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On Monochromatic Paths in Edge-Coloured Digraphs*

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Let G be a directed graph whose edges are coloured with two colours. Call a set S of vertices of G *independent* if no two vertices of S are connected by a monochromatic directed path. We prove that if G contains no monochromatic infinite outward path, then there is an independent set S of vertices of G such that, for every vertex x not in S, there is a monochromatic directed path from x to a vertex of S. In the event that G is infinite, the proof uses Zorn's lemma. The last part of the paper is concerned with the case when G is a tournament.

In this paper G will denote a directed graph, possibly infinite, possibly with multiple edges. A *directed path*, or simply a *path*, is a (finite or infinite) sequence $x_1x_2x_3 \cdots$ of distinct vertices of G such that there is a directed edge from x_i to x_{i+1} for each i. If the sequence $x_1x_2x_3 \cdots$ is infinite we call the path an *infinite outward path.* Let the edges of G be coloured with two colours. A set S of vertices of G is called *independent* if no two vertices of S are connected by a monochromatic directed path.

The following is our main result. It arose from consideration of an unpublished problem of I. Rival and the first author.

THEOREM 1. *Let G be a directed graph whose edges are coloured with two colours, such that G contains no monochromatic infinite outward path. Then there is an independent set S of vertices of G such that, for every vertex x not in S, there is a monochromatic path from x to a vertex of S.*

Proof. Let the two colours be red and blue. We first introduce some notation. For distinct vertices x, y of G, $x \to^{\text{red}} y$ will mean that there is a

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directed path from x to y, all of whose edges are coloured red. If S is a set of vertices of G and x is a vertex of G, $x \to^{\text{red}} S$ ($S \to^{\text{red}} x$) will mean that $x \to^{\text{red}} s$ ($s \to^{\text{red}} x$) for *some* $s \in S$. The negation of, for instance, $x \to^{\text{red}} s$ will be denoted $x \not\to^{\text{red}} s$. Similarly we define, e.g., $x \to^{\text{blue}} y$ and $x \to^{\text{mono}} y$, the latter meaning that there is a monochromatic path from x to y.

For sets S, T of vertices of G, put $S \leqslant T$ if for all $s \in S$ there is a $t \in T$ such that either $s = t$, or $s \to^{\text{blue}} t$ and $t \not\to^{\text{blue}} s$. In particular, $S \subseteq T$ implies $S \leqslant T$. It is easy to see that the collection of all *independent* sets of vertices of G is partially ordered by $\leqslant$.

Let $\mathscr{S}$ be the family of all nonempty independent sets S of vertices of G such that $S \to^{\text{red}} y$ implies $y \to^{\text{mono}} S$ for all vertices y of G. Note that $\mathscr{S}$ is nonempty; there is a vertex v such that $v \to^{\text{red}} y$ implies $y \to^{\text{red}} v$ for all y (since otherwise we can construct an infinite outward red path), whence $\{v\} \in \mathscr{S}$.

We claim that $(\mathscr{S}, \leqslant)$ has maximal elements. To see this, let $\mathscr{C}$ be a chain in $(\mathscr{S}, \leqslant)$, and define

$$S^\infty = \left\{ s \in \bigcup \mathscr{C} : \exists S \in \mathscr{C} \text{ such that } s \in T \text{ whenever } T \in \mathscr{C} \text{ and } T \geqslant S \right\}.$$

(S^∞ consists of all vertices of G that belong to every member of $\mathscr{C}$ from some point on.) Let $S \in \mathscr{C}$ and $s \in S$. If $s \notin S^\infty$, there is $S_1 \in \mathscr{C}$ such that $S_1 \geqslant S$ and $s \notin S_1$. Thus there must be $s_1 \in S_1$ such that $s \to^{\text{blue}} s_1$ and $s_1 \not\to^{\text{blue}} s$. If $s_1 \notin S^\infty$, then there is $S_2 \in \mathscr{C}$ such that $S_2 \geqslant S_1$ and $s_1 \notin S_2$. Thus there must be $s_2 \in S_2$ such that $s_1 \to^{\text{blue}} s_2$ and $s_2 \not\to^{\text{blue}} s_1$. It follows that $s \to^{\text{blue}} s_2$ and $s_2 \not\to^{\text{blue}} s$. Now if $s_2 \notin S^\infty$ we may continue, but, since G has no infinite outward blue path, this procedure must eventually terminate. That is, we obtain some $S_n \in \mathscr{C}$ and $s_n \in S_n$ such that $s \to^{\text{blue}} s_n$, $s_n \not\to^{\text{blue}} s$, and $s_n \in S^\infty$. We have proven that S^∞ is nonempty and that $S^\infty \geqslant S$ for all $S \in \mathscr{C}$. To show S^∞ is independent, let $s, t \in S^\infty$, and suppose without loss of generality that $S, T \in \mathscr{C}$ are such that $s \in S$, $s \in U$ whenever $U \in \mathscr{C}$ and $U \geqslant S$, $t \in T$, and $S \leqslant T$. Then $s \in T$, and since T is independent s and t cannot be connected by a monochromatic path. Thus S^∞ is independent. Finally, to show $S^\infty \in \mathscr{S}$ let $s \to^{\text{red}} y$ for $s \in S^\infty$ and y a vertex of G. There is $S \in \mathscr{C}$ such that $s \in S$, and from the definition of $\mathscr{S}$ there is $t \in S$ such that $y \to^{\text{mono}} t$. If $t \in S^\infty$, then we are done. Suppose $t \notin S^\infty$, so that in particular $t \neq s$. Then $y \not\to^{\text{red}} t$, since S is independent and $s \to^{\text{red}} y$; thus $y \to^{\text{blue}} t$, and since $S^\infty \geqslant S$ there is $t^\infty \in S^\infty$ such that $t \to^{\text{blue}} t^\infty$. Hence $y \to^{\text{blue}} S^\infty$, showing that $S^\infty \in \mathscr{S}$. We have proven that any chain in $\mathscr{S}$ has an upper bound in $\mathscr{S}$, and so, by Zorn's lemma, $(\mathscr{S}, \leqslant)$ contains maximal elements.

