A Multiplicity Result for the Yamabe Problem on $S^n$

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Communicated by H. Brezis

Received May 19, 1999; accepted May 27, 1999

We prove a multiplicity result for the Yamabe problem on the manifold $(S, \hat{g})$, where $\hat{g}$ is a perturbation of the standard metric $g_0$ of $S^n$. Solutions are found by variational methods via an abstract perturbation result.

Key Words: Yamabe problem; elliptic equations; positive solutions; multiples solutions; critical exponent.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Let $(M^n, g)$, $n \geq 3$, be a compact Riemannian $n$-dimensional manifold with scalar curvature $R_g$. Let $[g]$ denote the class of metric conformally equivalent to $g$. The Yamabe problem consists in looking for a metric $g' \in [g]$ such that for its scalar curvature $R_{g'}$ there results $R_{g'} \equiv 1$. This problem amounts to finding a positive solution $u \in H^1(S^n)$ of

$$-2c_n \Delta_g u + R_g u = u^{(n+2)/(n-2)}, \quad c_n = \frac{2(n - 1)}{(n - 2)}, \quad (1)$$

where $\Delta_g$ denotes the Laplace–Beltrami operator. Actually, if $u$ is such a solution then $g' = u^4/(n-2) g$ is a conformal metric satisfying $R_{g'} \equiv 1$.

A positive answer to the Yamabe problem has been given by T. Aubin [1, 2], see also the review [13], who has shown that if $n \geq 6$ and $(M^n, g)$ is not locally conformally flat then the Yamabe problem has a solution. The locally conformally flat case and dimensions $n = 3, 4, 5$ have been handled by R. Schoen and S. T. Yau [17] and by R. Schoen [15], respectively.

Much less is known about the existence of multiple solutions. One case is rather trivial: if $g_0$ denotes the standard metric on $S^n$ and $\phi: S^n \to S^n$ is any conformal nonisometric map, then problem (1) with $g = \phi^* g_0$ has infinitely many solutions. Other examples can be given on some product.

$^1$Supported by M.U.R.S.T., Variational Methods and Nonlinear Differential Equations.
manifolds, see [2, 16], or on manifolds that are invariant under some group action, see [11, 12]. See also Section 5.7 of [2].

Apart from these cases, we do not know any further multiplicity result and the purpose of this paper is to discuss a new class of metrics on $S^n$, close to the standard one, such that the Yamabe problem has at least two solutions.

Let us consider the unit sphere $S^n$ endowed with the standard metric $g_0$. It is convenient to use the stereographic projection $\sigma$ and transform (1) into an equivalent equation on $\mathbb{R}^n$. Precisely, let $E = D^{1,2}(\mathbb{R}^n)$ and let

$$z_\sigma(x) = \kappa_n \cdot \frac{1}{(1 + |x|^2)^{(\alpha - 2)/2}}, \quad \kappa_n = (4\alpha(n - 1))(n - 2)/4$$

(2)

denote the unique (up to dilations and translations) solution to the problem

$$\begin{cases} -2\kappa_n A u = u^{(n + 2)/(n - 2)} & \text{in } \mathbb{R}^n \\ u > 0, & u \in E. \end{cases}$$

(P$_0$)

Setting in stereographic coordinates, $\tilde{g} = z_\sigma^{-4/(n-2)}g$, problem (1) is equivalent to finding a solution of

$$\begin{cases} -2\kappa_n A_{\tilde{g}} u + R_{\tilde{g}} u = u^{(n + 2)/(n - 2)} & \text{in } \mathbb{R}^n \\ u > 0, & u \in E. \end{cases}$$

(3)

Solutions of (3) are the critical points of the functions $f: E \to \mathbb{R}$,

$$f(u) = \int_{\mathbb{R}^n} \left( c_n |\nabla u|^2 + \frac{1}{2} R_{\tilde{g}} u^2 - \frac{1}{2^*} |u|^{2^*} \right) dV_{\tilde{g}}.$$  

(4)

Let us recall that the isometry $\iota$ between $H^1(S^n)$ and $E$ given by $\iota(u)(x) = z_\sigma(x) u(\sigma^{-1}(\cdot))$ transforms the functional $f$ into the corresponding functional $J: H^1(S^n) \to \mathbb{R}$,

$$J(u) = \int_{S^n} \left( c_n |\nabla u|^2 + \frac{1}{2} R_{\tilde{g}} u^2 - \frac{1}{2^*} |u|^{2^*} \right) dV_{\tilde{g}}.$$  

In particular, $f$, including the term $\int R_{\tilde{g}} u^2$, is well defined and smooth on $E$.

In the sequel we shall be interested in a class of metrics $\tilde{g}$ of the form

$$\tilde{g}_\delta = \delta + \epsilon h$$

with components

$$\tilde{g}_\delta(x) = \delta_\sigma + \epsilon h_\sigma(x),$$

(5)
where $\delta_{ij}$ are the Kronecker symbols, and $h_{ij}: \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \ldots, n$ are smooth functions. More specifically, to give an idea of our nonuniqueness results, let us take

$$h_{11} = a(x), \quad h_{ij} \equiv 0 \quad \text{for } i, j > 1. \tag{6}$$

With this choice of $\hat{g}$, Eq. (1) becomes

$$-2c_nAu = u(n+2)(n-2) + K(x, u, Du, D^2u), \tag{7}$$

where $K$ is a suitable perturbation term depending also on $h$. Equation (7) can be handled by means of a perturbation method in critical point theory introduced in [3–5]. This approach has been used in [6] in connection with the scalar curvature problem on $\mathbb{R}^n$ to improve some preceding results by [9, 14]. The abstract setting consists, roughly, in seeking for critical points of a functional $f_\varepsilon(u) = f_0(u) + \varepsilon G(u)$ where $f_0$ possesses a finite dimensional manifold $Z$ of critical points satisfying a suitable non-degeneracy condition. It is shown that near $Z$ there exists a perturbed manifold $Z_\varepsilon$ such that the stationary points of $f_\varepsilon$ can be found by looking for the critical points of $f_\varepsilon$ constrained on $Z_\varepsilon$. Actually, in the case of the Yamabe problem this abstract approach needs to be modified because $G_{\mu\varepsilon} \equiv 0$. Precisely, here $f_\varepsilon$ has the form

$$f_\varepsilon(u) = f_0(u) + \varepsilon G_1(u) + \varepsilon^2 G_2(u) + o(\varepsilon^2),$$

where

$$f_0(u) = \frac{1}{2} \int \left( c_n |Vu|^2 - \frac{1}{2^*} |u|^{2^*} \right) dx$$

and

$$Z = \left\{ z_{\mu, \xi} = \mu^{-\frac{n-2}{2^*}} \left( \frac{x-\xi}{\mu} \right) | \mu > 0, \xi \in \mathbb{R}^n \right\}. \tag{8}$$

It turns out that $G_1(z) = 0$ for all $z \in Z$ and then for $z_\varepsilon \in Z_\varepsilon$ there results

$$f_\varepsilon(z_\varepsilon) = b + \varepsilon^2 \Gamma(z) + o(\varepsilon^2).$$

Here $b = f_0(z)$ is a constant and $\Gamma: Z \to \mathbb{R}$ is a suitable finite dimensional functional which depends on $G_1$ and $G_2$. The explicit expression of $\Gamma$ shows that it can be extended to $\mu = 0$ and there result $\Gamma(0, \xi) \equiv 0$. Moreover, for $n \geq 6$ one finds that all the derivatives with respect to $\mu$ up to order 3 evaluated at $(0, \xi)$ are zero. Furthermore, if $a$ is non constant there results $\lim_{\mu \to 0^+} \mu^{-1} \Gamma(\mu, \xi) < 0$ (if $n = 6$) for some $\xi$. Since $\Gamma(\mu, \xi) \to 0$ as
\[ \mu + |\xi| \to \infty, \Gamma \text{ achieves a minimum at some } (\mu_1, \xi_1) \text{ and (7) has a solution } u_{\varepsilon}, \text{ such that } u_{\varepsilon} \to u_{\varepsilon_0}, \xi_1, \varepsilon \to 0. \] By the way, this is in accordance with the Aubin result in which a minimum of the Sobolev quotient is found by choosing appropriate test functions concentrated near points where the Weyl tensor \( W \neq 0 \). Actually the first order term \( \tilde{W} \) in the expansion of \( W \) does not vanish if \( \tilde{g} \) has the form (5–6) and \( a \) is non constant. The new feature here is that our approach is more precise because it allows us to locate the solutions \( u_{\varepsilon} \) of (7). Taking advantage of this fact we can show

**Theorem 1.1.** Let \( \tilde{g} \) have the form (5–6), with

\[ a(x) = \tau(x) + \omega(x - x_0). \]

Suppose that \( n \geq 6 \), and that \( \tau, \omega \neq 0, \tau, \omega \) have compact support and let \( |x_0| \) be large enough. Then the Yamabe problem (7) has at least two distinct solutions provided \( \varepsilon \) is small enough.

