ELSEVIER

Discrete Mathematics 254 (2002) 259–274

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

On large sets of Kirkman systems with holes ☆

Jian-guo Lei

Institute of Mathematics, Hebei Normal University, Shijiazhuang 050016, People's Republic of China

Received 16 September 1998; revised 3 July 2000; accepted 9 April 2001

Abstract

We study the large sets of generalized Kirkman systems. The purpose of introducing the structure is to construct the large sets of Kirkman triple systems (briefly LKTS). Our main result is that there exists an LKTS(v) for $v \in \{6 \cdot 4^n 25^m + 3;\ m, n \geqslant 0\}$. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Large set of Kirkman triple systems; t-Wise balanced design; Generalized frame; Large set of generalized Kirkman systems

1. Introduction

The existence problem of a large set of disjoint Kirkman triple systems (briefly LKTS) comes from "Sylvester's problem of the 15 schoolgirls", which was posed in 1850. In 1973, Denniston gave a solution of Sylvester's problem of the 15 schoolgirls, i.e., there exists an LKTS(15) [3]. By the end of 1979, Denniston had given the direct constructions for several small values and a recursive construction of LKTS, where an LKTS was called a "double resolvable complete 3-designs" (see [4–6]). Meanwhile, Schreiber had also given the existence of an LKTS(33) (see [7]). Furthermore, some infinite classes can be obtained by using the Denniston's recursive construction (they are summarized in Lemma 3). The research on LKTS has not advanced very far since then. Especially, new recursive constructions were not found. Recently, the author has given a recursive construction which is a generalization of Denniston's tripling construction (see [11]). In this article, we will give a new method for constructing LKTS. Firstly, we introduce some definitions needed in this article.

☆ Research supported by the National Natural Science Youth Foundation of China with grant 19901008.
E-mail address: leijg@sj-user.he.cninfo.net or leijg@heinfo.net (J.-g. Lei).

PII: S0012-365X(01)00295-3

A *group-divisible design* $\mathrm{GDD}(t,K,v;r_1\{k_1\},\ldots,r_s\{k_s\})$, where $\sum_{i=1}^{s} r_ik_i=v$, $K\subseteq N$, and for any $k\in K$, $k\geqslant t$, is a triple $(X,\mathscr{G},\mathscr{B})$ such that

(1) X is a set of v points.
(2) $\mathscr{G}$ is a partition of X into r_i sets of k_i points (called groups), $i=1,2,\ldots,s$.
(3) $\mathscr{B}$ is a set of subsets of X (called blocks), such that $|B|\in K$, $|B\cap G|\leqslant 1$ for all $B\in\mathscr{B}$ and $G\in\mathscr{G}$, and such that any t-subset T of X with $|T\cap G|\leqslant 1$ for all $G\in\mathscr{G}$, is contained in exactly one block.

A $\mathrm{GDD}(t,\ K,v;v\{1\})$ $(X,\mathscr{G},\mathscr{B})$ is denoted by an $S(t,K,v)$; it is often called *t-wise balanced design*. We usually omit $\mathscr{G}$ in the triple $(X,\mathscr{G},\mathscr{B})$, and write $(X,\mathscr{B})$ instead of $(X,\mathscr{G},\mathscr{B})$. An $S(2,K,v)$ is called a *pairwise balanced design* or PBD. An $S(t,\{k\},v)$ is denoted by $S(t,k,v)$; it is called a *Steiner system*.

Lemma 1 (Beth et al. [1], Blanchard [2], Hartman [8]). *(i) There exists an* $S(3,q+1,q^n+1)$ *for any prime power q and any integer* $n\geqslant 2$.
(ii) There exists an $S(3,5,26)$.
(iii) If there exists an $S(3,q+1,v+1)$*, where q is a prime power, then there exists an* $S(3,q+1,qv+1)$.
(iv) If an $S(3,q+1,v+1)$ *and an* $S(3,q+1,w+1)$ *both exist, then there exists an* $S(3,q+1,vw+1)$*, where q is a prime power.*

Let $(X,\mathscr{B})$ be an $S(2,K,v)$. If there exists a set Γ which consists of subsets of $\mathscr{B}$, $\Gamma=\{P_1,P_2,\ldots,P_h\}$, such that each P_i is itself a partition of X, and $\bigcup_{i=1}^{h} P_i=\mathscr{B}$, and for each $B\in\mathscr{B}$, there are exactly $(|B|-1)/2$ elements P_i of Γ such that $B\in P_i$, then the set Γ is called a *resolution* of $\mathscr{B}$. Each element P_i of Γ is called a *parallel class*. If the block set $\mathscr{B}$ of an $S(2,K,v)$ has a resolution, then this $S(2,K,v)$ with the resolution is called a *modified resolvable* $S(2,K,v)$, briefly an $\mathrm{MRS}(K,v)$.

It is obvious that if $(X,\mathscr{B})$ is a $\mathrm{MRS}(K,v)$, then for any $k\in K$, k must be odd and $k\geqslant 3$.

Lemma 2. *The resolution of an* $\mathrm{MRS}(K,v)$ *must contain* $(v-1)/2$ *parallel classes. Thus* $v\equiv 1 \pmod 2$ *is the necessary condition for the existence of an* $\mathrm{MRS}(K,\ v)$.

Proof. Let $(X,\mathscr{B})$ be an $\mathrm{MRS}(K,v)$, and let $\Gamma=\{P_1,P_2,\ldots,P_h\}$ be a resolution of $\mathscr{B}$. Then we have $\bigcup_{i=1}^{h} P_i=\mathscr{B}$. Since each P_i is a parallel class and $(X,\mathscr{B})$ is an $S(2,K,v)$, we have

$$\sum_{B\in P_i}|B|=v \quad \text{and} \quad \sum_{B\in\mathscr{B}}\frac{|B|(|B|-1)}{2}=\frac{v(v-1)}{2}.$$

Furthermore,

$$hv=\sum_{i=1}^{h}\sum_{B\in P_i}|B|=\sum_{B\in\mathscr{B}}\frac{|B|(|B|-1)}{2}=\frac{v(v-1)}{2}.$$

Thus, $h=(v-1)/2$. $\square$

An MRS($\{3\}, v$) is called a *Kirkman triple system*, briefly a KTS(v). The resolution of KTS(v) $(X, \mathscr{B})$ must be a partition of $\mathscr{B}$. It is well known that there exists a KTS(v) if and only if $v \equiv 3 \pmod 6$ (see [12]).

In fact, we can regard an MRS(K, v) as a KTS(v) with "holes". The holes are the blocks which have more than three elements. We can get a KTS(v) by breaking up the long blocks into triples (see Corollary 1). And a Kirkman frame can also be regarded as a special MRS(K, v), if we add a new point into each group (see Corollary 2). By introducing the MRS(K, v), some methods for constructing KTS(v) can be used to construct large sets of Kirkman triple systems. In this article, our main result is that:

Main result. *There exists an* LKTS(v) *for* $v \in \{6 \cdot 4^n 25^m + 3; n, m \geqslant 0\}$.

2. LGKS

In this section, we introduce a kind of design which is called large sets of generalized Kirkman systems. In fact, it is a generalization of the large sets of disjoint Kirkman triple systems. Some ideas come from Luc Teirlinck's "large sets with holes" (see [15]). Firstly, we introduce the definition of generalized frame (see [14]).

A *generalized frame* $F(t, k, v\{m\})$ is a collection $\{(X, \mathscr{G}, \mathscr{B}_r); r \in R\}$, where X is a vm set, $\mathscr{G}$ is a partition of X into v sets of m points (called groups), such that each $(X \backslash G, \mathscr{G} \backslash G, \mathscr{B}_r)$, $G \in \mathscr{G}$, is a GDD($t-1, k, (v-1)m$; $(v-1)\{m\}$), and such that $(X, \mathscr{G}, \bigcup_{r \in R} \mathscr{B}_r)$ is a GDD($t, k, vm; v\{m\}$), and such that all $\mathscr{B}_r$, $r \in R$, are pairwise disjoint. It is known that an $F(t, k, v\{m\})$ contains $vm/(k-t+1)$ GDD($t-1, k$, $(v-1)m; (v-1)\{m\}$)s, i.e. $|R| = vm/(k-t+1)$. Let $R_G = \{r \in R; \mathscr{B}_r$ have the same group set $\mathscr{G} \backslash G\}$, it is known that $|R_G| = m/(k-t+1)$.

