) is as large as possible, say \( u \in B_1 \). We will show that \( u \) satisfies the conditions of the theorem.

First, suppose there is a vertex \( w \) not in \( B_1 \) such that \( w \rightarrow^{\text{mono}} u \). Then \( u \rightarrow^{\text{red}} w \) (via a single edge), and moreover \( x \rightarrow^{\text{red}} w \) for all vertices \( x \) such that \( x \rightarrow^{\text{red}} u \). Since \( N(w) \leq N(u) \) by the choice of \( u \), there must be a vertex \( v' \in B_1 \) such that \( v' < u \) and \( v' \rightarrow^{\text{red}} w \), and we may choose \( v' \) so that \( v'' \rightarrow^{\text{red}} w \) for all \( v'' \in B_1 \) satisfying \( v'' < v' \). Thus \( w \rightarrow^{\text{red}} v' \). But now \( v \rightarrow^{\text{red}} v' \), and \( x \rightarrow^{\text{red}} v' \) whenever \( x \rightarrow^{\text{red}} v \). Also, for all \( v'' \in B_1 \) satisfying \( v'' < v'' < v \) we have \( v'' \rightarrow^{\text{red}} w \) and so \( v'' \rightarrow^{\text{red}} v' \). It follows that \( N(v') > N(u) \), contradicting the choice of \( u \).

Now suppose \( w \not\rightarrow^{\text{mono}} u \) for some vertex \( w \in B_1 \), \( w \neq u \). Then \( v < w \) (so \( v \rightarrow^m w \)) and \( w \not\rightarrow^{\text{red}} v \). From above, \( x \rightarrow^{\text{red}} v \) and thus \( w \rightarrow^{\text{red}} x \) holds for all vertices \( x \) not in \( B_1 \), and so \( x \rightarrow^{\text{red}} w \) for all \( x \in B_1 \). Since \( N(w) \leq N(u) \) by the choice of \( v \), there must be a vertex \( u \in B_1 \) such that \( u \rightarrow^m v \) and \( u \not\rightarrow^m w \), and it follows that \( w < u \). Thus \( u \rightarrow^{\text{red}} v \), and there must be some vertex \( x \) not in \( B_1 \) such that \( u \rightarrow^{\text{red}} x \). But since \( x \rightarrow^{\text{red}} w \) this implies \( u \rightarrow^{\text{red}} w \), a contradiction. Thus \( u \) is the required vertex. \( \square \)

Recall that \( T_3 \) denotes the three-element three-coloured tournament mentioned earlier. No tournament of the sort described in Theorem 3 can contain \( T_3 \).

PROBLEM. Let \( T \) be a tournament whose edges are coloured with three colours and which does not contain \( T_3 \). Must \( T \) contain a vertex \( u \) such that for every other vertex \( x \) of \( T \) there is a monochromatic path from \( x \) to \( u \)?

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