Let S be a maximal element of $(\mathscr{S}, \leqslant)$; we claim that S is the set we seek. Suppose to the contrary that there is a vertex x not in S such that $x \not\to^{\text{mono}} S$.

Since G has no infinite outward red path, we may choose x so that $y \to^{\text{red}} x$ for each vertex y not in S satisfying $y \not\to^{\text{mono}} S$ and $x \to^{\text{red}} y$. Also note that, by the definition of $\mathscr{S}$, $S \not\to^{\text{red}} x$.

Let $T = \{t \in S \mid t \not\to^{\text{blue}} x\}$, so that (if $T \neq S$) $S - T \to^{\text{blue}} x$. Then $T \cup \{x\}$ is independent and $T \cup \{x\} > S$. By the maximality of S, there is a vertex y such that $T \cup \{x\} \to^{\text{red}} y$ and $y \not\to^{\text{mono}} T \cup \{x\}$. Clearly, $y \notin S$. If $T \to^{\text{red}} y$, then $S \to^{\text{red}} y$, and it follows from the definition of $\mathscr{S}$ that $y \to^{\text{mono}} S - T$. But this is impossible; $y \not\to^{\text{red}} S - T$ because $T \to^{\text{red}} y$ and S is independent, and $y \not\to^{\text{blue}} S - T$ because $S - T \to^{\text{blue}} x$ and $y \not\to^{\text{blue}} x$. Hence we must have $x \to^{\text{red}} y$. Now $y \not\to^{\text{red}} S$ because $x \not\to^{\text{red}} S$, and $y \not\to^{\text{blue}} S$ because $S - T \to^{\text{blue}} x$ and $y \not\to^{\text{blue}} T \cup \{x\}$. Also $y \not\to^{\text{red}} x$, and this contradicts the choice of x. Thus S satisfies the conditions of the theorem. ∎

We point out a special case.

COROLLARY 2. *Let T be a finite tournament whose edges are coloured with two colours. Then there is a vertex v of T such that for every other vertex x of T there is a monochromatic path from x to v.*

If the edges of the tournament are not coloured (or all coloured the same), then this result is fairly well known. In fact, *every finite tournament T contains a vertex v such that for every other vertex x of T there is a directed path of length at most two from x to v* (e.g., [2]; see [1] for an extension to directed graphs in general). On the other hand, no such bounding of path lengths is possible in Corollary 2. For example, consider the tournament with vertices $\{t_1, t_2, \dots, t_n\}$ and coloured directed edges as follows:

$$\begin{array}{lll} (t_i, t_{i+1}) & \text{coloured red for} & 1 \leqslant i \leqslant n-1 \\ (t_j, t_i) & \text{coloured blue for} & j > i+1. \end{array}$$

Then t_n is the only vertex satisfying the conclusion of Corollary 2; moreover, the only monochromatic directed path from t_1 to t_n passes through all the other vertices.

Interestingly, if we use *more* than two colours the corollary is false. A simple counterexample is the three-element tournament $\{a, b, c\}$ with directed edges (a, b), (b, c), and (c, a), all coloured differently (for future reference, we call this edge-coloured tournament T_3). It is not hard to add vertices to this tournament to construct arbitrarily large finite counterexamples.

Now consider the tournament with vertices

$$a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$$

and coloured directed edges

(a_1, a_2),	(b_1, b_2),	(c_1, c_2)	coloured red,	
(a_2, a_3),	(b_2, b_3),	(c_2, c_3)	coloured blue,	
(a_3, a_1),	(b_3, b_1),	(c_3, c_1)	coloured green,	
(a_i, b_j)			coloured red for all	i, j,
(b_i, c_j)			coloured blue for all	i, j,
(c_i, a_j)			coloured green for all	i, j.

