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A Remark on Rosenbloom's Paper "The Fix-Points of Entire Functions"

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1. INTRODUCTION

Some years ago Rosenbloom [3] proved the following theorem which was the first one giving a quantitative measure of the fix-points of entire functions.

THEOREM. *Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$ be a polynomial of degree $k \geqslant 2$, and f be a transcendental entire function. Then*

$$\varlimsup_{r\to\infty} \frac{N\left(r, \dfrac{1}{p(f) - z}\right)}{T(r, f)} \geqslant 1. \tag{1.1}$$

In this note we shall show that the above result (1.1) still holds when the coefficients a_i are replaced by meromorphic functions whose growth rates are much smaller than the given function f, and that in (1.1), $N(r, 1/p(f) - z)$ can be replaced by $\bar{N}(r, 1/p(f) - z)$. As a by-product of our arguments, some known results have also been improved.

The methods employed here are Nevanlinna's fundamental theorems for meromorphic functions and a technique used by Hayman [1, pp. 68–73].

It is assumed that the reader is familiar with the fundamental concepts of Nevanlinna's theory of meromorphic functions and the symbols $m(r, f)$, $N(r, f)$, $\bar{N}(r, f)$, $T(r, f)$, etc. [1].

2. NOTATIONS AND PRELIMINARY LEMMAS

In what follows f will always be a meromorphic function which is not constant in the plane, and $S(r, f)$ will be any quantity satisfying

$$S(r, f) = o\{T(r, f)\} \tag{2.1}$$

as $r \to \infty$, possibly outside a set of r values of finite measure. Also throughout this note we shall denote by $a(z)$, $a_1(z), \ldots$, functions which are meromorphic in the plane and satisfy

$$T\{r, a(z)\} = S(r, f) \tag{2.2}$$

as $r \to \infty$.

DEFINITION. *A differential polynomial $p(f)$ is a polynomial in f and the derivatives of f with coefficients $b(z)$ satisfying*

$$m(r, b(z)) = S(r, f). \tag{2.3}$$

A differential polynomial in f of degree at most n is denoted by $p_n(f)$. Here we note that in [1, p. 68] a differential polynomial $p(f)$ is defined as a polynomial in f and its derivatives, with meromorphic functions $a(z)$ as the coefficients which satisfy conditions (2.2).

LEMMA 1 (Clunie [1, p. 68]). *Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n(z)\, p(f) = Q(f), \tag{2.4}$$

where $p(f)$ and $Q(f)$ are differential polynomials in f and the degree of $Q(f)$ is at most n. Then

$$m\{r, p(f)\} = S(r, f) \tag{2.5}$$

as $r \to \infty$.

Remark. In the original statement the coefficients $b(z)$ in $p(f)$ and $Q(f)$ are assumed to satisfy $T(r, b(z)) = S(r, f)$. It is clear that the same argument does work when we only assume that $m\{r, b(z)\} = S(r, f)$.

LEMMA 2. *Let $p(f)$ be a differential polynomial in f. Suppose that the coefficients $a(z)$ in $p(f)$ satisfy $T(r, a(z)) = S(r, f)$. Then*

$$T(r, p(f)) = O(1)\{T(r, f)\} + S(r, f). \tag{2.6}$$

Proof. First of all, we note that $p(f)$ can have poles only at poles of $f(z)$ or the coefficients $a_i(z)$. Assume that the degree of $p(f)$ is n and the highest derivative of f occurring in $p(f)$ is l. Then at a pole of $f(z)$ of order k, $f^{(l)}(z)$ has a pole of order at most $n(l+1)\,k$. Thus from the above observation and Nevanlinna's first fundamental theorem we have

$$\begin{aligned} N(r, p(f)) &\leqslant n(l+1)\, N(r, f) + \Sigma N\{r, a_i(z)\} \\ &= O(1)\{T(r, f)\} + S(r, f). \end{aligned} \tag{2.7}$$

Also by a result of Milloux (see [1, p. 55]) we have

$$\begin{aligned} m(r, p(f)) &= O\left[\sum_{i=1}^{l} m(r, f^{(i)}(z)) + \Sigma m(r, a_j(z))\right] \\ &= O(1)\{T(r,f)\} + S(r,f). \end{aligned} \tag{2.8}$$

The lemma follows from (2–7) and (2–8).