For more general results, see Theorems 6.3 and 6.5.

The paper is organized as follows. Section 2 contains some preliminary calculation that lead to the explicit form of \( f, G_1, G_2 \). In Section 3 we outline the abstract setting adapted to our situation. Section 4 is devoted to show how this abstract set up applies to the Yamabe problem. In Section 5 the proof of Theorem 1.1 is given and in Section 6 some improvements are discussed. Since most of the arguments rely on heavy, but straight, calculation we have postponed in an Appendix some of them. Even so, it would have been too long to insert a detailed proof of our existence results in the general case. For this reason the proof of Theorem 1.1 is carried out in some details for a specific choice of \( a \). The general case does not require new ideas and hence is sketched, only.

**Notation**

We will work mainly in the functional space \( E = D^{1,2}(\mathbb{R}^n) \), which is the completion of \( C_0^\infty(\mathbb{R}^n) \) with respect to the Dirichlet norm \( ||u|| = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \). \( (u, v) \) is the standard scalar product \( \int_{\mathbb{R}^n} \langle Vu, \nabla v \rangle \, dx \), for \( u, v \in E \).

If \( E \) is an Hilbert space and \( f \in C^2(\mathbb{R}^n) \) is a functional, we denote by \( f' \) or \( \nabla f \) its gradient; \( f''(u): E \to E \) is the linear operator defined by duality in the following way

\[ (f''(u) \, v, w) = D^2(f(u)) \, [v, w], \quad \forall v, w \in E. \]

If \( u \in \text{crit}(f) \), we denote by \( m(f, u) \) the Morse index of \( u \).
Given \( u: \mathbb{R}^n \to \mathbb{R} \), and given \( \mu \in \mathbb{R}, \ \xi \in \mathbb{R}^n \), we set \( u_{\mu, \xi} = \mu^{-(n-2)/2}(x - \xi)/\mu \); it is worth noticing that
\[
\|u_{\mu, \xi}\|_{2^*} = \|u\|_{2^*}, \quad \|u_{\mu, \xi}\|_E = \|u\|_E, \quad \text{for all } u \in E.
\]

\( \sigma \) denotes the stereographic projection \( \sigma: S^n = \{ x \in \mathbb{R}^{n+1} | |x| = 1 \} \to \mathbb{R}^n \) through the north pole, where we identify \( \mathbb{R}^n \) with \( \{ x \in \mathbb{R}^{n+1} | x_{n+1} = 0 \} \).

For \( i, j \) integers we set
\[
I_j = \int_{\mathbb{R}^n} \frac{|x|^j}{(1 + |x|^2)^{j/2}},
\]
whenever this integral is defined.

Moreover, for every homogeneous monomial \( P(x) \) and for every integer \( j \) we set
\[
I_j(P(x)) = \int_{\mathbb{R}^n} \frac{P(x)}{(1 + |x|^2)^{j/2}},
\]
whenever this integral makes sense.

2. PRELIMINARY RESULTS

In this section we will prove some lemmas which will enable us to write \( f \) in a form suitable for using the perturbation method sketched before. Let \( R_\varepsilon \) denote the scalar curvature of \( \tilde{g} \) given by (5).

**Lemma 2.1.** We have
\[
R_\varepsilon(x) = \varepsilon R_1(x) + \varepsilon^2 R_2(x) + o(\varepsilon^2),
\]
where
\[
R_1 = \sum_{k, j} D^2_{ij} h_{ij} - \text{trace } h;
\]
and
\[
R_2 = -2 \sum_{k, j, l} h_{ij} D^2_{kl} h_{kl} + \sum_{k, j, l} h_{ij} D^2_{kl} h_{ij} + \sum_{k, j, l} h_{ij} D^2_{kl} h_{kl}
+ \frac{3}{4} \sum_{k, j, l} D_j h_{ij} D_k h_{kj} - \sum_{k, j, l} D_j h_{ij} D_k h_{jk} + \sum_{k, j, l} D_j h_{ij} D_j h_{kk}
- \frac{1}{4} \sum_{k, j, l} D_j h_{ij} D_j h_{kk} - \frac{1}{2} D_j h_{ij} D_j h_{jk}.
\]
Proof. First of all we need to express the matrix \( \tilde{g}_{ij} = (\tilde{g})_{ij}^{-1} \) as an expansion in terms of \( \varepsilon \): if we write \( \tilde{g}^{-1} = I + \varepsilon A + \varepsilon^2 B \), from the relation

\[
(I + \varepsilon h)(I + \varepsilon A + \varepsilon^2 B) = I + o(\varepsilon^2),
\]

we obtain immediately

\[
\tilde{g}^{ij} = \delta_{ij} - \varepsilon h_{ij} + \varepsilon^2 \sum_s h_{is} h_{sj} + o(\varepsilon^2). \tag{9}
\]

The Christoffel symbols are given by

\[
\Gamma^i_{jk} = \frac{1}{2} \left[ D_j h_{ik} + D_k h_{ij} - D_i h_{jk} \right] \tilde{g}^{kl},
\]

so, using (5) and (9) we obtain

\[
\Gamma^i_{jk} = \frac{\varepsilon}{2} \sum_s \left( \delta_{is} - \varepsilon h_{is} + \varepsilon^2 \sum_s h_{is} h_{sl} \right) \left( D_j h_{ik} + D_k h_{ij} - D_i h_{jk} \right)
= \varepsilon \left( D_j h_{ik} + D_k h_{ij} - D_i h_{jk} \right)
- \frac{\varepsilon^2}{2} \sum_s h_{is} \left( D_j h_{ks} + D_k h_{js} - D_s h_{kj} \right) + o(\varepsilon^2). \tag{10}
\]

The components of the Riemann curvature tensor are

\[
R^n_{ijk} = D_i \Gamma^l_{jk} - D_j \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik}.
\]

Hence, using (10) we obtain

\[
R^n_{ijk} = \frac{\varepsilon}{2} \left( D^2_{jk} h_{li} + D^2_{il} h_{jk} - D^2_{jl} h_{ik} - D^2_{ik} h_{jl} \right)
+ \varepsilon^2 \left[ \frac{1}{2} \sum_s D_j h_{is} (D_i h_{js} + D_s h_{js} - D_j h_{is}) 
- \frac{1}{2} \sum_s D_i h_{is} (D_j h_{is} + D_s h_{ij} - D_j h_{is}) 
+ \frac{1}{2} \sum_s h_{is} (D^2_{jk} h_{is} - D^2_{ij} h_{ik}) 
- \frac{1}{2} \sum_s h_{is} (D^2_{il} h_{js} - D^2_{il} h_{js}) 
+ \frac{1}{4} \sum_m (D_i h_{lm} + D_m h_{li} - D_j h_{lm})(D_j h_{mk} + D_k h_{mj} - D_m h_{jk}) 
- \frac{1}{4} \sum_m (D_j h_{lm} + D_m h_{lj} - D_i h_{lm})(D_i h_{mk} + D_k h_{mi} - D_m h_{ik}) \right]
+ o(\varepsilon^2). \tag{11}
\]
The components of the Ricci tensor are given by
\[ R_{ij} = R^l_{kij}, \]
and then using (11) we get
\[ R_{ij} = R^l_{kij} = 2 \sum_l \left[ \frac{1}{2} D_l h_{ml} (D_l h_{mk} - D_m h_{lk}) ight] 
+ \frac{1}{2} D_l h_{ml} (D_l h_{mk} - D_m h_{lk}) 
- \frac{1}{2} D_l h_{ml} (D_l h_{mk} - D_m h_{lk}) 
+ \frac{1}{4} h_{ml} (D_l h_{mk} - D_m h_{lk}) 
- \frac{1}{4} (D_l h_{lm} + D_l h_{ml}) (D_l h_{mk} - D_m h_{lk}) 
+ o(\varepsilon^2). \] (12)

In conclusion, since the scalar curvature is given by
\[ R = R_{ij} \hat{\delta}^i_j, \]
then, using (12) we recover the Lemma.