An $F(2, 3, v\{m\})$ is called a *Kirkman frame*, briefly a KF(m^v), and each element (i.e. a GDD($1, 3, (v-1)m; (v-1)\{m\}$)) of the $F(2, 3, v\{m\})$ is called a *partial parallel class* (or *holey parallel class*). In fact, each partial parallel class is a partition of $X \backslash G$ for some group G. It is well known that there exists a KF(m^v) if and only if $m \equiv 0 \pmod 2$, $v \geqslant 4$ and $m(v-1) \equiv 0 \pmod 3$ (see [13]). If each element (a GDD($2, 3, (v-1)m; (v-1)\{m\}$)) of an $F(3, 3, v\{m\})$ is also a KF(m^{v-1}), then this $F(3, 3, v\{m\})$ is denoted by OLKF(m^v); it is called an *overlarge set of Kirkman frames*.

Theorem 1. *There exists an* OLKF(6^5).

Proof. An OLKF(6^5) is given in appendix. $\square$

Now, we introduce the definition of large set of generalized Kirkman systems.

Let Y be an mu set and Y_1 be a w set, where $mu > w \geqslant 2$, $Y \cap Y_1 = \phi$, $X = Y \cup Y_1$. A *large set of generalized Kirkman systems* LGKS($K, \{m+w\}, m^u, mu+w, w$), is a collection $\{(X, \mathscr{B}_i);\ 1 \leqslant i \leqslant mu\} \cup \{(Y, \mathscr{G}, \mathscr{A}_j);\ 1 \leqslant j \leqslant w-2\}$, satisfying the following conditions:

(1) Each $(Y, \mathscr{G}, \mathscr{A}_j)$ is a KF(m^u). And if $w=2$, let $m=1$.
(2) Each $(X, \mathscr{B}_i)$ is an MRS$(K \cup \{m+w\}, mu+w)$ and $\mathscr{B}_i$ contains a block $G \cup Y_1$ for some $G \in \mathscr{G}$.
(3) For any 3-subset (or triple) T of X with $|T \cap Y_1| \leqslant 2$, T is contained in exactly one block of $(\bigcup_{i=1}^{mu} \mathscr{B}_i) \cup (\bigcup_{j=1}^{w-2} \mathscr{A}_j)$.
(4) For any block $B \in (\bigcup_{i=1}^{mu} \mathscr{B}_i)$, such that if $Y_1 \not\subset B$, then

$$|\{\mathscr{B}_i; B \in \mathscr{B}_i\}| = |B| - 2$$

and $Y_1 \subset B$ if and only if $B \backslash Y_1 \in \mathscr{G}$ and $|\{\mathscr{B}_i; B \in \mathscr{B}_i\}| = m$.

Note. If $w=2$, it is obvious that the elements of an LGKS$(K, \{3\}, 1^u, u+2, 2)$ are only some MRS$(K \cup \{3\}, u+2)$s, thus, we simply write it as LGKS$(K \cup \{3\}, u+2)$.

An LGKS$(\{3\}, u+2)$ is called a *large set of disjoint Kirkman triple systems*, briefly an LKTS$(u+2)$. So far, known results of LKTS(u) is very limited (see [3–7,11,16,17]). They are summarized as follows:

Lemma 3. *There exists an* LKTS(u) *for* $u \in \{3^n m(2 \cdot 13^k + 1)^t;\ n \geqslant 1,\ t=0,1,\ m \in \{1,5,11,17,25,35,43,67\}\} \cup \{369\}$.

An MRS(K, v) can be considered as an KTS(v) "with holes". Similarly, an LGKS$(K, \{m+w\}, m^u, mu+w, w)$ can be considered as an LKTS$(mu+w)$ "with holes". In this article, we will obtain some new results of LKTS by constructing LGKS.

3. Basic constructions

Construction 1. *If there exist an* MRS(K, u) *and an* MRS(K', k) *for any* $k \in K$, *then there exists an* MRS(K', u).

Proof. Let X be an u set, and let $(X, \mathscr{B})$ be an MRS(K, u) with the resolution $\Gamma = \{P_i; 1 \leqslant i \leqslant (u-1)/2\}$. For any $B \in \mathscr{B}$, $|B| \in K$, let $R_B = \{i; B \in P_i\}$, then $|R_B| = (|B|-1)/2$. Suppose that $(B, \mathscr{A}(B))$ is an MRS$(K', |B|)$ with its resolution $\{P(B,i); i \in R_B\}$. Let

$$\mathscr{A} = \bigcup_{B \in \mathscr{B}} \mathscr{A}(B),$$

$$Q_i = \bigcup_{B \in P_i} P(B,i), \quad 1 \leqslant i \leqslant \frac{u-1}{2}.$$

Then $(X, \mathscr{A})$ is an MRS(K', u), and its resolution is $\{Q_i; 1 \leqslant i \leqslant (u-1)/2\}$. $\square$

In fact, Construction 1 breaks up long blocks into small blocks. The condition "each block B is contained in $(|B|-1)/2$ parallel classes" guarantees that the resolution of $(X,\mathscr{B})$ is still a resolution of $(X,\mathscr{A})$.

Corollary 1. *If there exist an* MRS(K,u) *and a* KTS(k) *for any* $k\in K$, *then there exists a* KTS(u).

Construction 2. *If there exist a* KF(m^u) *and an* MRS$(K\cup\{w\},m+w)$, *then there exists an* MRS$(K\cup\{w,3\},mu+w)$.

Proof. Let Y be an mu set, and let $(Y,\mathscr{G},\mathscr{B})$ be a KF(m^u), such that $\mathscr{B}$ is partitioned into partial parallel classes $\{A(G,i); G\in\mathscr{G},\ 1\leqslant i\leqslant m/2\}$, where each $A(G,i)$ is a partition of $Y\backslash G$. Let $Y_1=\{\infty_1,\infty_2,\ldots,\infty_w\}$, where $Y\cap Y_1=\phi$. And let $X=Y\cup Y_1$. For any $G\in\mathscr{G}$, let $(G\cup Y_1,\mathscr{B}(G))$ be an MRS$(K\cup\{w\},m+w)$, such that $\mathscr{B}(G)$ contains the block Y_1. And each $\mathscr{B}(G)$ has a resolution $\{P(G,j); 1\leqslant j\leqslant(m+w-1)/2\}$ (note that w must be odd by the definition of an MRS), such that $Y_1\in P(G,j)$ for $(m/2)+1\leqslant j\leqslant(m+w-1)/2$. Let

$$\mathscr{A}=\left(\bigcup_{G\in\mathscr{G}}\mathscr{B}(G)\right)\cup\mathscr{B},$$

$$\mathscr{P}(G,i)=P(G,i)\bigcup A(G,i),\quad G\in\mathscr{G},\ 1\leqslant i\leqslant\frac{m}{2},$$

$$\mathscr{P}(j)=\bigcup_{G\in\mathscr{G}}P(G,j),\quad \frac{m}{2}+1\leqslant j\leqslant\frac{m+w-1}{2}.$$

Then $(X,\mathscr{A})$ is an MRS$(K\cup\{w\},mu+w)$, and it has a resolution $\{\mathscr{P}(G,i); G\in\mathscr{G}, 1\leqslant i\leqslant m/2\}\cup\{\mathscr{P}(j); m/2+1\leqslant j\leqslant(m+w-1)/2\}$. $\square$

Construction 2 adds w points of Y_1 into each group before breaking these groups.

Corollary 2. *If there exists a* KF(m^u), *then there exists an* MRS$(\{3,m+1\},mu+1)$.

The methods of Constructions 1 and 2 can be applied to construct the large sets of generalized Kirkman systems.

Construction 3. *If there exists an* LGKS$(K,\{m+w\},m^u,mu+w,w)$, *and there exist both an* LGKS$(K'\cup\{w\},m+w)$ *and an* LGKS(K',k) *for any* $k\in K$, *then there exists an* LGKS$(K'\cup\{w,3\},mu+w)$.

Proof. Let Y be an mu set and let Y_1 be a w set, where $mu>w\geqslant 2$, $Y\cap Y_1=\phi$. Let $X=Y\cup Y_1$, and an LGKS$(K,\{m+w\},m^u,mu+w,w)=\{(X,\mathscr{B}_i);\ 1\leqslant i\leqslant mu\}\cup\{(Y,\mathscr{G},\mathscr{A}_j);\ 1\leqslant j\leqslant w-2\}$.

