Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 221 (2008) 47–51
www.elsevier.com/locate/cam

Lagrange interpolation for continuous piecewise smooth functions

Sergio Amat*, Sonia Busquier, Antonio Escudero, J. Carlos Trillo

Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Paseo Alfonso XIII, 52, 30203 Cartagena (Murcia), Spain

Received 24 January 2007; received in revised form 19 September 2007

Abstract

This note is devoted to Lagrange interpolation for continuous piecewise smooth functions. A new family of interpolatory functions with explicit approximation error bounds is obtained. We apply the theory to the classical Lagrange interpolation.

MSC: 41A05; 41A10; 65D05; 65D17

Keywords: Lagrange interpolation; Smoothness; Error estimates

1. Introduction

High order linear interpolation procedures associated with large supports are affected by the presence of singularities in the signal considered. These kinds of techniques produce the Gibbs-like phenomenon in the presence of jump discontinuities [1–6]. However, we have not seen such problems for continuous functions. The goal of this paper is to provide a simple, but rigorous, mathematical justification of this fact. We present a class of interpolatory functions verifying certain approximation error bounds for continuous piecewise smooth functions. We apply this study to the classical Lagrange interpolation [10] introduced just for smooth functions. For uniform bounds associated with specific nodes, similar to those obtained when the nodes are the zeros of Chebyshev polynomials in Lagrange interpolation of smooth functions, we refer the reader to [7–9,11,12]. However, these papers are of a technical nature. Our goal is to provide a rigorous but simple justification.

2. Lagrange interpolation in the Newton form

Let $f : [a, b] \to \mathbb{R}$ be a continuous function in a finite closed interval $[a, b]$. Let $x_0, x_1, \ldots, x_n \in [a, b]$ $(x_i \neq x_j)$, $n \in \mathbb{N}$, be the interpolatory nodes. Let $P^n : [a, b] \to \mathbb{R}$ be the unique interpolatory polynomial of degree n associated with the nodes $x_0, x_1, \ldots, x_n$ and the function f. Then we can write the polynomial as

$$P^n(x) = \sum_{j=0}^{n} f[x_0, \ldots, x_j](x - x_0) \ldots (x - x_{j-1}), \quad x \in [a, b]. \tag{1}$$

* Corresponding author.
E-mail addresses: sergio.amat@upct.es (S. Amat), sonia.busquier@upct.es (S. Busquier), antonio.escudero@upct.es (A. Escudero), jc.trillo@upct.es (J. Carlos Trillo).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.011

This is the Newton form of the polynomial.

The constant coefficients $f[x_0,\dots,x_j]$, $j=0,\dots,n$, are called jth-order *divided differences* associated with f.

Using the uniqueness of the polynomial we can prove the following proposition.

Proposition 1. *For each permutation σ of the set $\{0,\dots,k\}$, we have*

$$f[x_0,\dots,x_k]=f[x_{\sigma(0)},\dots,x_{\sigma(k)}].$$

As a consequence we arrive at the principal property of the divided differences:

Proposition 2.

$$f[x_0,\dots,x_k]=\frac{f[x_1,\dots,x_k]-f[x_0,\dots,x_{k-1}]}{x_k-x_0}. \tag{2}$$

Using expression (2) and considering the data of the following table:

x_0	x_1	x_2	$\dots$	x_n
f_0	f_1	f_2	$\dots$	f_n

the divided differences are obtained via the following scheme:

$$
\begin{array}{ccccccc}
x_0 & f[x_0] & & & & & \\
 & & \searrow & & & & \\
x_1 & f[x_1] & \rightarrow & f[x_0,x_1] & & & \\
 & & \searrow & & \searrow & & \\
x_2 & f[x_2] & \rightarrow & f[x_1,x_2] & \rightarrow & & \\
\vdots & \vdots & & \vdots & & & \\
 & & & & & & \\
\vdots & \vdots & & \vdots & & \searrow & \\
x_n & f[x_n] & \rightarrow & f[x_{n-1},x_n] & \rightarrow & \rightarrow & f[x_0,\dots,x_n]
\end{array}
$$

where $f[x_j]=f(x_j)$, $j=0,\dots,n$.

Error for Lagrange interpolation

Using the Rolle Theorem several times the following error formula holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function in the finite interval $[a,b]$ and $n+1$ times differentiable in (a,b). Let $x_0,\dots,x_n$ be different points in $[a,b]$ $(x_i\neq x_j)$ and let $P^n(x)$ be the associated Lagrange interpolatory polynomial of degree n. Then, for each $x\in(a,b)$ there exists ξ_x such that*

$$f(x)-P^n(x)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-x_0)\cdots(x-x_n) \tag{3}$$

where

$$\min(x,x_0,\dots,x_n)<\xi_x<\max(x,x_0,\dots,x_n). \tag{4}$$

For non-smooth functions is clear that we cannot use the above theorem. In the case of discontinuous functions, the Gibbs-like phenomenon appears in practice. However, we do not have such problems for continuous functions. The goal of this paper is to provide, following the ideas of the present section, a simple, but rigorous, mathematical justification of this fact.