**Lemma 2.2.** There results
\[ f_\varepsilon(u) = f_0(u) + \varepsilon G_1(u) + \varepsilon^2 G_2(u) + o(\varepsilon^2), \]
where
\[ f_0(u) = \int \left( c_n |\nabla u|^2 - \frac{1}{2\varepsilon} |u|^{2*} \right) dx; \quad u \in E, \] (13)
\[ G_1(u) = \int \left( - c_n \sum_{i,j} h_{ij} D_i u D_j u + \frac{1}{2} R_i u^2 
+ \left( c_n |\nabla u|^2 - \frac{1}{2\varepsilon} |u|^{2*} \right) \frac{1}{2} \text{tr} \ h \right) dx, \] (14)
\[ G_2(u) = \int \left[ c_n \sum_{i,j} h_{ij} D_i u D_j u + \frac{1}{2} R_2 u^2 + \left( c_n |\nabla u|^2 - \frac{1}{2} |u|^2^* \right) \right] \times \left( \frac{1}{8} (\text{tr} h)^2 - \frac{1}{4} \text{tr}(h^2) \right) \]

+ \frac{1}{2} \text{tr} \left( \frac{1}{2} R_1 u^2 - c_n \sum_{i,j} h_{ij} D_i u D_j u \right) dx. \quad (15)

**Proof.** First we expand \(|\nabla u|^2|\) in terms of \(\varepsilon\). There results

\[ |\nabla u|^2 = \sum_{i,j} \tilde{g}_{ij} D_i u D_j u. \]

Thus, using (9) we obtain

\[ |\nabla u|^2 = |\nabla u|^2 - \varepsilon \sum_{i,j} h_{ij} D_i u D_j u + \varepsilon^2 \sum_{i,j,l} h_{ij} h_{lj} D_i u D_j u D_l u + o(\varepsilon^2). \quad (16) \]

In order to evaluate \(dV_{\tilde{g}} = |\tilde{g}|^{1/2} \) dx, let us expand \(|\tilde{g}|\) in power series of \(\varepsilon\).

Consider the determinant of the matrix

\[
\begin{pmatrix}
1 + \varepsilon h_{11} & \varepsilon h_{12} & \cdots \\
\varepsilon h_{21} & 1 + \varepsilon h_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

Its linear part in \(\varepsilon\) is \(\text{tr} \, h\), while its quadratic part is \(\frac{1}{2} (\varepsilon \sum_{i,j} h_{ij}^2 - \sum_{i,j} h_{ij}^2)\), which coincides with \(\frac{1}{2} (\text{tr}(h)^2 - \text{tr}(h^2))\). Then we obtain

\[ |\tilde{g}|^{1/2} = 1 + \frac{\varepsilon}{2} \text{tr} \, h + \varepsilon^2 \left( \frac{1}{8} (\text{tr} h)^2 - \frac{1}{4} \text{tr}(h^2) \right) + o(\varepsilon^2). \quad (17) \]

Now, using (16) and (17), we can write

\[
\begin{align*}
f_{\varepsilon}(u) &= \int \left( c_n \left( |\nabla u|^2 - \varepsilon \sum_{i,j} h_{ij} D_i u D_j u + \varepsilon^2 \sum_{i,j,l} h_{ij} h_{lj} D_i u D_j u D_l u \right) \right) \\
&\quad + \frac{1}{2} (\varepsilon R_1 + \varepsilon^2 R_2) u^2 - \frac{1}{2} |u|^2^* \times \left( 1 + \frac{\varepsilon}{2} \text{tr} \, h + \varepsilon^2 \left( \frac{1}{8} (\text{tr} h)^2 - \frac{1}{4} \text{tr}(h^2) \right) \right) dx + o(\varepsilon^2).
\end{align*}
\]

Factorizing with respect to \(\varepsilon\) and \(\varepsilon^2\) the conclusion follows. \(\blacksquare\)
3. THE ABSTRACT SETTING

In this section we recall the abstract perturbation method developed in [3, 4, 6]. The specific form of the abstract set up is motivated by the calculation in the preceding section. We want to find critical points of a functional of the form

$$f_\epsilon(u) = f_0(u) + \epsilon G_1(u) + \epsilon^2 G_2(u) + o(\epsilon^2), \quad u \in E.$$  (18)

It is always understood that $E$ is a Hilbert space and $f_\epsilon(u)$, $f_0$, $G_1$, $G_2 \in C^2(E, \mathbb{R})$. The fundamental tool is the following theorem (see [4, Lemmas 2 and 4]).

Let $B_R = \{u \in E | \|u\| \leq R\}$.

**Theorem 3.1.** Suppose $f_0$ satisfies

1. $f_0$ has a finite dimensional manifold of critical points $Z$; let $b = f_0(z)$, for all $z \in Z$;
2. $f_0(z)$ is a Fredholm operator of index zero for all $z \in Z$;
3. for all $z \in Z$ it is $T_zZ = \text{Ker} f_0'(z)$.

Then, given $R > 0$, there exist $\epsilon_0$ and a smooth function

$$w = w(z, \epsilon) : M = Z \cap B_R \times (-\epsilon_0, \epsilon_0) \to E$$

such that

1. $w(z, 0) = 0$ for all $z \in Z \cap B_R$;
2. $w(z, \epsilon)$ is orthogonal to $T_zZ$, $\forall (z, \epsilon) \in M$;
3. the manifold $Z_\epsilon = \{z + w(z, \epsilon) : (z, \epsilon) \in M\}$

is a natural constrain for $f_\epsilon'$, namely, if $u \in Z_\epsilon$ and $f'_\epsilon|_{Z_\epsilon}(u) = 0$, then $f'_\epsilon(u) = 0$.

For future reference let us recall that $w$ satisfies (ii) above and $f'_\epsilon(z + w) \in T_zZ$, namely $f'_0(z)[w] + \epsilon G_1(z) + o(\epsilon) \in T_zZ$. As a consequence, if $G_1(z) \perp T_zZ$, one finds

$$w(\epsilon, z) = -\epsilon L_zG_1'(z) + o(\epsilon),$$  (19)

where $L_z$ denotes the inverse of the restriction to $(T_zZ)^+$ of $f_0'(z)$.

In our applications, $G_1 \equiv 0$ on $Z$. This motivates the following Lemma.
Lemma 3.2. Suppose that

\[ G_1(z) = 0, \quad \forall z \in Z \]  

and let \( \Gamma : Z \to \mathbb{R} \) be defined by setting

\[ \Gamma(z) = G_2(z) - \frac{1}{2}(L_z G_1(z), G_1(z)). \]  

Then we have

\[ f_{\varepsilon}(z + w(\varepsilon, z)) = b + \varepsilon^2 \Gamma(z) + o(\varepsilon^2). \]

Proof. Since \( G_1|_Z \equiv 0 \), then \( G_1'(z) \in (T_z Z)^\perp \) so (19) holds. Then one finds

\[
\begin{align*}
  f_{\varepsilon}(z + w(\varepsilon, z)) &= f_0(z + w(\varepsilon, z)) + \varepsilon G_1(z + w(\varepsilon, z)) + \varepsilon^2 G_2(z + w(\varepsilon, z)) + o(\varepsilon^2) \\
  &= f_0(z) + \varepsilon f_0'(z)[w, w] + \varepsilon G_1(z) + \varepsilon G_1'(z)[w] + \varepsilon^2 G_2(z) + o(\varepsilon^2).
\end{align*}
\]

Using (20) and (19) the lemma follows.  