For any $B\in\bigcup_{i=1}^{mu}\mathscr{B}_i$, let $R_B=\{i; B\in\mathscr{B}_i\}$. Then if $Y_1\subset B$, we have $|R_B|=m$ and $|B|=m+w$; if $Y_1\not\subset B$, we have $|R_B|=|B|-2$ and $|B|\in K$. By the known conditions, there exist both an LGKS$(K' \cup \{w\}, m+w)$ and an LGKS(K', k) for any $k \in K$. Thus, for any $B \in \bigcup_{i=1}^{mu} \mathscr{B}_i$, if $Y_1 \subset B$, then by the definition, $B \backslash Y_1 \in \mathscr{G}$ (or $B = G \cup Y_1$ for some $G \in \mathscr{G}$), furthermore, there exists an LGKS$(K' \cup \{w\}, m+w) = \{(B, \mathscr{B}(B,i));\ i \in R_B\} \cup \{(B, \mathscr{A}(B,j));\ 1 \leqslant j \leqslant w-2\}$, such that each $\mathscr{A}(B,j)$ contains common block Y_1; if $Y_1 \not\subset B$, then there exists an LGKS$(K', |B|) = \{(B, \mathscr{B}(B,i));\ i \in R_B\}$. Define

$$\mathscr{C}_i = \bigcup_{B \in \mathscr{B}_i} \mathscr{B}(B,i), \quad 1 \leqslant i \leqslant mu,$$

$$\mathscr{C}_{mu+j} = \left(\bigcup_{G \in \mathscr{G}} \mathscr{A}(G \cup Y_1, j) \right) \cup \mathscr{A}_j, \quad 1 \leqslant j \leqslant w-2.$$

By Constructions 1 and 2, each $(X, \mathscr{C}_l)$ $(1 \leqslant l \leqslant mu+w-2)$ is an MRS$(K' \cup \{w, 3\}, mu+w)$. And it is easy to show that the collection $\{(X, \mathscr{C}_l); 1 \leqslant l \leqslant mu+w-2\}$ satisfies the other conditions of the definition of an LGKS. Thus the collection $\{(X, \mathscr{C}_l); 1 \leqslant l \leqslant mu+w-2\}$ is an LGKS$(K' \cup \{w,3\}, mu+w)$. $\square$

Corollary 3. *If there exist an* LGKS(K,u) *and an* LGKS(K',k) *for any* $k \in K$, *then there exists an* LGKS(K',u).

Corollary 4. *If there exist an* LGKS(K,u) *and an* LKTS(k) *for any* $k \in K$, *then there exists an* LKTS(u).

4. The construction of LGKS I

Let X be a u set; an LR *design* of order u (briefly LR(u)) is a collection $\{(X, \mathscr{A}_k^j);\ 1 \leqslant k \leqslant (u-1)/2,\ j = 0, 1\}$ of KTS(u) with following properties:

(i) Let the resolution of $\mathscr{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h);\ 1 \leqslant h \leqslant (u-1)/2\}$. There is an element in each Γ_k^j, which without loss of generality, we may suppose is $A_k^j(1)$, such that
$$\bigcup_{k=1}^{(u-1)/2} A_k^0(1) = \bigcup_{k=1}^{(u-1)/2} A_k^1(1) = \mathscr{A}$$
and $(X, \mathscr{A})$ is a KTS(u).

(ii) For any triple $T = \{x, y, z\} \subset X$, $x \neq y \neq z \neq x$, there exist k, j such that $T \in \mathscr{A}_k^j$.

The LR design was introduced by the Lei [11], and the following results are known:

Lemma 4 (Lei [11]). *An* LR(v) *exists for* $v \in \{3^m(2 \cdot 13^k + 1)^t;\ m, k$ *is nonnegative integer,* $t = 0, 1,\ m + t \geqslant 1\}$.

Theorem 2. *If there exists an* LR$(2u+1)$, *then there exists an* LGKS$(\{3\}, \{9\}, 6^u, 6u+3, 3)$.

Construction. Let X be a $2u$ set, $\infty \notin X$, and let X have linear order "$>$" (i.e. for any $x \neq y$, $x, y \in X$, there is either $x > y$ or $y > x$), and suppose that $\infty > x$ for any $x \in X$. Let an LR$(2u+1) = \{(X \cup \{\infty\}, \mathscr{A}_i^j);\ 1 \leqslant i \leqslant u,\ j = 0,1\}$. And each $\mathscr{A}_i^j$ has a resolution $\Gamma_i^j = \{A_i^j(h);\ 1 \leqslant h \leqslant u\}$, such that $\bigcup_{i=1}^u A_i^0(1) = \bigcup_{i=1}^u A_i^1(1) = \mathscr{A}$, and $(X \cup \{\infty\}, \mathscr{A})$ is a KTS$(2u+1)$. Let $Y = Z_3 \times (X \cup \{\infty\})$. We will construct an LGKS$(\{3\},\{9\},6^u,6u+3,3)$ on set Y. The collection will be $\{(Y, \mathscr{C}_i^j(l));\ 1 \leqslant i \leqslant u,\ j = 0,1,\ l \in Z_3\} \cup \{(Z_3 \times X, \mathscr{G}, \mathscr{C}_\infty)\}$, where each $(Y, \mathscr{C}_i^j(l))$ is an MRS$(\{3,9\},6u+3)$ and each $(Z_3 \times X, \mathscr{G}, \mathscr{C}_\infty)$ is a KF(6^u) with $\mathscr{G} = \{Z_3 \times \{x,y\};\ \{x,y,\infty\} \in \mathscr{A},\ x,y \in X\}$.

Let $B \in \bigcup_{i=1}^u \bigcup_{j=0}^1 \mathscr{A}_i^j$.

(1) If $B \in \mathscr{A}$, let $R_B = \{(i,j); B \in A_i^j(1)\}$, then $|R_B| = 2$.

(i) If $B = \{x,y,\infty\} \in \mathscr{A}$, let $\mathscr{C}(i,j,l,1) = \{Z_3 \times B\}$, where $(i,j) \in R_B$, $l \in Z_3$. (Note: the long block $Z_3 \times B$ will appear in 6 $\mathscr{C}_i^j(l)$s and $(Z_3 \times B) \backslash (Z_3 \times \{\infty\}) \in \mathscr{G}$.)

(ii) If $B = \{x,y,z\} \in \mathscr{A}$, and $\infty \notin B$, then we can construct an LKTS(9) on set $Z_3 \times B$ (it is known that an LKTS(9) exists). Let the LKTS$(9) = \{(Z_3 \times B, \mathscr{C}_B(i,j,l,2));\ (i,j) \in R_B,\ l \in Z_3\} \cup \{(Z_3 \times B, \mathscr{C}_B(\infty))\}$, and suppose that $\mathscr{C}_B(\infty)$ contains parallel class $P_B(\infty) = \{\{(0,t),(1,t),(2,t)\};\ t \in B\}$. (Note: In fact, $(Z_3 \times B, P_B(\infty), \mathscr{C}_B(\infty) \backslash P_B(\infty))$ is an RGDD$(2,3,9;3\{3\})$.)

Let

$$\mathscr{C}(i,j,l,2) = \bigcup_{\substack{B \in A_i^j(1) \\ \infty \notin B}} \mathscr{C}_B(i,j,l,2), \quad l \in Z_3.$$

(2) If $B = \{x,y,z\} \in \mathscr{A}_i^j \backslash A_i^j(1)$ (i.e. $B \notin \mathscr{A}$), let $\mathscr{C}_B(i,j,l,3)$ $(l \in Z_3)$ contains following blocks:

$$\{(a,x),(a+b,y),(a+2b+l,z)\},$$

where $a,b \in Z_3$ and $x < y < z$. (In fact, each $\mathscr{C}_B(i,j,l,3)$ is an RGDD$(2,3,9;3\{3\})$ on set $Z_3 \times B$ with group set $\{Z_3 \times \{s\};\ s \in B\}$.)

Let

$$\mathscr{C}(i,j,l,3) = \bigcup_{B \in \mathscr{A}_i^j \backslash A_i^j(1)} \mathscr{C}_B(i,j,l,3), \quad l \in Z_3.$$

Define

$$\mathscr{C}_i^j(l) = \bigcup_{m=1}^3 \mathscr{C}(i,j,l,m),$$

$$\mathscr{C}_\infty = \bigcup_{\substack{B \in \mathscr{A} \\ \infty \notin B}} (\mathscr{C}_B(\infty) \backslash P_B(\infty)),$$

where $1 \leqslant i \leqslant u$, $j = 0,1$, $l \in Z_3$. Then the collection $\{(Y, \mathscr{C}_i^j(l));\ 1 \leqslant i \leqslant u,\ j = 0,1,\ l \in Z_3\} \cup \{(Z_3 \times X, \mathscr{G}, \mathscr{C}_\infty)\}$ will be an LGKS$(\{3\},\{9\},6^u,6u+3,3)$.