3. Auxiliary family of continuous piecewise smooth functions

Let us consider a set of points

$$\{y_0, y_1, \ldots, y_m\}$$

of a given real interval (a, b), where the signal can be non-smooth.

For a fixed collection of increasing continuous functions

$$\{\gamma_i : [a, b] \subset \mathbb{R} \to \mathbb{R}\}_{i=0}^{m},$$

such that $(\gamma_i)'(x) \in \mathbb{R}$ for all $x \in (a, b) \setminus \{y_i\}$ and $(\gamma_i)'(y_i) = +\infty$, we introduce the following space of continuous functions:

$$\mathcal{F}_{\gamma_i}([a, b]) := \left\{ f \in \mathcal{C}([a, b]) / \exists \wp_{\gamma_i}(f)(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{\gamma_i(x+h) - \gamma_i(x)} \forall x \in (a, b) \right\}.$$

An example of γ_i functions is

$$\gamma_i(x) = \begin{cases} \sqrt{x - y_i} & x \geq y_i \\ -\sqrt{-x + y_i} & \text{otherwise.} \end{cases}$$

Notice that this generalized derivative in the definition of $\mathcal{F}_{\gamma_i}([a, b])$ is linear and zero for constants and that these spaces include the smooth functions. For these spaces we generalize some classical theorems.

Lemma 1. *Let $f, g \in \mathcal{F}_{\gamma_i}([a, b])$; then $fg \in \mathcal{F}_{\gamma_i}(\mathbb{R})$ and*

$$\wp_{\gamma_i}(fg)(x) = f(x)\wp_{\gamma_i}(g)(x) + \wp_{\gamma_i}(f)(x)g(x).$$

Proposition 3 (*Generalization of the Rolle Theorem*). *Let $f \in \mathcal{F}_{\gamma_i}([a, b])$. Let a, b be such that $f(a) = f(b) = 0$. Then there exists $\xi \in (a, b)$ such that $\wp_{\gamma_i}(f)(\xi) = 0$.*

Proof. If $f(x) = 0$ for all $x \in [a, b]$ then $\wp_{\gamma_i}(f)(x) = 0$ for all $x \in (a, b)$.

Let ξ be such that $f(\xi) := \max_{x \in [a,b]} |f(x)|$. Then, using that γ_i is an increasing function, we have

$$\frac{f(\xi + h) - f(\xi)}{\gamma_i(\xi + h) - \gamma_i(\xi)} \cdot \frac{f(\xi - h) - f(\xi)}{\gamma_i(\xi - h) - \gamma_i(\xi)} \leq 0$$

and from definition $\wp_{\gamma_i}(f)(\xi) = 0$. □

Definition 1. Define $f_{\gamma_i}^{(0)} := f$, $f_{\gamma_i}^{(1)} := \wp_{\gamma_i}(f), \ldots, f_{\gamma_i}^{(m)} := \wp_{\gamma_i}(f_{\gamma_i}^{(m-1)})$, a function $f \in \mathcal{F}_{\gamma_i}^{m}([a, b])$ if and only if there exists $f_{\gamma_i}^{(m-1)} \in \mathcal{F}_{\gamma_i}([a, b])$, $m \in \mathbb{N}$.

Let us define by induction the following generalized divided differences:

$$f_{\gamma_i}[x_{j-1}, x_j] := \frac{f(x_j) - f(x_{j-1})}{\gamma_i(x_j) - \gamma_i(x_{j-1})}$$

$$f_{\gamma_i}[x_{j-p}, \ldots, x_{j-p+s}] := \frac{f_{\gamma_i}[x_{j-p+1}, \ldots, x_{j-p+s}] - f_{\gamma_i}[x_{j-p}, \ldots, x_{j-p+s-1}]}{\gamma_i(x_{j-p+s}) - \gamma_i(x_{j-p})} \quad 0 \leq p \leq j,\ 0 \leq s.$$

Proposition 4 (*Generalized Lagrange Interpolation*). *Let $f \in \mathcal{F}_{\gamma_i}^{n+1}([a, b])$ and let $x_0, x_1, \ldots, x_n$ be different points in $[a, b]$. Then*

$$f(x) - P_{\gamma_i}^{n}(x) = \frac{f_{\gamma_i}^{(n+1)}(\xi)}{(n+1)!}(\gamma_i(x) - \gamma_i(x_0)) \ldots (\gamma_i(x) - \gamma_i(x_n)), \quad x \in (a, b)$$

where

$$P^n_{\gamma_i}(x) = f(x_0) + f_{\gamma_i}[x_0, x_1](\gamma_i(x) - \gamma_i(x_0)) + \cdots + f_{\gamma_i}[x_0, x_1, \ldots, x_n](\gamma_i(x) - \gamma_i(x_0)) \ldots (\gamma_i(x) - \gamma_i(x_{n-1}))$$

and $\xi \in (\min\{x, x_0, x_1, \ldots, x_n\}, \max\{x, x_0, x_1, \ldots, x_n\})$.