3. Statements and the Proofs of the Main Results

THEOREM 1. *Let $f(z)$ be meromorphic and transcendental in the plane, and let $h(z) = f^k(z) + a_1(z) f^{k-1}(z) + \cdots + a_k(z)$, $k \geq 2$. Assume that*

$$h(z) - z \not\equiv (cf(z) - a(z))^k,$$

c, a constant. Then

$$\varlimsup_{r\to\infty} \frac{\bar{N}\left(r, \frac{1}{h-z}\right) + N(r,f)}{T(r,f)} \geq 1. \tag{3.1}$$

Proof. Set

$$F(z) = f(z) + \frac{a_1(z)}{k}. \tag{3.2}$$

Then

$$g(z) = h(z) - z = F^k(z) + p_{k-2}(F), \tag{3.3}$$

where $p_{k-2}(F)$ is a differential polynomial in F of degree $\leq k-2$ with coefficients $b(z)$ satisfying

$$T(r, b(z)) = S(r,f). \tag{3.4}$$

Differentiating (3.3) we obtain

$$g'(z) = kF^{k-1}(z) F' + Q_{k-2}(F). \tag{3.5}$$

We now multiply (3.3) by g'/g and subtract (3.3) from (3.5), thus obtaining

$$F^{k-2}F\left(F\frac{g'}{g} - kF'\right) = -p_{k-2}(F)\frac{g'}{g} + Q_{k-2}(F). \tag{3.6}$$

By Lemma 2 we see that $m(r, g'/g) = S(r,g) = S(r,F)$. Thus we may apply Lemma 1 to both $p(f) = Fg'/g - kF'$ and $p(f) = F(Fg'/g - kF')$, deducing that

$$m\left(r, F\frac{g'}{g} - kF'\right) = S(r,f) \tag{3.7}$$

and

$$m\left(r, F\left(F\frac{g'}{g} - kF'\right)\right) = S(r, F).$$

We cannot have $Fg'/g - kF \equiv 0$, since this yields

$$k\frac{F'}{F} = \frac{g'}{g} \tag{3.8}$$

or

$$g(z) = c\{F(z)\}^k,$$

which contradicts our hypothesis. Alternatively,

$$F\frac{g'}{g} - kF' \not\equiv 0. \tag{3.9}$$

Now from (3.2) we see that

$$T(r, f) = (1 + o(1))\, T(r, F) = (1 + o(1))\, [N(r, F) + m(r, F)] \tag{3.10}$$

as $r \to \infty$, possibly outside a set of finite measure.

Combining (3.7) with (3.9) we deduce that

$$\begin{aligned} m(r, F) &\leqslant m\left(r, F\left(F\frac{g'}{g} - kF'\right)\right) + m\left(r, \frac{1}{F(g'/g) - kF'}\right) + O(1) \\ &\leqslant S(r, F) + T\left(r, F\frac{g'}{g} - kF'\right) + O(1) \\ &= S(r, F) + m\left(r, F\frac{g'}{g} - kF'\right) + N\left(r, F\frac{g'}{g} - kF'\right) + O(1) \\ &= S(r, F) + S(r, F) + N\left(r, F\frac{g'}{g} - kF'\right) + O(1). \end{aligned} \tag{3.11}$$

In view of the forms of $p_{k-2}(F)$ and $Q_{k-2}(F)$ and comparing the multiplicities of the poles of F on both sides of (3.6), one can find that all the poles of $Fg'/g - kF'$ come from the zeros of g, poles of f or the poles of all the coefficients of $g(z)$. Hence

$$N\left(r, F\frac{g'}{g} - kF'\right) \leqslant \bar{N}\left(r, \frac{1}{g}\right) + S(r, F). \tag{3.12}$$

Combining (3.12) with (3.11) and (3.10) we find

$$\begin{aligned} T(r, f) &\leqslant (1 + o(1))\, N(r, F) + (1 + o(1))\, \bar{N}\left(r, \frac{1}{g}\right) + S(r, F) \\ &= (1 + o(1))\, N(r, f) + (1 + o(1))\, \bar{N}\left(r, \frac{1}{g}\right) + S(r, f). \end{aligned} \tag{3.13}$$

It follows that

$$\frac{T(r,f)+S(r,f)}{T(r,f)} \leqslant \frac{(1+o(1))\,\bar{N}\left(r,\frac{1}{g}\right)+(1+o(1))\,N(r,f)}{T(r,f)}, \tag{3.14}$$

and the assertion (3.1) follows from this.