Proof. (1) Each $(Y,\mathscr{C}_i^j(l))$ (where $1\leqslant i\leqslant u$, $j=0,1$ and $l\in Z_3$) is an MRS($\{3,9\}$, $6u+3$).

(i) For any pair $P=\{(i_1,x),(i_2,y)\}\subset Y$.

If $x\neq y$, since $\mathscr{A}_i^j$ is a KTS($2u+1$), there is exactly one block $B\in\mathscr{A}_i^j$, such that $\{x,y\}\subset B$. If $B\in A_i^j(1)$, then P is contained in exactly one block of $\mathscr{C}(i,j,l,1)$ or $\mathscr{C}(i,j,l,2)$. If $B\in\mathscr{A}_i^j\backslash A_i^j(1)$, then P is contained in exactly one block of $\mathscr{C}_B(i,j,l,3)$.

If $x=y$, since $A_i^j(1)$ is a parallel class, there is exactly one block $B\in A_i^j(1)$ such that $x\in B$. Thus, P is contained in one block of $\mathscr{C}(i,j,l,1)$ or $\mathscr{C}(i,j,l,2)$.

Therefore, $(Y,\mathscr{C}_i^j(l))$ is an $S(2,\{3,9\},6u+3)$.

(ii) The $S(2,\{3,9\},6u+3)$ has a resolution.

Let $B\in\mathscr{A}_i^j$. If $B\in A_i^j(1)$ and $\infty\notin B$, let the resolution of $\mathscr{C}_B(i,j,l,2)$ (it is a LKTS(9)) be $\{P_B(i,j,l,d,1);\ 1\leqslant d\leqslant 4\}$. If $B=\{x,y,z\}\in A_i^j(h)$, $1<h\leqslant u$, let $P_B(i,j,l,b,h)=\{\{(a,x),(a+b,y),(a+2b+l,z)\};\ x<y<z,\ a\in Z_3\}$, $b\in Z_3$. Define

$$P(i,j,l,d,1)=\left(\bigcup_{\substack{B\in A_i^j(1)\\ \infty\notin B}} P_B(i,j,l,d,1)\right)\cup\mathscr{C}(i,j,l,1),\quad 1\leqslant d\leqslant 4,$$

$$P(i,j,l,b,h)=\bigcup_{B\in A_i^j(h)} P_B(i,j,l,b,h),\quad b\in Z_3,\ 2\leqslant h\leqslant u.$$

It is easy to check that each $P(i,j,l,k,h)$ is a parallel class, and $\{P(i,j,l,d,1);\ 1\leqslant d\leqslant 4\}\cup\{P(i,j,l,b,h);\ b\in Z_3,\ 2\leqslant h\leqslant u\}$ is a resolution of $\mathscr{C}_i^j(l)$.

Therefore, $(Y,\mathscr{C}_i^j(l))$ is an MRS($\{3,9\},6u+3$).

(2) The $(Z_3\times X,\mathscr{G},\mathscr{C}_\infty)$ is a KF(6^u).

(i) For any pair $P=\{(i_1,x),(i_2,y)\}\subset Z_3\times X$ with $P\not\subset G$ for any $G\in\mathscr{G}$, we have $x\neq y$. Since $\mathscr{A}$ is a KTS($2u+1$), there exists a $z\in X$ such that $B=\{x,y,z\}\in\mathscr{A}$ (note here $z\neq\infty$, otherwise $P\subset G$ for some $G\in\mathscr{G}$). Furthermore, since $(Z_3\times B,P_B(\infty),\mathscr{C}_B(\infty)\backslash P_B(\infty))$ is a GDD($2,3,9;3\{3\}$), P is contained in exactly one block of $\mathscr{C}_B(\infty)\backslash P_B(\infty)$. Therefore, $(Z_3\times X,\mathscr{G},\mathscr{C}_\infty)$ is a GDD($2,3,6u;u\{6\}$).

(ii) For any $B\in\mathscr{A}$, $\infty\notin B$. Let the resolution of $\mathscr{C}_B(\infty)$ be $\{P_B(\infty,d);\ 1\leqslant d\leqslant 4\}$ such that $P_B(\infty,4)=P_B(\infty)$. Define

$$P_\infty(d,i)=\bigcup_{\substack{B\in A_i^0(1)\\ \infty\notin B}} P_B(\infty,d)$$

where $1\leqslant i\leqslant u$, $1\leqslant d\leqslant 3$. Since $P_\infty(d,i)\cup\mathscr{C}(i,0,l,1)$ is a partition of $Z_3\times(X\cup\{\infty\}),P_\infty(d,i)$ is exactly one holey parallel class, and $\{P_\infty(d,i);\ 1\leqslant i\leqslant u,\ 1\leqslant d\leqslant 3\}$ is a partition of $\mathscr{C}_\infty$.

Therefore, $(Z_3\times X,\mathscr{G},\mathscr{C}_\infty)$ is a KF(6^u).

(3) The collection $\{(Y,\mathscr{C}_i^j(l));\ 1\leqslant i\leqslant u,\ j=0,1,\ l\in Z_3\}\cup\{(Z_3\times X,\ \mathscr{G},\mathscr{C}_\infty)\}$ is an LGKS$(\{3\},\{9\},6^u,6u+3,3\}$.

(i) By (1) and (2), the collection satisfies the conditions (1) and (2) of the definition of an LGKS.

(ii) Let $T=\{(i_1,x_1),(i_2,x_2),(i_3,x_3)\}$ be an arbitrary triple of Y and $T\neq\{(0,\infty),(1,\infty),(2,\infty)\}$. All the possibilities of T are exhausted as follows:

(a) If $T=\{(0,x_1),(1,x_1),(2,x_1)\}$, $x_1\in X$, then there exists $x_2\in X$ such that $\{x_1,x_2,\infty\}\in\mathscr{A}$. Thus, T is contained in the long block $Z_3\times\{x_1,x_2,\infty\}$.

(b) If $T=\{(i_1,x_1),(i_2,x_2),(i_3,x_2)\}$, where $x_1\neq x_2\in X\cup\{\infty\}$, $i_2\neq i_3\in Z_3$, since $\mathscr{A}$ is a KTS$(2u{+}1)$, there exists an x_3 such that $B=\{x_1,x_2,x_3\}\in\mathscr{A}$. Furthermore, there exists an $\mathscr{A}_i^j$ such that $B\in A_i^j(1)\subset\mathscr{A}_i^j$. If $\infty\in B$, then T is contained in the long block $Z_3\times B$. If $\infty\notin B$, since $\{(Z_3\times B,\mathscr{C}_B(i,j,l,2));\ (i,j)\in R_B,\ l\in Z_3\}\cup\{(Z_3\times B,\mathscr{C}_B(\infty))\}$ is an LKTS(9), T is in exactly one block of $(\bigcup_{l\in Z_3}\mathscr{C}_B(i,j,l,2))\cup(\mathscr{C}_B(\infty)\backslash P_B(\infty))$.

(c) If $T=\{(i_1,x_1),(i_2,x_2),(i_3,x_3)\}$, where $x_1\neq x_2\neq x_3\neq x_1\in X\cup\{\infty\}$, let $x_1<x_2<x_3$. Since $\bigcup_{j=0}^{1}\bigcup_{i=1}^{u}\mathscr{A}_i^j$ contains all triples of set $X\cup\{\infty\}$, there is some $\mathscr{A}_i^j$ such that $B=\{x_1,x_2,x_3\}\in\mathscr{A}_i^j=\bigcup_{h=1}^{u}A_i^j(h)$. If $B\in A_i^j(1)$ and $\infty\in B$, then T is contained in the long block $Z_3\times B$. If $B\in A_i^j(1)$ and $\infty\notin B$, then T is in exactly one block of $(\bigcup_{l\in Z_3}\mathscr{C}_B(i,j,l,2))\cup(\mathscr{C}_B(\infty)\backslash P_B(\infty))$. If $B\in A_i^j(h)$, $2\leqslant h\leqslant u$, then $T=\{(i_1,x_1),(i_1+(i_2-i_1),x_2),(i_1+2(i_2-i_1)+(i_3-2i_2+i_1),x_3)\}$ is in exactly one block of $\mathscr{C}_B(i,j,i_3-2i_2+i_1,3)$.

(iii) It is obvious that each $\mathscr{C}_i^j(l)$ contains only one size 9 block $D=Z_3\times\{x,y,\infty\}$, where $B=\{x,y,\infty\}\in\mathscr{A}$, and all others are triples. Let $B\in\mathscr{A}_i^j$, $\{i,j\}\in R_B$ (where $|R_B|=2$), then D is contained in $\mathscr{C}_i^j(l)$, $l\in Z_3$, $\{i,j\}\in R_B$, and $Z_3\times\{x,y\}\in\mathscr{G}$.