We are now in position to state the abstract result that we will use to find the critical points of \( f_{\varepsilon} \).

Theorem 3.3. Assume that we are in the hypotheses of Theorem 3.1, and Lemma 3.2 and that there exist a set \( A \subseteq Z \) with compact closure and \( z_0 \in A \) such that

\[
\Gamma(z_0) < \inf_{z \in \partial A} \Gamma(z) \quad (\text{resp. } \Gamma(z_0) > \sup_{z \in \partial A} \Gamma(z)).
\]

Then, for \( \varepsilon \) small enough, \( f_{\varepsilon} \) has at least a critical point \( u_{\varepsilon} \in Z_{\varepsilon} \) such that

\[
\begin{align*}
  b + \varepsilon^2 \inf_{A} \Gamma + o(\varepsilon^2) &\leq f_{\varepsilon}(u_{\varepsilon}) \leq b + \varepsilon^2 \sup_{A} \Gamma + o(\varepsilon^2) \\
  (\text{resp. } b + \varepsilon^2 \inf_{\partial A} \Gamma + o(\varepsilon^2) &\leq f_{\varepsilon}(u_{\varepsilon}) \leq b + \varepsilon^2 \sup_{A} \Gamma + o(\varepsilon^2)).
\end{align*}
\]

Furthermore, up to a subsequence, there exists \( \tilde{z} \in A \) such that \( u_{\varepsilon_n} \to \tilde{z} \) in \( E \) as \( \varepsilon_n \to 0 \).
4. APPLICATION TO THE YAMABE PROBLEM

In this section we apply Theorem 3.3 to find multiple solutions of the Yamabe problem. In Section 2 we have shown that $f_0$ has the form (18) with $f_0, G_1, G_2$ given in Lemma 2.2. Let

$$z_{\mu, \xi}(x) = \mu^{-(n-2)/2} z_0\left(\frac{x - \xi}{\mu}\right), \quad \mu > 0, \quad \xi \in \mathbb{R}^n,$$

and

$$Z = \{z_{\mu, \xi} | \mu > 0, \xi \in \mathbb{R}^n\}.$$ 

$Z$ is an $n + 1$ dimensional manifold homeomorphic to $\mathbb{R}^{n+1} = \{x \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$ and every $z \in Z$ is a critical point of $f_0$. In particular, hypothesis (f1) in Theorem 3.1 is satisfied with $b = f_0(w_0)$. Assumption (f2) holds too, since $f_0'(z) = I - C, C$ compact for every $z \in Z$, and (f3) follows from the following lemma (see [6]).

**Lemma 4.1.** $T_{z_{\mu, \xi}} Z = \text{Ker} f_0'(z_{\mu, \xi})$, namely if $u \in E$ solves

$$-2c_n Au = \frac{n + 2}{n - 2} z_{\mu, \xi}^{-2} u,$$

then

$$u = \alpha D_{\mu} z_{\mu, \xi} + \langle \nabla_{z_{\mu, \xi}} \beta \rangle, \quad \text{for some} \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}^N.$$

Furthermore one has

**Lemma 4.2.** There results

$$G_1(z) = 0, \quad z \in Z.$$

**Proof.** From (2) we deduce

$$D_i z_{\mu, \xi} = (2 - n) \mu^{-(n-2)/2} \frac{\kappa_n}{(1 + |(x - \xi)/\mu|^2)^{n/2}} (x_i - \xi),$$

and

$$D_{ij} z_{\mu, \xi} = (2 - n) \mu^{-(n-2)/2} \frac{\kappa_n \delta_{ij}}{(1 + |(x - \xi)/\mu|^2)^{n/2}}$$

$$+ n(n - 2) \mu^{-(n-2)/2} \frac{\kappa_n (x_i - \xi)(x_j - \xi)}{(1 + |(x - \xi)/\mu|^2)^{n/2}}, \quad \text{for some} \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}^N.$$

Furthermore, one has

$$G_1(z) = 0, \quad z \in Z.$$
Therefore
\[
z_{\mu, \xi} D_{\eta} z_{\mu, \xi} = (2 - n) \mu^{-n} \frac{\kappa_n^2 \delta_{\eta}}{(1 + |(x - \xi)/\mu|^2)^{n-1}} \\
+ \frac{n}{n-2} D_{\mu, \xi} D_{\eta} z_{\mu, \xi}.
\] (24)

Using (8) and integrating by parts, we obtain
\[
\int R_1(x) z_{\mu, \xi}^2(x) \, dx = \int \sum_{i,j} h_i(x) (2 D_i z_{\mu, \xi} D_j z_{\mu, \xi} + 2 z_{\mu, \xi} D_{\mu, \xi} D_{\eta} z_{\mu, \xi}) \, dx
\]
\[
+ \int \text{tr} \, h(x) (2 z_{\mu, \xi} D_{\mu, \xi} + 2 |\nabla z_{\mu, \xi}|^2) \, dx.
\]

From the fact that \( z_0 \) solves \((P)\), and from (24) we deduce the equality
\[
\int R_1(x) z_{\mu, \xi}^2(x) \, dx = \int \sum_{i,j} h_i(x) \left( 2 \left( 1 + \frac{n}{n-2} \right) D_{\mu, \xi} D_{\eta} z_{\mu, \xi} \right.
\]
\[
+ \frac{2(2 - n) \mu^{-n} \kappa_n^2 \delta_{\eta}}{(1 + |(x - \xi)/\mu|^2)^{n-1}} \right) \, dx
\]
\[
+ \int \text{tr} \, h(x) \left( \frac{1}{\kappa_n} z_{\mu, \xi}^2 - 2 |\nabla z_{\mu, \xi}|^2 \right) \, dx,
\]

which inserted in (14) yields
\[
G_1(z_{\mu, \xi}) = \frac{1}{2} \int h \left( \frac{2}{n-2} |\nabla z_{\mu, \xi}|^2 + \frac{n-2}{2n(n-1)} |z_{\mu, \xi}|^{2^*} \right.
\]
\[
+ \frac{2(2 - n) \kappa_n^2 \mu^{-n}}{(1 + |(x - \xi)/\mu|^2)^{n-1}} \right) \, dx
\]
\[
= \frac{1}{2} \int h \left( \frac{n-2}{2n(n-1)} \kappa_n^2 \mu^{-n} \right.
\]
\[
\times \left( 2 \left( \frac{|x - \xi|^2}{\mu} + \frac{4n(n-1)}{2n(n-1)} + \frac{2(2 - n) \mu^{-n}}{(1 + |(x - \xi)/\mu|^2)^{n-1}} \right) \right) \, dx = 0.
\]

This completes the proof. \( \square \)

In order to find critical points of \( F \) it is convenient to study the behaviour of \( F \) as \( \mu \to 0 \) and as \( \mu + |\xi| \to \infty \). In the sequel we set \( F(\mu, \xi) = F(z_{\mu, \xi}) \), etc. Our goal will be to show
Proposition 4.3. \( \Gamma \) can be extended smoothly to the hyperplane \( \{ (\mu, \zeta) | \mu = 0 \} \) by setting
\[
\Gamma(0, \zeta) = 0.
\] (25)

Moreover there results
\[
\Gamma(\mu, \zeta) \to 0, \quad \text{as} \quad \mu + |\zeta| \to +\infty.
\] (26)

Recall that \( \Gamma = G_2(z) + \frac{1}{2} (G_1(z), \bar{w}) \), where \( z \) stands for \( z_{\mu, \zeta} \) and \( \bar{w} = \lim_{\epsilon \to 0} e^{-\epsilon w} \). First of all we handle \( G_2 \).