Proof. Applying the above Generalized Rolle's Theorem $n+1$ times to the function

$$\Psi(t) = f(t) - P^n_{\gamma_i}(t) + \mathcal{K} \prod_{j=0}^{n} (\gamma_i(t) - \gamma_i(x_j))$$

where $\mathcal{K}$ is such that $\Psi(x) = 0$, and remarking that

$$(P^n_{\gamma_i})^{(n+1)}(t) = 0$$

and

$$\left(\prod_{j=0}^{n} (\gamma_i(t) - \gamma_i(x_j)) \right)^{(n+1)} (t) = (n+1)!$$

the proposition holds. □

Application to the classical Lagrange interpolation

Let $f \in \mathcal{F}^{n+1}_{\gamma_i}([a, b])$, $x \in (a, b)$ and let $x_0, x_1, \ldots, x_n$ be different points in $[a, b]$. Then

$$f(x) - P^n(x) = \frac{(f_{\gamma_i} - P^n_{\gamma_i})^{(n+1)}(\xi)}{(n+1)!} (\gamma_i(x) - \gamma_i(x_0)) \ldots (\gamma_i(x) - \gamma_i(x_n))$$

where $P^n(x)$ and $P^n_{\gamma_i}(x)$ are, respectively, the Lagrange polynomial and the polynomial defined in Proposition 4.

A final remark

If $f \in \mathcal{C}([a, b])$ is sufficiently smooth at all points $x \in (a, b) \setminus \{y_0, y_1, \ldots, y_m\}$ having finite lateral derivatives at each y_i, then taking

$$\gamma = \sum_{i=0}^{m} \gamma_i$$

and considering

$$\mathcal{F}_\gamma([a, b]) := \left\{ f \in \mathcal{C}([a, b]) / \exists \wp(f)(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{\gamma(x+h) - \gamma(x)}, \forall x \in (a, b) \right\},$$

we will obtain that

$$f \in \mathcal{F}_\gamma([a, b]).$$

Acknowledgements

The first and second authors' research was supported in part by the Spanish grants MTM2004-07114 and 00675/PI/04.

References

[1] J.H. Foster, F.B. Richards, The Gibbs phenomenon for piecewise-linear approximation, Amer. Math. Monthly. 98 (1) (1991) 47–49.

[2] D. Gottlieb, J.S. Hesthaven, Spectral methods for hyperbolic problems, J. Comput. Appl. Math. 128 (1–2) (2001) 83–131.

[3] D. Gottlieb, C.W. Shu, A. Solomonoff, H. Vandeven, On the Gibbs phenomenon. I. Recovering exponential accuracy from Fourier partial sum of a nonperiodic analytic function. Orthogonal polynomials and numerical methods, J. Comput. Appl. Math. 43 (1-2) (1992) 81–98.

[4] D. Gottlieb, C.W. Shu, On the Gibbs phenomenon. III. Recovering exponential accuracy in a sub-interval from a spectral partial sum of a piecewise analytic function, SIAM J. Numer. Anal. 33 (1) (1996) 280–290.

[5] D. Gottlieb, C.W. Shu, On the Gibbs phenomenon. V. Recovering exponential accuracy from collocation point values of a piecewise analytic function, Numer. Math. 71 (4) (1995) 511–526.
[6] D. Gottlieb, C.W. Shu, On the Gibbs phenomenon and its resolution, SIAM Rev. 39 (4) (1997) 644–668.
[7] X. Li, On the Lagrange interpolation for a subset of C^k functions, Constr. Approx. 11 (3) (1995) 287–297.
[8] G. Mastroianni, J. Szabados, Jackson order of approximation by Lagrange interpolation, Rend. Circ. Mat. Palermo 2 (33) (1993) 375–386.
[9] G. Mastroianni, J. Szabados, Jackson order of approximation by Lagrange interpolation. II, Acta Math. Hungar. 69 (1–2) (1995) 73–82.
[10] J. Stoer, R. Burlirsch, Introduction to Numerical Analysis, Springer-Verlag, New York, 1980.
[11] S.P. Zhou, On approximation by Lagrange interpolating polynomials for a subset of the space of continuous functions, Colloq. Math. 75 (1) (1998) 1–5.
[12] S.P. Zhou, L.Y. Zhu, Convergence of Lagrange interpolation polynomials for piecewise smooth functions, Acta Math. Hungar. 93 (1–2) (2001) 71–76.