Therefore, the collection $\{(Y,\mathscr{C}_i^j(l));1\leqslant i\leqslant u,\ j=0,1,\ l\in Z_3\}\cup\{(Z_3\times X,\mathscr{G},\mathscr{C}_\infty)\}$ is an LGKS$(\{3\},\{9\},6^u,6u+3,3)$. $\square$

Theorem 3. *There exists an* LGKS$(\{3\},\{9\},6^{(v-3)/6},v,3\}$ *for* $v\in\{3^m(2\cdot 13^k+1)^t;\ t=0,1,\ k\geqslant 0,\ m\geqslant 1\}$.

Proof. By Lemma 4, for any $u\in\{3^m(2\cdot 13^k+1)^t;\ t=0,1,\ k\geqslant 0,\ m\geqslant 0,\ m+t\geqslant 1\}$, there exists an LR$(u)$. Thus by Theorem 2, there exists an LGKS$(\{3\},\{9\},6^{(u-1)/2},3u,3\}$. Let $v=3u$, then our result is true. $\square$

5. The construction of LGKS II

In this section, we will construct the large sets of generalized Kirkman triple systems using 3-wise balanced designs. Let $(X,\mathscr{B})$ be an $S(3,K,v)$, and let $\mathscr{B}_x=\{B\backslash\{x\};\ x\in B\in\mathscr{B}\}$, $\mathscr{B}_{x,y}=\{B\backslash\{x,y\};\ \{x,y\}\subset B\in\mathscr{B}\}$ for any $x\neq y\in X$. Then we have that $(X\backslash\{x\},\mathscr{B}_x)$ is an $S(2,K',v-1)$ for any $x\in X$, where $K'=\{k-1;\ k\in K\}$, and $\mathscr{B}_x\cap$

$\mathscr{B}_y = \phi$ for any $x \neq y \in X$. And we still have that $(X \backslash \{x,y\}, \mathscr{B}_{x,y})$ is an $S(1, K'', v-2)$ for any $x \neq y \in X$, where $K'' = \{k-2;\ k \in K\}$, i.e. $\mathscr{B}_{x,y}$ is a partition of $X \backslash \{x,y\}$. We use above characteristics of a 3-wise balanced designs to construct large sets of generalized Kirkman systems. There are some analogous constructions presenting in Qingde Kang and Jianguo Lei [10], Lei [11], Teirlinck [15], respectively, which were used in constructing different kinds of large sets. Now, we introduce our construction.

If $(X \cup \{\infty\}, \mathscr{B})$, $|X| = u$, $\infty \notin X$, is an $S(3, K, u+1)$, let $K_0 = \{|B|;\ B \in \mathscr{B},\ \infty \notin B\}$, $K_1 = \{|B|;\ B \in \mathscr{B},\ \infty \in B\}$, then we denote this $S(3, K, u+1)$ by an $S(3, K_0, K_1, u+1)$.

Theorem 4. *If there exists an* $S(3, K_0, K_1, u+1)$, *and there exist both an* OLKF(m^k) *for all* $k \in K_0$ *and an* LGKS$(K', \{m+w\}, m^{k-1}, m(k-1)+w, w)$ *for all* $k \in K_1$, *then there exists an* LGKS$(K' \cup \{3\}, \{m+w\}, m^u, mu+w, w)$.

Construction. Let X be a u set, $\infty \notin X$, and let $(X \cup \{\infty\}, \mathscr{B})$ be an $S(3, K_0, K_1, u+1)$. We will construct an LGKS over the set $Y = (X \times Z_m) \cup \{\infty_1, \infty_2, \ldots, \infty_w\}$, where $\{\infty_1, \infty_2, \ldots, \infty_w\} \cap (X \times Z_m) = \phi$, by following steps:

Step 1: For any $B \in \mathscr{B}$, $\infty \in B$, (i.e. $|B| \in K_1$), there exists an LGKS$(K', \{m+w\}, m^{|B|-1}, m(|B|-1)+w, w)$ on set $X_B = ((B \backslash \{\infty\}) \times Z_m) \cup \{\infty_1, \infty_2, \ldots, \infty_w\}$. (Note: By the existing condition of a KF(m^k), we have $m \equiv 0 \pmod 2$ Furthermore, by Lemma 2, we also have $w \equiv 1 \pmod 2$.) Let an LGKS$(K', \{m+w\}, m^{|B|-1}, m(|B|-1)+w, w) = \{(X_B, \mathscr{A}^i_B(x));\ i \in Z_m,\ x \in B \backslash \{\infty\}\} \cup \{((B \backslash \{\infty\}) \times Z_m, \mathscr{G}_B, \mathscr{A}_B(\infty, n));\ 1 \leqslant n \leqslant w-2\}$, where $\mathscr{G}_B = \{G_x = \{x\} \times Z_m;\ x \in B \backslash \{\infty\}\}$, and such that each $\mathscr{A}^i_B(x)$ $(i \in Z_m)$ contains the block $G_x \cup \{\infty_1, \infty_2, \ldots, \infty_w\}$. (Note: this condition can be satisfied by the definition of LGKS.) And let each $\mathscr{A}^i_B(x)$ have a resolution

$$\left\{A^i_B(x,y,t);\ y \in B \backslash \{\infty, x\}, 1 \leqslant t \leqslant \frac{m}{2}\right\} \cup \left\{A^i_B(x,x,v);\ 1 \leqslant v \leqslant \frac{m+w-1}{2}\right\},$$

where each parallel class $A^i_B(x,x,v)$ contains the block $(\{x\} \times Z_m) \cup \{\infty_1, \infty_2, \ldots, \infty_w\}$. And suppose each $\mathscr{A}_B(\infty, n)$ $(1 \leqslant n \leqslant w-2)$ can be partitioned into partial classes $\{A_B(\infty, n, y, t);\ y \in B \backslash \{\infty\},\ 1 \leqslant t \leqslant m/2\}$, where each $A_B(\infty, n, y, t)$ is a partition of $(B \backslash \{y, \infty\}) \times Z_m$.

A parallel class in $\mathscr{A}^i_B(x)$ is marked with $A^i_B(x,y,t)$ by following method:

By the definition of an LGKS and $G_x \cup \{\infty_1, \ldots, \infty_w\} \in \mathscr{A}^i_B(x)$, if block $C = \{\infty_j, (z_1, i_1), \ldots, (z_l, i_l)\} \in \mathscr{A}^i_B(x)$, where $z_1, z_2, \ldots, z_l \in B \backslash \{x, \infty\}$, $i_1, \ldots, i_l \in Z_m$, then we have $z_h \neq z_s$ for $h \neq s$, (since the triple $\{\infty_j, (x, i'_1), (x, i'_2)\} \subset G_x \cup \{\infty_1, \ldots, \infty_w\}$), and $l \equiv 0 \pmod 2$ (since $l+1 \in K'$, and K' only contains odd numbers). Thus, we can take a 1-factor of complete graph K_l on set $\{z_1, z_2, \ldots, z_l\}$. Specifically, let $\{D^i_{(B,x)}(r) \in \mathscr{A}^i_B(x);\ \infty_1 \in D^i_{(B,x)}(r),\ \infty_j \notin D^i_{(B,x)}(r),\ r \in R\}$, i.e. it is the set of all blocks which only contains ∞_1 in $\mathscr{A}^i_B(x)$, then $(\bigcup_{r \in R} D^i_{(B,x)}(r)) \backslash \{\infty_1\} = (B \backslash \{\infty, x\}) \times Z_m$ and $D^i_{(B,x)}(r_1) \cap D^i_{(B,x)}(r_2) = \phi$ for any $r_1 \neq r_2$. For any $r \in R$, let $F^i_{(B,x)}(r)$ be a 1-factor of complete graph on set $\{z_s;(z_s,i_s)\in D^i_{(B,x)}(r)\setminus\{\infty_1\}\}$. Then $\bigcup_{r\in R}F^i_{(B,x)}(r)=E^i(B,x)$ is an m-regular graph on $B\setminus\{x,\infty\}$ (may be the $E^i(B,x)$ has double edges). Furthermore, we can get an $m/2$-regular digraph $\bar{E}^i(B,x)$ on $B\setminus\{x,\infty\}$. For each $y\in B\setminus\{x,\infty\}$, $m/2$ out-arcs of y are labeled by $y_1,y_2,\dots,y_{(m/2)}$. And if $y_t=(y,z)$, $(1\leqslant t\leqslant m/2)$, we suppose that the block which contains triple $\{\infty_1,(y,i_1),(z,i_2)\}$ is included in the parallel class $A^i_B(x,y,t)$. (Note: If there exist double arcs, without loss of generality, let $y_1=y_2=(y,z)$, then there must be two blocks C_1,C_2 which contain the triple $\{\infty_1,(y,i_1),(z,i_2)\}$ and $\{\infty_1,(y,i'_1),(z,i'_2)\}$ respectively (of course, where $i_1\neq i'_1$ and $i_2\neq i'_2$). Thus, there are two different parallel classes containing C_1 and C_2, respectively. Furthermore, we can suppose that one of the two parallel classes is $A^i_B(x,y,1)$ as you like, and the other is $A^i_B(x,y,2)$).