Lemma 4.4. There holds
\[
\lim_{\mu \to 0} G_2(\mu, \zeta) = \kappa_2 \frac{n(n-1)(n-2)}{n} \left( \text{tr}(h^2) - \frac{1}{2} (\text{tr} h)^2 \right)^2.
\] (27)

Proof. Using the change of variables \( x = \mu y + \zeta \), we can write
\[
G_2(\mu, \zeta) = \int c_n \sum_{i, j} h_{ij}(\mu y + \zeta) D_i z_0(y) D_j z_0(y) + \mu^2 \frac{1}{2} R_2(\mu y + \zeta) z_0^2(y) + \left( c_n |\nabla z_0(y)|^2 - \frac{1}{2} |z_0(y)|^2 \right)
\times \left( \frac{1}{8} (\text{tr} h^2) - \frac{1}{4} (\text{tr} h^2) \right) (\mu y + \zeta) + \frac{1}{2} \text{tr} h(\mu y + \zeta)
\times \frac{1}{2} \mu^2 R_2(\mu y + \zeta) z_0^2(y)
- c_n \sum_{i, j} h_{ij}(\mu y + \zeta) D_i z_0(y) D_j z_0(y)) \right) dy.
\]

Passing to the limit for \( \mu \to 0 \) we obtain
\[
G_2(0, \zeta) = \frac{c_n}{n} \frac{\text{tr}(h^2)(\zeta)}{\int |\nabla z_0(y)|^2 dy} + \frac{1}{8} (\text{tr} h^2) - \frac{1}{4} (\text{tr} h^2) \left( \zeta \right)
\times \frac{1}{2} \left( c_n |\nabla z_0(y)|^2 - \frac{1}{2} |z_0(y)|^2 \right)
- \frac{1}{2} \frac{c_n}{n} (\text{tr} h^2) (\zeta) \left( |\nabla z_0(y)|^2 \right) dy.
\]
Using the expression of $z_0$ we can write
\[
G_2(0, \xi) = \frac{2}{n} \kappa_n^2 \text{tr}(h^2)(\xi) I_n^2 + \left(\frac{1}{8} \text{tr}(h)^2 - \frac{1}{4} \text{tr}(h^2)\right) 2(n-1)(n-2) \kappa_n^2 (I_n^2 - I_n) - \frac{(n-1)(n-2)}{n} \kappa_n^2 \text{tr}(h)^2 (\xi) I_n^2.
\]

Using (43) we conclude
\[
G_2(0, \xi) = (n-1)(n-2) \kappa_n^2 \times \left(\frac{2}{n} \text{tr}(h^2) + \frac{1}{n} \left(\frac{1}{2} \text{tr}(h)^2 - \text{tr}(h^2)\right) - \frac{1}{n} \text{tr}(h)^2\right) I_n^2,
\]
and the lemma follows.

As for the second term in $\Gamma$ we have
\[
(G_1'(z), \bar{w}) = \alpha_1 + \alpha_2,
\]
where
\[
\alpha_1 = \int \frac{1}{2} \text{tr}(2c_n \langle \nabla z, \nabla \bar{w} \rangle - |z|^{2^*_n - 1} \bar{w}) \, dx
\]
\[
\alpha_2 = \int \left(-2c_n \sum_y h_y D_z z_0 D_y \bar{w} + R_1 z_0 \bar{w}\right) \, dx.
\]

It is convenient to introduce $w^*(y) = w^*_n \zeta(y)$ by setting
\[
w^*(y) = \mu^{n^2-1} \bar{w}(\mu y + \zeta).
\]

Then, a change of variable yields
\[
\alpha_1 = \int \frac{1}{2} \text{tr}(\mu y + \zeta)(2c_n \langle \nabla z_0(y), \nabla w^*(y) \rangle - |z_0(y)|^{2^*_n} w^*(y)) \, dy
\]
\[
\alpha_2 = \int \left(-2c_n \sum_y h_y(\mu y + \zeta) D_z z_0(y) D_y w^*(y)\right) \, dy + \mu^2 \int R_1(\mu y + \zeta) z_0(y) w^*(y) \, dy.
\]
The following formulas (A1)–(A2) are proved in the Appendix

\[ w_{\mu, \zeta}(y) \to w_0(y) \quad \text{as} \quad \mu \to 0, \]  

(A1)

where, setting \( c_n = c_n \kappa_n ((n-2)^2/4(n-1)), \)

\[ w_0(y) = w_{0, \zeta}(y) = -\frac{c_n}{(1 + |y|^2)^{n/2}} \sum h_{jk}(\zeta) y_j y_k, \]  

(29)

\[ \sum h_{jk} \int D_i z_0 D_j w_0 = \frac{\kappa^2(n-2)^2}{2\pi} \left( \frac{1}{2} \left( \text{tr}(h^2) - \frac{1}{2} \left( \text{tr} h \right)^2 \right) \right) I_n^2. \]  

(A2)

We are now in position to prove (25):

**Proof of (25).** From (28) and (A1) we infer

\[ \lim_{\mu \to 0} \int (2\epsilon_n \langle \nabla z_0, \nabla w^* \rangle - |z_0|^{2^* - 1} w^*) \, dy = 0, \]

because \( z_0 \) satisfies \((P_0)\).

As for \( z_2 \) one finds

\[ \lim_{\mu \to 0} \sum h_{jk}(\zeta) \int D_i z_0 D_j w^* \, dy. \]

Using (A2) it follows that

\[ \lim_{\mu \to 0} \sum h_{jk}(\zeta) \int D_i z_0 D_j w^* \, dy. \]

This, together with (27) implies \( I(0, \xi) = 0. \)

**Proof of Proposition 4.3 Completed.** Let

\[ \hat{g}_q(x) = \hat{g}(x) \left( \frac{x}{|x|^2} \right), \]

and consider the corresponding functional \( \hat{f} \), obtained by substituting in (4) \( \hat{g} \) with \( \hat{g} \). Similarly, let us consider \( \hat{G}_i(\mathbf{u}), i = 1, 2 \), etc. Letting

\[ u^*(x) = |x|^{2 - \alpha} u \left( \frac{x}{|x|^2} \right) \]
it is easy to check that one has
\[ f_q(u) = \hat{f}_q(u^*), \quad G_q(u) = \hat{G}_q(u^*). \]

Moreover, one also has
\[ \Gamma(z) = \hat{\Gamma}(z^*). \]

This in terms of coordinates \((\mu, \zeta)\) becomes
\[ \Gamma(\mu, \zeta) = \hat{\Gamma}\left( \frac{\mu}{\mu^2 + |\zeta|^2}, \frac{\zeta}{\mu^2 + |\zeta|^2} \right). \]

Finally one finds
\[ \lim_{\mu + |\zeta| \to \infty} \Gamma(\mu, \zeta) = \hat{\Gamma}(0, 0) = 0, \]
proving (26).

In the next Proposition we take \( \hat{g} \) of the form (5)-(6).

**Proposition 4.5.** For \( n > 6 \), and for \( \hat{g} \) of the form (5)-(6) we have
\[ \frac{\partial \Gamma}{\partial \mu}(0, \zeta) = 0, \quad \frac{\partial^2 \Gamma}{\partial \mu^2}(0, \zeta) = 0, \quad \frac{\partial^3 \Gamma}{\partial \mu^3}(0, \zeta) = 0, \quad \forall \zeta \in \mathbb{R}^n; \quad (30) \]
\[ \frac{\partial^4 \Gamma}{\partial \mu^4}(0, \zeta) = - \sum_{i, j > 1} k_{ij} |D^2_{ij}\partial a(\zeta)|^2, \quad \forall \zeta \in \mathbb{R}^n; \quad (31) \]
where \( k_{ij} > 0 \).

The proof of this proposition will be sketched in the Appendix.

**Remark 4.6.** If \( n = 6 \) it turns out that
\[ \lim_{\mu \to 0} \frac{\Gamma(\mu, \zeta)}{\mu^{3/2}} = -\infty \]
for all \( \zeta \) such that \( |D^2_{ij}\partial a(\zeta)|^2 \neq 0 \), for some \( i, j > 1 \).

From Propositions 4.3, 4.5, and Theorem 3.3 we can immediately deduce

**Corollary 4.7.** Suppose that \( \hat{g} \) has the form (5)-(6), \( n \geq 6 \) and that \( a \) is not constant. Then \( \Gamma \) achieves a minimum and hence the Yamabe problem (3) has a positive solution.
For a relationship between this and the Aubin result, see Remark 6.2 later on.