This is a tournament whose edges are coloured with three colours and for which no *pair* of vertices satisfies the conclusion of Corollary 2. More precisely, there is no set S of exactly two vertices such that for every other vertex v there is a monochromatic path from v to a vertex of S. For example, $S = \{a_1, b_2\}$ does not work because there is no monochromatic path from b_3 to either a_1 or b_2. The following question (due also to Erdös) is still open.

PROBLEM. For each n, is there a (least) positive integer $f(n)$ so that every finite tournament whose edges are coloured with n colours contains a set S of $f(n)$ vertices with the property that for every vertex v not in S there is a monochromatic path from v to a vertex of S? In particular, is $f(3) = 3$?

Corollary 2, of course, just says that $f(2) = 1$.

Under certain circumstances Corollary 2 may still hold for tournaments whose edges are coloured with three colours. For instance, if the tournament is transitive (i.e., acyclic), Corollary 2 will hold no matter how many colours are used. Our final theorem gives a less trivial example.

THEOREM 3. *Let T be a tournament whose edges are coloured with three colours, and whose vertices can be partitioned into disjoint blocks such that*

(i) *two vertices in different blocks are always connected by a red edge;*

(ii) *two vertices in the same block are always connected by a blue or a green edge.*

Then there is a vertex v of T such that for every other vertex x of T there is a monochromatic path from x to v.

Proof. Let the blocks of T be $B_1, B_2, \ldots, B_n$. We first define a partial order on T consisting simply of a linear order on each block. The order on block B_i is determined as follows. The edges of B_i are two-coloured, so by Corollary 2 we may choose a vertex v_1 of B_i such that $x \rightarrow^{\text{mono}} v_1$ for all $x \in B_i - \{v_1\}$. Then $B_i - \{v_1\}$ is still a tournament whose edges are two-coloured, so we may choose a vertex v_2 of $B_i - \{v_1\}$ such that $x \to^{\text{mono}} v_2$ for all $x \in B_i - \{v_1, v_2\}$. Put $v_2 < v_1$. By continuing this process we construct a linear order on each B_i such that if v and w are vertices of B_i with $v < w$, then there is a monochromatic path in B_i from v to w.

For vertices x and v of T let us write $x \to^m v$ if

(i) x and v are in different blocks of T and $x \to^{\text{red}} v$, or

(ii) x and v are in the same block of T and $x < v$, or

(iii) x and v are in the same block of T, $v < x$, and $x \to^{\text{red}} v$.

Note that $x \to^m v$ implies $x \to^{\text{mono}} v$, but the converse need not hold. For each v, let $N(v)$ denote the number of vertices x for which $x \to^m v$. Choose a vertex v for which $N(v)$ is as large as possible, say $v \in B_1$. We will show that v satisfies the conditions of the theorem.

First, suppose there is a vertex w not in B_1 such that $w \not\to^{\text{mono}} v$. Then $v \to^{\text{red}} w$ (via a single edge), and moreover $x \to^{\text{red}} w$ for all vertices x such that $x \to^{\text{red}} v$. Since $N(w) \leqslant N(v)$ by the choice of v, there must be a vertex $v' \in B_1$ such that $v' < v$ and $v' \not\to^{\text{red}} w$, and we may choose v' so that $v'' \to^{\text{red}} w$ for all $v'' \in B_1$ satisfying $v' < v'' < v$. Thus $w \to^{\text{red}} v'$. But now $v \to^{\text{red}} v'$, and $x \to^{\text{red}} v'$ whenever $x \to^{\text{red}} v$. Also, for all $v'' \in B_1$ satisfying $v' < v'' < v$ we have $v'' \to^{\text{red}} w$ and so $v'' \to^{\text{red}} v'$. It follows that $N(v') > N(v)$, contradicting the choice of v.

Now suppose $w \not\to^{\text{mono}} v$ for some vertex $w \in B_1$, $w \neq v$. Then $v < w$ (so $v \to^m w$) and $w \not\to^{\text{red}} v$. From above, $x \to^{\text{red}} v$ and thus $w \not\to^{\text{red}} x$ holds for all vertices x not in B_1, and so $x \to^{\text{red}} w$ for all $x \notin B_1$. Since $N(w) \leqslant N(v)$ by the choice of v, there must be a vertex $u \in B_1$ such that $u \to^m v$ and $u \not\to^m w$, and it follows that $w < u$. Thus $u \to^{\text{red}} v$, and there must be some vertex x not in B_1 such that $u \to^{\text{red}} x$. But since $x \to^{\text{red}} w$ this implies $u \to^{\text{red}} w$, a contradiction. Thus v is the required vertex. ∎

Recall that T_3 denotes the three-element three-coloured tournament mentioned earlier. No tournament of the sort described in Theorem 3 can contain T_3.

PROBLEM. Let T be a tournament whose edges are coloured with three colours and which does not contain T_3. Must T contain a vertex v such that for every other vertex x of T there is a monochromatic path from x to v?

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