Step 2: For any $B\in\mathscr{B}$, $\infty\notin B$, (i.e. $|B|\in K_0$), there exists an OLKF($m^{|B|}$) on set $B\times Z_m$. Let an OLKF($m^{|B|}$) $=\{(B\times Z_m,\mathscr{G}_B,\mathscr{A}^i_B(x));\ x\in B,\ i\in Z_m\}$, where $\mathscr{G}_B=\{G_x=\{x\}\times Z_m;\ x\in B\}$, and each $((B\setminus\{x\})\times Z_m,\mathscr{G}_B\setminus\{G_x\},\mathscr{A}^i_B(x))$ be a KF($m^{|B|-1}$). And each $\mathscr{A}^i_B(x)$ can be partitioned into partial classes $\{A^i_B(x,y,t);\ y\in B\setminus\{x\},\ 1\leqslant t\leqslant m/2\}$, where each $A^i_B(x,y,t)$ is a partition of $(B\setminus\{x,y\})\times Z_m$. (Note: Here $x\neq y$.)

Define

$$\mathscr{A}^i_x=\bigcup_{x\in B\in\mathscr{B}}\mathscr{A}^i_B(x),\quad x\in X,\ i\in Z_m,$$

$$\mathscr{A}(n)=\bigcup_{\infty\in B\in\mathscr{B}}\mathscr{A}_B(\infty,n),\quad 1\leqslant n\leqslant w-2,$$

$$\mathscr{G}=\{G_x=\{x\}\times Z_m;\ x\in X\}.$$

Then $\{(Y,\mathscr{A}^i_x);\ x\in X,\ i\in Z_m\}\cup\{(X\times Z_m,\mathscr{G},\mathscr{A}(n));\ 1\leqslant n\leqslant w-2\}$ will be an LGKS($K'\cup\{3\},\{m+w\},m^u,mu+w,w$).

Proof. (1) Each $(Y,\mathscr{A}^i_x)$, $x\in X$, $i\in Z_m$, is an MRS($K'\cup\{m+w,3\},mu+w$).

(i) For any pair $P\subset Y$, all the possibilities of P are exhausted as follows:

(a) $P=\{\infty_j,(x,i_1)\}$ or $\{(x,i_1),(x,i_2)\}$ or $\{\infty_j,\infty_l\}$, $i\in Z_v$, where $i_1\neq i_2\in Z_m$, $1\leqslant j,l\leqslant w$. Then $P\subset(\{x\}\times Z_m)\cup\{\infty_1,\infty_2,\dots,\infty_w\}$.

(b) $P=\{\infty_j,(y,i_2)\}$ or $\{(x,i_1),(y,i_2)\}$ or $\{(y,i_1),(y,i_2)\}$, $y\neq x$, $y\in X$, $i_1,i_2\in Z_m$, $1\leqslant j\leqslant w$. Since $(X\cup\{\infty\},\mathscr{B})$ is an $S(3,K_0\cup K_1,u+1)$, there is exactly one block B, such that $\{\infty,x,y\}\subseteq B$. And since $(X_B,\mathscr{A}^i_B(x))$ is an MRS($K'\cup\{m+w\},m(|B|-1)+w$), there is exactly one block containing pair P.

(c) $P=\{(y,i_1),(z,i_2)\}$, $x\neq y\neq z\neq x$, $y,z\in X$, $i_1,i_2\in Z_m$. Then there exists a block $B\in\mathscr{B}$, such that $\{x,y,z\}\subseteq B$. If $\infty\in B$, then $|B|\in K_1$. Since $(X_B,\mathscr{A}^i_B(x))$ is an MRS($K'\cup\{m+w\},m(|B|-1)+w$), there is exactly one block containing pair P. If $\infty\notin B$, then $|B|\in K_0$. Since $((B\setminus\{x\})\times Z_m,\mathscr{A}^i_B(x),\mathscr{G}_B\setminus\{G_x\})$ is a KF($m^{|B|-1}$), P is contained in exactly one block of $\mathscr{A}^i_B(x)$.

Thus, $(Y,\mathscr{A}_x^i)$ is an $S(2,K'\cup\{m+w,3\},mu+w)$.

(ii) Let

$$A^i(x,y,t)=\bigcup_{\{x,y\}\subset B\in\mathscr{B}} A_B^i(x,y,t),\ y\in X\backslash\{x\},\quad 1\leqslant t\leqslant\frac{m}{2},$$

$$A(x,x,v)=\bigcup_{\{x,\infty\}\subset B\in\mathscr{B}} A_B^i(x,x,v),\quad 1\leqslant v\leqslant\frac{m+w-1}{2}.$$

Since for any fixed $y\in X$, the set $\{B\backslash\{x,y\};\ \{x,y\}\subset B\in\mathscr{B}\}$ is a partition of $(X\cup\{\infty\})\backslash\{x,y\}$, and there exists a unique block $B\in\mathscr{B}$ such that the triple $\{\infty,x,y\}\subset B$, it is easy to check that each $A^i(x,y,t)$ (and also each $A(x,x,v)$) is a partition of $Y=(X\times Z_m)\cup\{\infty_1,\ldots,\infty_w\}$, and

$$\left\{A^i(x,y,t);\ y\in X\backslash\{x\},1\leqslant t\leqslant\frac{m}{2}\right\}\cup\left\{A(x,x,v);\ 1\leqslant v\leqslant\frac{m+w-1}{2}\right\}$$

is a resolution of $\mathscr{A}_x^i$.

Therefore, each $(Y,\mathscr{A}_x^i)$, $x\in X, i\in Z_m$, is an MRS$(K'\cup\{m+w,3\},um+w)$.

(2) Each $(X\times Z_m,\mathscr{G},\mathscr{A}(n))$, $1\leqslant n\leqslant w-2$, is a KF$(m^u)$.

(i) For any pair $P=\{(x_1,i_1),(x_2,i_2)\}\subset X\times Z_m$, and $P\not\subset G_x$, $G_x\in\mathscr{G}$, we must have $x_1\neq x_2\in X$. Thus, there is exactly one block $B\in\mathscr{B}$, such that $\{\infty,x_1,x_2\}\in B$. Since $((B\backslash\{\infty\})\times Z_m,\mathscr{G}_B,\mathscr{A}_B(\infty,n))$ is a KF$(m^{|B|-1})$, thus P is contained in exactly one block of $\mathscr{A}_B(\infty,n)\subset\mathscr{A}(n)$.

(ii) Let $A(n,y,t)=\bigcup_{\{\infty,y\}\subset B\in\mathscr{B}} A_B(\infty,n,y,t)$, $y\in X$, $1\leqslant t\leqslant m/2$. Since the set $\{B\backslash\{\infty,y\};\{\infty,y\}\subset B\in\mathscr{B}\}$ is a partition of $X\backslash\{y\}$, and each $A_B(\infty,n,y,t)$ is a partition of $(B\backslash\{\infty,y\})\times Z_m$, each $A(n,y,t)$ exactly forms a partition of $(X\backslash\{y\})\times Z_m$. Furthermore, each $\mathscr{A}(n)$ can be partitioned into partial parallel classes $\{A(n,y,t); y\in X, 1\leqslant t\leqslant m/2\}$.

Therefore, each $(X\times Z_m,\mathscr{G},\mathscr{A}(n))$, $1\leqslant n\leqslant w-2$, is a KF$(m^u)$.

(3) The collection $\{(Y,\mathscr{A}_x^i);x\in X,i\in Z_m\}\cup\{(X\times Z_m,\mathscr{G},\mathscr{A}(n));1\leqslant n\leqslant w-2\}$ is an LGKS$(K'\bigcup\{3\},\{m+w\},m^u,mu+w,w)$.

(i) By (1) and (2), the collection satisfies the conditions (1) and (2) of the definition of an LGKS.