5. PROOF OF THEOREM 1.1

We first prove some auxiliary lemmas

**Lemma 5.1.** There holds

$L_z$ is uniformly bounded for $z \in Z. \quad (UB)$

**Proof.** Since $f_0'(z_0)$ is invertible on $(T_z Z)^\perp$, there exists $c_0$ such that for every $u \in (T_z Z)^\perp$, there exist $v(u) \in (T_z Z)^\perp$ with $\|v(u)\| = 1$ and

$$(f_0'(z_0) u, v(u)) \geq c_0 \|u\|.$$ 

If $u \in (T_z Z)^\perp$, then $u_{\mu, \xi} \in (T_{z_{\mu, \xi}} Z)^\perp$; by using the change of variables $x = \mu y + \xi$, we deduce

$$(f_0'(z_{\mu, \xi}) u_{\mu, \xi}, v_{\mu, \xi}) = (f_0'(z_0) u, v), \quad \forall u, v \in (T_z Z)^\perp.$$ 

Let us set $(\mu', \xi') = (\mu^{-1}, -\xi/\mu)$; then one can easily see that

$$(u_{\mu', \xi'})_{\mu, \xi} = u, \quad \forall u \in E^1.$$ 

Now take $u \in (T_{z_{\mu, \xi}} Z)^\perp$. From the last equalities we have

$$(f_0'(z_{\mu, \xi}) u, (v(u_{\mu', \xi'}))_{\mu, \xi}) = (f_0'(z_0) u_{\mu', \xi'}, v(u_{\mu', \xi'}))$$

$$\geq c_0 \|u_{\mu', \xi'}\| = c_0 \|u\|,$$

where the inequality holds by the definition of $v(u_{\mu', \xi'})$. The last formula implies that $L_z$ is uniformly bounded for $z \in Z.$

In the sequel we shall consider two smooth functions $\tau$ and $\omega$ satisfying the assumptions of Theorem 1.1. Let $G_1^\tau$, $G_\omega^\tau$, etc., be the corresponding functionals.

**Lemma 5.2.** We have

$$\|\nabla G_1^\tau(z)\| \to 0 \quad \text{as} \quad \mu \to +\infty \text{ uniformly in } \xi. \quad (32)$$
Proof. We denote by $K_1$ the support of $\tau_1(\cdot)$. By (14) there holds

$$
\left| (\nabla G_1^i(z), v) \right| = \left| -2c_n \int_{K_i} \tau \, D_1 z \, D_1 v + \int_{K_i} R_1^i \, z v \right |
$$

$$
+ \int \left( 2\mu \langle \nabla z, \nabla v \rangle - |z|^{\alpha-1} v \right) \right|
$$

$$
\leq 2c_n \| \tau \|_\infty \| \nabla z \|_\infty \int_{K_i} |D_1 v| + \| R_1^i \|_\infty \| z \|_\infty \int_{K_i} |v|
$$

$$
+ \frac{1}{2} \| \tau \|_\infty \left( 2c_n \| \nabla z \|_\infty \int_{K_i} |D_1 v| + \| z \|^{\alpha-1} \int_{K_i} |v| \right).
$$

Using the Hölder and the Sobolev inequalities we obtain for some $C_1 > 0$

$$
\left| (\nabla G_1^i(z), v) \right| \leq C_1 \left( \| \nabla z \|_\infty + \| z \|_\infty + \| z \|_\infty^{\alpha-1} \| v \|_E. \right.
$$

Since $|\nabla z|_\infty$, $||z||_\infty \rightarrow 0$ when $\mu \rightarrow +\infty$, we obtain $\|\nabla G_1^i(z)\| \rightarrow 0$. 

Now we want to consider the metric of the form (5)–(6) with $a(x) = \tau(x) + \omega(x - x_0)$. It is easy to check that there results, for $i = 1, 2$,

$$
G^{i = \omega}(z, \xi) = G^{\omega}(z, \xi - x_0).
$$

As a consequence one finds

$$
\Gamma^{i = \omega}(\mu, \xi) = \Gamma^{\omega}(\mu, \xi - x_0).
$$

Furthermore, if $|x_0|$ is large enough so that the supports of $\tau$ and $\omega(\cdot - x_0)$ are disjoint, one also has

$$
G^{i = \omega}(z, \xi) = G^{\omega}(z, \xi) + G^{\omega}(z, \xi - x_0).
$$

Similar results hold for $\nabla G_i$. In order to find a similar expression for $\Gamma$, the following lemma is in order

**Lemma 5.3.** Given $M > 0$, there results

$$
\| \nabla G_1^i(z) \| + \| \nabla G_1^i(\cdot - x_0) \| \rightarrow 0, \quad \text{as} \quad |x_0| \rightarrow \infty,
$$

uniformly in $(\mu, \xi), \mu \leq M$. 

Proof. We have the estimate
\[
|(\nabla G^1_1(z), v)| \leq 2c_n \|\tau\| \int_{K_1} |\nabla z| |\nabla v| + \|R^1_1\| \int_{K_1} |v| |z|
\]
\[+ \|\tau\| \int_{K_1} |\nabla z| |\nabla v| + \frac{1}{2} \|\tau\| \int_{K_1} |z|^{2^* - 1} |v|.
\]
Using again the Hölder and the Sobolev inequalities, we obtain
\[
|(\nabla G^1_1(z), v)| \leq C_2 \|z\|_E \|v\|_E
\]
for some \(C_2 > 0\), so it is sufficient to show that
\[
\min\{\|\nabla G^1_1(z)\|, \|\nabla G^1_1(z)\|\} \to 0 \quad \text{as} \quad |x_0| \to \infty,
\]
uniformly in \((\mu, \zeta), \mu \leq M\). Looking at the expression of \(z_{\mu, \zeta}\) we deduce that for every \(\eta > 0\) there exists \(R > 0\) such that
\[
|\nabla z_{\mu, \zeta}(x)|, |z_{\mu, \zeta}(x)| \leq \eta \quad \text{for} \quad |x| \geq R, \quad \mu \leq M.
\]
Using the change of variables \(y = x - \zeta\), we have
\[
\nabla G^1_1(z) = -2c_n \int_{K_1 - \zeta} \tau(y + \zeta) D_1 z_{\mu, \zeta}(y) D_1 v(y + \zeta)
\]
\[+ \int_{K_1 - \zeta} R^1_1(y + \zeta) z_{\mu, \zeta}(y) v(y + \zeta) dy
\]
\[+ \frac{1}{2} \int_{K_1 - \zeta} \tau(y + \zeta)(2c_n \langle \nabla z_{\mu, \zeta}(y), \nabla v(y) \rangle
\]
\[+ |z_{\mu, \zeta}(y)|^{2^* - 2} z_{\mu, \zeta}(y) v(y)) dy.
\]
If \(\text{dist}(\xi, K_1) \geq R\), and if \(\mu \leq M\), then, using (38), the Hölder and the Sobolev inequalities we get
\[
|(\nabla G^1_1(z), v)| \leq C_3 (\eta + \eta^{2^* - 1}) \|v\|_E
\]
for some \(C_3 > 0\). Since the above estimate is uniform in \(v\), it is
\[
\|\nabla G^1_1(z)\| \leq C_3 (\eta + \eta^{2^* - 1}), \quad \text{for} \quad \text{dist}(\zeta, K_1) \geq R, \quad \mu \leq M.
\]
Similarly, if \(K_2\) denotes the support of \(\omega\) we obtain
\[
\|\nabla G^1_1(\cdot - x_0)(z)\| \leq C_3 (\eta + \eta^{2^* - 1}), \quad \text{for} \quad \text{dist}(\zeta - x_0, K_2) \geq R, \quad \mu \leq M.
\]
When $|x_0|$ is large enough, it is always \( \text{dist}(\xi, K_1) \geq R \) or \( \text{dist}(\xi - x_0, K_2) \geq R \), so

\[
\min \{ \| \nabla G_1^*(z) \|, \| \nabla G_1^{\omega \cdot - \omega}(z) \| \} \leq C_3 (\eta + \eta^{2\omega - 1}).
\]

By the arbitrary of \( \eta \), (37) follows.