As an immediate consequence of Theorem 1 we have the following theorem.

THEOREM 2. *Let $f(z)$ be meromorphic and transcendental in the plane, with $N(r,f)=o\{T(r,f)\}$ as $r\to\infty$. Assume that*

$$h(z)=f^k(z)+a_1(z)f^{k-1}(z)+\cdots+a_k(z),$$

where the $a_i(z)$ are meromorphic functions satisfying condition (2.2) and k is an integer $\geqslant 2$. Assume that $h(z)-z\not\equiv(cf(z)-a(z))^k$. Then

$$\varlimsup_{r\to\infty}\frac{\bar{N}\left(r,\frac{1}{h-z}\right)}{T(r,f)}\geqslant 1. \tag{3.15}$$

In particular, if f is transcendental and entire and if all the coefficients $a(z)$ in $h(z)$ are constants, then we obtain Rosenbloom's theorem cited in the introduction.

If we say that $a(z)$ is a k-ramified defect function of $f(z)$ whenever $T(r,a(z))=o\{T(r,f)\}$ as $r\to\infty$ such that all except finitely many of the roots of the equation $f(z)=a(z)$ have multiplicity at least k, then we have the following theorem.

THEOREM 3. *If f is a transcendental entire function, then there exists at most one 3-ramified defect function of $f(z)$.*

Proof. Suppose that f has two 3-ramified defect functions $a_1(z)$ and $a_2(z)$. Then by setting $h(z)=(f(z)-a_1(z))(f(z)-a_2(z))+z$ in Theorem 2 we obtain from (3.15) that

$$\varlimsup_{r\to\infty}\frac{\bar{N}\left(r,\frac{1}{f-a_1}\right)+\bar{N}\left(r,\frac{1}{f-a_2}\right)}{T(r,f)}\geqslant 1. \tag{3.16}$$

On the other hand, since $a_1(z)$ and $a_2(z)$ are 3-ramified defect functions, we have

$$\varlimsup_{r\to\infty}\frac{\bar{N}\left(r,\frac{1}{f-a_i}\right)}{T(r,f)}\leqslant\frac{1}{3} \tag{3.17}$$

for $i=1,2$. This contradicts (3.16), and the theorem is thus proved.

Remark. This result also follows from a result of Nevanlinna [1, p. 47]. Along similar lines we can obtain the following improved result [4].

THEOREM 4. *Let $f(z)$ be a transcendental meromorphic function with $N(r,f) = S(r,f)$. Let $\Psi(z)$ be a differential polynomial in f (in the sense that the coefficients $a(z)$ satisfy $T(r, a(z)) = S(r,f)$. Suppose that*

$$\Psi(z) = f^n(z) + p_{n-2}(f)$$

with $n \geqslant 2$. Then

$$\delta(c, \Psi) < 1 - \frac{1}{n} \qquad \text{for } c \not\equiv 0, \infty.$$

Finally we remark that the argument used in proving Theorem 1, especially inequality (3.13), reveals that we can obtain the theorem of Tumura–Clunie [1, p. 68], if the condition

$$N(r,f) + N\left(r, \frac{1}{g}\right) = S(r,f)$$

is replaced by

$$N(r,f) + N\left(r, \frac{1}{f}\right) = S(r,f)$$

and

$$\bar{N}\left(r, \frac{1}{g}\right) \leqslant (1 - \epsilon)\, T(r,f), \qquad 1 > \epsilon > 0$$

as $r \to \infty$, possibly outside a set of r values of finite measure. From this we can obtain the following theorem which is an improvement of Hayman's result [1, p. 74].

THEOREM 5. *Suppose that $f(z)$ is meromorphic and not constant in the plane and that for some $l \geqslant 2$ and $1 > \epsilon > 0$,*

$$N(r,F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F^{(l)}}\right) \leqslant (1 - \epsilon)\, T\left(r, \frac{F'}{F}\right)$$

as $r \to \infty$, possibly outside a set of r values of finite measure. Then $F(z) = e^{az+b}$, where a and b are constants.

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