(ii) For any 3-subset $T\subset Y$, and $T\not\subset\{\infty_1,\ldots,\infty_w\}$, all the possibilities of T are exhausted as follows:

(a) $T=\{\infty_j,\infty_l,(x_1,i_1)\}$, $j\neq l$, then T is contained in the long block $(\{x_1\}\times Z_m)\cup\{\infty_1,\ldots,\infty_w\}$.

(b) $T=\{\infty_j,(x_1,i_1),(x_2,i_2)\}$ or $\{(x_1,i_1),(x_1,i_2),(x_2,i_3)\}$, $x_1,x_2\in X$, $i_1,i_2,i_3\in Z_m$. If $x_1=x_2$, then T is contained in exactly one block $(\{x_1\}\times Z_m)\cup\{\infty_1,\ldots,\infty_w\}$. If $x_1\neq x_2$, then there is exactly one block $B\in\mathscr{B}$, such that $\{\infty,x_1,x_2\}\subset B$, thus T is contained in exactly one block of set $\bigcup_{i\in Z_m}\bigcup_{x\in B\backslash\{\infty\}}\mathscr{A}_B^i(x)\subset\bigcup_{i\in Z_m}\bigcup_{x\in X}\mathscr{A}_x^i$.

(c) $T=\{(x_1,i_1),(x_2,i_2),(x_3,i_3)\}$, $x_1\neq x_2\neq x_3\neq x_1$, then there is exactly one block $B\in\mathscr{B}$ such that $\{x_1,x_2,x_3\}\subset B$. If $\infty\in B$, since $\{(X_B,\mathscr{A}_B^i(x));$

$i\in Z_m,\ x\in B\backslash\{\infty\}\}\cup\{((B\backslash\{\infty\})\times Z_m,\ \mathcal{G}_B,\ \mathcal{A}_B(\infty,n));\ 1\leqslant n\leqslant w-2\}$ is an LGKS$(K',\{m+w\},m^{|B|-1},m(|B|-1)+w,w)$ on set $X_B=((B\backslash\{\infty\})\times Z_m)\cup\{\infty_1,\infty_2,\ldots,\infty_w\}$, T is contained in exactly one block of $(\bigcup_{i\in Z_m}\bigcup_{x\in B\backslash\{\infty\}}\mathcal{A}^i_B(x))\cup(\bigcup_{n=1}^{w-2}\mathcal{A}(\infty,n))$. If $\infty\notin B$, since $\{(B\times Z_m,\mathcal{G}_B,\mathcal{A}^i_B(x));i\in Z_m,\ x\in B\}$ is an OLKF$(m^{|B|-1})$, T is contained in exactly one block of set $(\bigcup_{i\in Z_m}\bigcup_{x\in B}\mathcal{A}^i_B(x))$.

By the construction, we have following facts. For any $B_1\neq B_2\in\mathcal{B}$,

1. $\mathcal{A}^{i_1}_{B_1}\cap\mathcal{A}^{i_2}_{B_2}\subset\{(\{x\}\times Z_m)\cup\{\infty_1,\infty_2,\ldots,\infty_w\};\ x\in X\},\ i_1,i_2\in Z_m$.
2. If $\infty\in B_2$, then

$$\mathcal{A}^{i_1}_{B_1}\cap\mathcal{A}_{B_2}(\infty,n)=\phi,\quad i_1\in Z_m,\ 1\leqslant n\leqslant w-2.$$

3. If $\infty\in B_1$ and $\infty\in B_2$, then

$$\mathcal{A}_{B_1}(\infty,n_1)\cap\mathcal{A}_{B_2}(\infty,n_2)=\phi,\ 1\leqslant n_1,n_2\leqslant w-2.$$

Therefore, T is contained in exactly one block of set $\bigcup_{i\in Z_m}\bigcup_{x\in X}\mathcal{A}^i_x$. Furthermore the condition (3) of LGKS is satisfied by the collection.

(iii) It is obvious that for any $G_x\in\mathcal{G}$, we have $G_x\cup\{\infty_1,\ldots,\infty_w\}\in\mathcal{A}^i_x,\ i\in Z_m$. And for any block $D\in(\bigcup_{i\in Z_w}\bigcup_{x\in X}\mathcal{A}^i_x)$ with $\{\infty_1,\ldots,\infty_w\}\not\subset D$, we have

$$|D\cap\{\infty_1,\infty_2,\ldots,\infty_w\}|\leqslant 1,\ \text{and}\ |D|\in K'.$$

Since

$$\left(\bigcup_{i\in Z_m}\bigcup_{x\in X}\mathcal{A}^i_x\right)=\left(\bigcup_{i\in Z_m}\bigcup_{x\in X}\bigcup_{x\in B\in\mathcal{B}}\mathcal{A}^i_B(x)\right),$$

there exists a $B\in\mathcal{B}$, such that $D\in\mathcal{A}^i_B(x)$. If $\infty\in B$, since $\{(X_B,\mathcal{A}^i_B(x));\ i\in Z_m,\ x\in B\backslash\{\infty\}\}\cup\{((B\backslash\{\infty\})\times Z_m,\mathcal{G}_B,\mathcal{A}_B(\infty,n));\ 1\leqslant n\leqslant w-2\}$ is an LGKS$(K',\{m+w\},m^{|B|-1},m(|B|-1)+w,w)$, then by the definition of an LGKS, we have

$$|\{\mathcal{A}^i_B(x);\ D\in\mathcal{A}^i_B(x)\}|=|D|-2.$$

If $\infty\notin B$, since $\mathcal{A}^i_B(x)$ is a KF$(m^{|B|-1})$, we must have $|D|=3$ and D is contained in exactly one $\mathcal{A}^i_B(x)$.

By fact 1–3, we have

$$|\{\mathcal{A}^i_x;\ D\in\mathcal{A}^i_x\}|=|\{\mathcal{A}^i_B(x);\ D\in\mathcal{A}^i_B(x)\}|=|D|-2.$$

Thus the condition (4) of LGKS is satisfied by the collection.

Therefore, the collection $\{(Y,\mathcal{A}^i_x);\ x\in X,i\in Z_m\}\cup\{(X\times Z_m,\mathcal{G},\mathcal{A}(n));\ 1\leqslant n\leqslant w-2\}$ is an LGKS$(K'\cup\{3\},\{m+w\},m^u,mu+w,w)$. $\square$

6. Main results

Theorem 5. *There exists an* LGKS$(\{3\},\{9\},6^u,6u+3,3)$ *for* $u\in\{4^n25^m;m,n\geqslant 0\}$.

Proof. By Lemma 1, there exist an $S(3,5,4^n+1)$ for $n\geqslant 2$ and an $S(3,5,26)$ thus there exists an $S(3,\{5\},\{5\},u+1)$ for any $u\in\{4^n25^m;\ m,n\geqslant 0\}$. And both an LGKS$(\{3\},\{9\},6^4,24+3,3)$ and an OLKF(6^5) exist by Theorems 1 and 3. Thus, there exists an LGKS$(\{3\},\{9\},6^u,6u+3,3)$ for $u\in\{4^n25^m;\ m,n\geqslant 0\}$ by Theorem 4. $\square$

Combining Theorem 5 and Construction 3, we have

Theorem 6. *There exists an* LKTS(v) *for* $v\in\{6\cdot 4^n25^m+3;\ m,n\geqslant 0\}$.

Acknowledgements

The author would like to thank Professor L. Zhu for helpful suggestions and the referees for their helpful comments.

Appendix

An OLKF$(6^5)=\{(X,\mathscr{G},\mathscr{B}_h^i);\ 0\leqslant i\leqslant 4, 1\leqslant h\leqslant 6\}$, where
point set: $X=Z_{10}\times Z_3$,
group set: $\mathscr{G}=\{G_i=\{i,i+5\}\times Z_3; 0\leqslant i\leqslant 4\}$.