Putting together (UB), Lemma 5.2, and Lemma 5.3 we infer

\[
(L_z \nabla G_1^*(z), \nabla G_1^{\omega \cdot - \omega}(z)) \to 0 \quad \text{as} \quad |x_0| \to \infty,
\]

uniformly for \( z \in Z \).

Finally, from (33), (35), and (39) we infer:

**Lemma 5.4. There results**

\[
\Gamma^{* + \omega \cdot - \omega}(\mu, \xi) = \Gamma^*(\mu, \xi) + \Gamma^{\omega}(\mu, \xi - x_0) + o(1),
\]

where \( o(1) \to 0 \) as \( |x_0| \to \infty \), uniformly in \( (\mu, \xi) \).

We are now in position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** From Corollary 4.7 it follows that both \( \Gamma^* \) and \( \Gamma^{\omega} \) achieve a minimum at \( (\mu_1, \xi_1) \), resp. \( (\mu_2, \xi_2) \). Using (34) it follows that \( \Gamma^{\omega}(\cdot - x_0) \) achieves the minimum at \( (\mu_2, \xi_2 - x_0) \). From Lemma 5.4 we infer that for \( |x_0| \) sufficiently large there exists \( \delta > 0 \) such that the sublevel \( \{ \Gamma^{* + \omega \cdot - \omega} < -\delta \} \) is disconnected, namely \( \{ \Gamma^{* + \omega \cdot - \omega} < -\delta \} = A_1 \cup A_2 \) with \( A_1 \cap A_2 = \emptyset \). Applying the abstract result, Theorem 3.3, it follows that \( \Gamma^{* + \omega \cdot - \omega} \) has two distinct minima that give rise to two distinct solutions of (3).

**Remark 5.5.** Since \( \Gamma \) has two local minima, it possesses also a mountain pass critical point \( (\mu^*, \xi^*) \). If \( \mu^* > 0 \) such a critical point gives rise to a solution of the Yamabe problem which on general will have Morse index greater than 1. On this topic see [7, 8]. This and other related questions will be addressed in a future paper.

6. **FURTHER RESULTS**

In this section we indicate some possible extension of Theorem 1.1. The proofs can be obtained using arguments similar to those carried out before. In view of their length they are omitted. The first result which needs to be
modified is Proposition 4.5. Given a metric $\tilde{g}$ of the form (5) let $\tilde{W}$ denote the corresponding Weyl tensor. Expanding $\tilde{W}$ with respect to $\varepsilon$ one finds

$$\tilde{W} = \varepsilon \tilde{W} + o(\varepsilon),$$

where $\tilde{W}(x)$ is a tensor depending on the second derivatives $D_{x}^{2} h_{\tilde{g}}(x)$, only.

One finds:

**Proposition 6.1.** For $n > 6$, Eq. (30) holds and moreover

$$\frac{1}{4!} \frac{\partial^{4} \Gamma}{\partial \mu^{4}}(0, \xi) = - \sum_{i,j,k,l} c_{i,j,k,l} |\tilde{W}_{\mu \nu}(\xi)|^{2} \quad \forall \xi \in \mathbb{R}^{n},$$

(40)

where $c_{i,j,k,l} > 0$. If $n = 6$ then

$$\lim_{\mu \to 0} \frac{\Gamma(\mu, \xi)}{\mu^{4}} = - \infty, \quad \forall \xi \in \mathbb{R}^{n}, \quad \tilde{W}(\xi) \neq 0.$$

**Remark 6.2.** Proposition 6.1, jointly with Lemma 4.3, implies that $\Gamma$ has a minimum provided $\tilde{W} \neq 0$. This can be related to a result of Aubin [1] where the existence of a solution of the Yamabe problem for a compact non-locally conformally flat manifold $(M, g)$, is proved by minimizing the Sobolev quotient

$$Q(u) = \frac{\int_{M} c_{n} |\nabla u|^{2} + (1/2) R_{g} u^{2}}{\|u\|^{2}_{2}}.$$

Roughly, this can be done by taking a test function $u \in H^{1}(M)$ which concentrates near points where $\tilde{W} \neq 0$.

Consider $\tilde{g}$ of the form (5) where $h, k$ have compact support and let

$$\tilde{W}_{\mu} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \tilde{W}_{\delta + \varepsilon h}, \quad \tilde{W}_{k} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \tilde{W}_{\delta + \varepsilon k}.$$

Using the same arguments carried over in the preceding section one can show:

**Theorem 6.3.** Let $n \geq 6$, consider the metric of the form (5)

$$\tilde{g}(x) = \delta + \varepsilon h(x) + \varepsilon k(x - x_{0}),$$

and suppose that $\tilde{W}_{\mu}, \tilde{W}_{k} \neq 0$. Then for $|x_{0}|$ large enough, there are two different metrics in $[\tilde{g}]$ with scalar curvature identically equal to 1, provided $\varepsilon$ is sufficiently small.
Remark 6.4. The condition $\hat{W}_h \neq 0$ is generic.

Theorem 6.3 can be further extended by dealing with metrics $h, k$ which do not have compact support. To do this, we can first consider a metric $\hat{h}$ such that $h_{ij}(P) = 0$ at some $P \in \mathbb{R}^n$. Then one takes a sequence $h_m$ of smooth metrics such that $h_m = \hat{h}$ outside the ball $B_{2/m}(P)$ and $h_m \equiv 0$ in $B_{1/m}(P)$. It is easy to see that one can also choose $h_m$ in such a way that

$$\|h_m - \hat{h}\|_{C^\ell} \leq c \cdot m, \quad \|h_m - \hat{h}\|_{C^{\ell+1}} \leq c', \quad \|h_m - \hat{h}\|_\infty \leq c''/m.$$ 

This implies that the corresponding $\Gamma_m \rightarrow \Gamma$ uniformly in $Z$. To prove this claim it suffices to consider the terms $G_{2,m} - G_2$ and $(VG_{1,m}, \tilde{w}_m) - (VG_1, \tilde{w})$. As for the former one has

$$|G_{2,m}(z) - G_2(z)| \leq c_1 \frac{1}{m} + c_2 \int (R_{2,m} - R_2) z^2 + (\text{tr } h_m R_{1,m} - \text{tr } h_m R_1) z^2.$$

Since $R_2$, resp. $R_1$, depends only on $h D^2 h$, $D_i h D_i h$, resp. $D^2 h$, the choice of $h_m$ implies that

$$\int (R_{2,m} - R_2) z^2 \leq c_3 \int R_{2,m}(P) z^2,$$

$$\int (\text{tr } h_m R_{1,m} - \text{tr } h R_1) z^2 \leq c_4 \int R_{2,m}(P) z^2.$$ 

Using the Hölder inequality one infers

$$|G_{2,m}(z) - G_2(z)| \leq c_1 \frac{1}{m} + c_3 \frac{1}{m^2}.$$ 

Taking into account (UB) a similar argument shows that $(VG_{1,m}, \tilde{w}_m) - (VG_1, \tilde{w}) \rightarrow 0$ uniformly, proving the claim.

Similarly, if $h(x) \rightarrow 0$ as $|x| \rightarrow \infty$ one can find a sequence $h_m$ having compact support and such that $\Gamma_m \rightarrow \Gamma$ uniformly in $Z$. This follows from the preceding arguments, by using the transformation $h \rightarrow \hat{h}$. Furthermore, if $h, k \rightarrow 0$ as $|x| \rightarrow \infty$ then $h(x) + k(x - x_0)$ will be approximated by the corresponding sequence $h_m(x) + h_m(x - x_0)$ and for $m$ and $|x_0|$ large enough there results

$$\Gamma_m^{h + k(x - x_0)} \sim \Gamma^{h + k(x - x_0)},$$

uniformly in $z$.