For convenience, we write t_j instead of $(t,j)\in Z_{10}\times Z_3$. We take $j,k\in Z_3$ in the following $\mathscr{B}_h^i, 1\leqslant h\leqslant 6$:

$\mathscr{B}_1^i$:

$$\begin{aligned}
&\{(1+i)_j,\ (2+i)_{j+k},\ (8+i)_{j-k}\}, && \{(6+i)_j,\ (7+i)_{j+k},\ (3+i)_{j-k}\},\\
&\{(1+i)_j,\ (4+i)_{j+k},\ (7+i)_{j-k}\}, && \{(6+i)_j,\ (9+i)_{j+k},\ (2+i)_{j-k}\},\\
&\{(1+i)_j,\ (3+i)_{j+k},\ (9+i)_{j-k}\}, && \{(6+i)_j,\ (8+i)_{j+k},\ (4+i)_{j-k}\},\\
&\{(2+i)_j,\ (3+i)_{j+k},\ (4+i)_{j-k}\}, && \{(7+i)_j,\ (8+i)_{j+k},\ (9+i)_{j-k}\}.
\end{aligned}$$

$\mathscr{B}_2^i$:

$$\begin{aligned}
&\{(1+i)_j,\ (2+i)_{j+k},\ (8+i)_{j-k+1}\}, && \{(6+i)_j,\ (7+i)_{j+k},\ (3+i)_{j-k+1}\},\\
&\{(1+i)_j,\ (4+i)_{j+k},\ (7+i)_{j-k+1}\}, && \{(6+i)_j,\ (9+i)_{j+k},\ (2+i)_{j-k+1}\},\\
&\{(1+i)_j,\ (3+i)_{j+k},\ (9+i)_{j-k+1}\}, && \{(6+i)_j,\ (8+i)_{j+k},\ (4+i)_{j-k+1}\},\\
&\{(2+i)_j,\ (3+i)_{j+k},\ (4+i)_{j-k+1}\}, && \{(7+i)_j,\ (8+i)_{j+k},\ (9+i)_{j-k+1}\}.
\end{aligned}$$

$\mathscr{B}_3^i$:

$$\begin{aligned}
&\{(1+i)_j,\ (2+i)_{j+k},\ (9+i)_{j-k-1}\}, && \{(6+i)_j,\ (7+i)_{j+k},\ (4+i)_{j-k-1}\},\\
&\{(1+i)_j,\ (3+i)_{j+k},\ (7+i)_{j-k-1}\}, && \{(6+i)_j,\ (8+i)_{j+k},\ (2+i)_{j-k-1}\},\\
&\{(8+i)_j,\ (1+i)_{j+k},\ (4+i)_{j-k-1}\}, && \{(3+i)_j,\ (6+i)_{j+k},\ (9+i)_{j-k-1}\},\\
&\{(2+i)_j,\ (3+i)_{j+k},\ (4+i)_{j-k-1}\}, && \{(7+i)_j,\ (8+i)_{j+k},\ (9+i)_{j-k-1}\}.
\end{aligned}$$

$$\begin{array}{lll}
\mathscr{B}_4^i: & \{(1+i)_j,\ (2+i)_{j+k},\ (4+i)_{j-k-1}\}, & \{(6+i)_j,\ (7+i)_{j+k},\ (9+i)_{j-k-1}\}, \\
& \{(1+i)_j,\ (7+i)_{j+k},\ (8+i)_{j-k-1}\}, & \{(6+i)_j,\ (2+i)_{j+k},\ (3+i)_{j-k-1}\}, \\
& \{(1+i)_j,\ (3+i)_{j+k},\ (9+i)_{j-k-1}\}, & \{(6+i)_j,\ (8+i)_{j+k},\ (4+i)_{j-k-1}\}, \\
& \{(2+i)_j,\ (8+i)_{j+k},\ (9+i)_{j-k}\}, & \{(7+i)_j,\ (3+i)_{j+k},\ (4+i)_{j-k}\}.
\end{array}$$

$$\begin{array}{lll}
\mathscr{B}_5^i: & \{(1+i)_j,\ (2+i)_{j+k},\ (4+i)_{j-k}\}, & \{(6+i)_j,\ (7+i)_{j+k},\ (9+i)_{j-k}\}, \\
& \{(3+i)_j,\ (4+i)_{j+k},\ (6+i)_{j-k+1}\}, & \{(8+i)_j,\ (9+i)_{j+k},\ (1+i)_{j-k+1}\}, \\
& \{(1+i)_j,\ (3+i)_{j+k},\ (7+i)_{j-k}\}, & \{(6+i)_j,\ (2+i)_{j+k},\ (8+i)_{j-k}\}, \\
& \{(2+i)_j,\ (3+i)_{j+k},\ (9+i)_{j-k-1}\}, & \{(7+i)_j,\ (8+i)_{j+k},\ (4+i)_{j-k-1}\}.
\end{array}$$

$$\begin{array}{lll}
\mathscr{B}_6^i: & \{(1+i)_j,\ (2+i)_{j+k},\ (9+i)_{j-k}\}, & \{(6+i)_j,\ (7+i)_{j+k},\ (4+i)_{j-k}\}, \\
& \{(3+i)_j,\ (4+i)_{j+k},\ (1+i)_{j-k+1}\}, & \{(8+i)_j,\ (9+i)_{j+k},\ (6+i)_{j-k+1}\}, \\
& \{(1+i)_j,\ (7+i)_{j+k},\ (8+i)_{j-k+1}\}, & \{(6+i)_j,(2+i)_{j+k},\ (3+i)_{j-k+1}\}, \\
& \{(2+i)_j,\ (4+i)_{j+k},\ (8+i)_{j-k+1}\}, & \{(7+i)_j,(9+i)_{j+k},\ (3+i)_{j-k+1}\}.
\end{array}$$

In the above, $(l+i)$ is calculated module 10, and subscripts $j+k, j-k, j-k+1, j-k-1$ are calculated module 3.

For each row of the above $\mathscr{B}_h^i$, when subscript j runs over Z_3 and subscript k is fixed, all blocks in the row consist of a partial parallel class. For example, $((X\backslash\{0,5\})\times Z_3, \mathscr{B}_1^0)$ is a KF(6^4). Taking the first row in $\mathscr{B}_1^0$, all blocks are as follows:

$$\begin{array}{lll}
k=0 & \{1_0,2_0,8_0\} & \{6_0,7_0,3_0\} \\
& \{1_1,2_1,8_1\} & \{6_1,7_1,3_1\} \\
& \{1_2,2_2,8_2\} & \{6_2,7_2,3_2\}, \\
k=1 & \{1_0,2_1,8_2\} & \{6_0,7_1,3_2\} \\
& \{1_1,2_2,8_0\} & \{6_1,7_2,3_0\} \\
& \{1_2,2_0,8_1\} & \{6_2,7_0,3_1\}, \\
k=2 & \{1_0,2_2,8_1\} & \{6_0,7_2,3_1\} \\
& \{1_1,2_0,8_2\} & \{6_1,7_0,3_2\} \\
& \{1_2,2_1,8_0\} & \{6_2,7_1,3_0\}.
\end{array}$$

For each fixed k, when subscript j runs over Z_3, all blocks form a partition of $((X\backslash\{0,5\})\times Z_3)\backslash(\{4,9\}\times Z_3)$.

References

[1] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Cambridge University Press, Cambridge, 1986.
[2] J.L. Blanchard, A construction for Steiner 3-designs, J. Combin. Theory Ser. A 71 (1995) 60–67.
[3] R.H.F. Denniston, Sylvester's problem of the 15 schoolgirls, Discrete Math. 9 (1974) 229–238.
[4] R.H.F. Denniston, Double resolvability of some complete 3-designs, Manuscripta Math. 12 (1974) 105–112.
[5] R.H.F. Denniston, Four double resolvable complete three-designs, Ars Combin. 7 (1979) 265–272.
[6] R.H.F. Denniston, Further cases of double resolvability, J. Combin. Theory Ser. A 26 (1979) 298–303.
[7] H. Hanani, Resolvable designs, Colloq. Internaz. Teorie Combin., Rome, 1976, pp. 249–252.
[8] A. Hartman, The fundamental construction for 3-designs, Discrete Math. 124 (1994) 107–132.

[9] Qingde Kang, Jianguo Lei, On large sets of resolvable and almost resolvable oriented triple systems, J. Combin. Designs 4 (1996) 95–104.
[10] Jianguo Lei, Completing the spectrum for LGDD(m^v), J. Combin. Designs 5 (1997) 1–11.
[11] Jianguo Lei, On the large sets of Kirkman triple systems, Discrete Math., to appear.
[12] D.K. Ray-Chandhuri, R.M. Wilson, Solution of Kirkman's school-girl problem, Amer. Math. Soc. Symp. Pure Math. 19 (1971) 187–204.
[13] D.R. Stinson, A survey of Kirkman triple systems and related designs, Discrete Math. 92 (1991) 371–393.
[14] Luc Teirlinck, Some new 2-resolvable $S(3,4,v)$s, Designs, Codes Cryptogr. 4 (1994) 5–10.
[15] L. Teirlinck, Large sets with holes, J. Combin. Designs 1 (1994) 69–94.
[16] Lisheng Wu, Large sets of KTS(v), Lecture Notes in Pure and Applied Mathematics, Vol.126, Marcel Dekker, New York, 1990, pp. 175–178.
[17] Chang Yanxun, Ge Gennian, On the existence of LKTS(v), Ars Combin. 51 (1999) 306–312.