Finally, given any $h$ coming from a regular metric on $S^n$, it tends at infinity to the constant metric $\hat{h}(0)$. Then we can consider $h \rightarrow \hat{h}(0)$ which
fits into the preceding set up. Since there results \( I^{k-h(0)} = I^k \) we can conclude with the following general result:

**Theorem 6.5.** Let \( n \geq 6 \) and consider the family of smooth metrics
g \( (x) = \delta + \epsilon h(x) + \epsilon k(x - x_0). \)

Suppose that \( \bar{W}_h, \bar{W}_k \neq 0 \). Then the same conclusion of Theorem 6.3 holds.

**APPENDIX**

1. **Some Integrals**

   From \( |x|^2 = x_1^2 + \cdots + x_n^2 \) it follows immediately that
   \[
   I_n(x_1^2) = \frac{1}{n} I_n^2.
   \]  

   From \( x_1^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \), and from the expansion of \( |x|^4 = (\sum_{i=1}^n x_i^2)^2 \), we deduce
   \[
   I_n(x_1^2) = \frac{1}{n(n+2)} I_n^4; \quad I_n(x_1^4) = \frac{1}{n(n+2)} I_n^4.
   \]  

   From \[10, p. 58\], we have the relationships
   \[
   \int_0^\infty x^n(1 + x^2)^m dx = \frac{2m}{2m+n+1} \int_0^\infty x^n(1 + x^2)^{m-1} dx,
   \]
   \[
   m, n \in \mathbb{R}, \quad 2m + n + 1 \neq 0,
   \]

   whenever the integrals are defined and \( \lim_{x \to 0, +\infty} x^{n+1}(1 + x^2)^m = 0 \); we also have
   \[
   \int_0^\infty x^n(1 + x^2)^m dx = \frac{n-1}{2m+n+1} \int_0^\infty x^{n-2}(1 + x^2)^m dx,
   \]
   \[
   m, n \in \mathbb{R}, \quad 2m + n + 1 \neq 0,
   \]

   whenever the integrals are defined and \( \lim_{x \to 0, +\infty} x^{n+1}(1 + x^2)^m = 0 \).

   It follows in particular that
   \[
   I_n = \frac{n-2}{n} I_n^2; \quad I_{n+1}^4 = \frac{n+2}{2n} I_n^2. \]
2. Proof of (A1)

**Lemma A.1.** If we set
\[ \tilde{w}_{\mu, \zeta}(x) = \lim_{\varepsilon \to 0} \frac{w_{\mu, \zeta}}{\varepsilon} = \mu^{-\frac{n+2}{2}} w_{\mu, \zeta} \left( \frac{x - \zeta}{\mu} \right), \] (44)
then there holds
\[ w_{\mu, \zeta}^*(x) \to w_0(x) \text{ in } E, \quad \text{as } \mu \to 0, \] (A1)
where \( w_0 \) is given by (29), namely
\[ w_0 = -c_n N_n \frac{(n-2)^2}{4(n-1)} \left[ \sum_{j,k} h_{jk} x_j x_k \right] \frac{1}{(1 + |x|^2)^{n/2}}. \]

**Proof.** We have
\[ (G^*_1(z), v) = \int \left( \frac{1}{2} \text{tr } h(2c_n \langle \nabla z, \nabla v \rangle - |z|^{2^* - 2} v) \right. \]
\[ \left. -2c_n \sum_{j} h_{jj} D_j z D_j v + R_1 z v \right) dx, \quad v \in E \] (45)
and
\[ (f^*_0(z_{\mu, \zeta}, \tilde{w}_{\mu, \zeta}, v) = \int \left( 2c_n \langle \nabla \tilde{w}_{\mu, \zeta}, \nabla v \rangle - (2^* - 1) |z_0|^{2^* - 2} \tilde{w}_{\mu, \zeta} v \right) dx, \]
\[ v \in E. \] (46)
We know that \( w_{\mu, \zeta} = -\varepsilon \tilde{L}_{z_{\mu, \zeta}} G^*_1(z_{\mu, \zeta}) + o(\varepsilon) \), and hence
\[ (f^*_0(z_{\mu, \zeta}, \tilde{w}_{\mu, \zeta}, v) = (G^*_1(z_{\mu, \zeta}), v), \]
for all \( v \in E \). This implies that \( \tilde{w}_{\mu, \zeta} \) solves
\[ 2c_n A \tilde{w}_{\mu, \zeta}(x) + \frac{n+2}{n-2} \tilde{w}_{\mu, \zeta}^{2^* - 2}(x) \tilde{w}_{\mu, \zeta}(x) \]
\[ = 2c_n \sum_{j,k} h_{jk} (x) D^2_{jk} z_{\mu, \zeta}(x) + 2c_n \sum_{j,k} D_j h_{jk}(x) D_k z_{\mu, \zeta}(x) \]
\[ -c_n \sum_{j,k} D_j h_{jk}(x) D_j z_{\mu, \zeta}(x) + R_1(x) z_{\mu, \zeta}(x). \]
From (44) we deduce that \( w^*_{\mu, \xi} \) is a solution of

\[
w^*_{\mu, \xi}(x) = -L_{z_0}k_{\mu, \xi}(x), \tag{47}\]

where

\[
k_{\mu, \xi}(x) = 2c_n \sum_{j,k} h_{jk}(\mu x + \xi) D^2_{jk} z_0(x) + 2\mu c_n \sum_{j,k} D_j h_{jk}(\mu x + \xi) D_k z_0(x) - \mu c_n \sum_{j,k} D_j h_{jk}(\mu x + \xi) D_j z_0(x) + \mu^2 R_1(\mu x + \xi) z_0(x). \tag{48}\]

We have that

\[
k_{\mu, \xi} \to k_0 \quad \text{in } E, \quad \text{as } \mu \to 0,
\]

where

\[
k_0 = 2c_n \sum_{j,k} h_{jk}(\xi) D^2_{jk} z_0(x).
\]

Since \( L_{z_0} \) is continuous we deduce that

\[
w_0(x) = -L_{z_0}k_0(x), \tag{49}\]

which implies that \( w_0 \) solves

\[
Aw_0 + \frac{n(n+2)}{(1 + |x|^2)^n} w_0 = \sum_{j,k} h_{jk}(\xi) D^2_{jk} z_0. \tag{50}\]

**Remark A.2.** Equation (50) has infinitely many solutions \( \hat{w} \) which are equal modulo \( T^*_Z \). The quantity \( df_1(z)[w] \), which we are interested in, is not affected by the translations of \( w \) by elements of \( T^*_Z \), so in the following it will be enough to find just one solution to (50).

Since Eq. (50) is linear in \( h \), we can solve it in some particular cases and sum up at the end: we treat therefore separately:

Case 1. \( (h_{ij}) = \delta_{11}\delta_{22} \). In this case \( w_0 \) solves

\[
Aw_0 + \frac{n(n+2)}{(1 + |x|^2)^n} w_0 = D^2_{12} z_0 = \kappa_n(n-2) \frac{x_1x_2}{(1 + |x|^2)^n x_1^2 + x_2^2}. \tag{51}\]
We try to solve (51) with a function of the type \( w_0 = x_1 x_2 f(|x|^2) \). With such a position we have

\[
Aw_0 + \frac{m(n+2)}{(1 + |x|^2)^2} w_0 = 4 |x|^2 f''(|x|^2) + 2nf'(|x|^2)
\]

\[
+ 2(x_2^2 x_1 + x_1^2 x_2) f'(|x|^2) + \frac{m(n+2)}{(1 + |x|^2)^2} x_1 x_2,
\]

so (51) becomes

\[
4f''(t) + (2n + 8) f'(t) + \frac{m(n+2)}{(1 + t)^2} f(t) = \kappa_n (n-2) \frac{1}{(1 + t)^{n/2}}.
\]

which is solved by

\[
f(t) = -\frac{n-2}{2} - \kappa_n \frac{1}{(1 + t)^{n/2}}.
\]

In conclusion we get

\[
w_0(x) = -\frac{n-2}{2} \kappa_n \frac{x_1 x_2}{(1 + |x|^2)^{n/2}}. \tag{52}
\]

**Case 2.** \((h) = \delta_{i_1} \delta_{j_3}\). In this case \( w_0 \) solves

\[
Aw_0 + \frac{m(n+2)}{(1 + |x|^2)^2} w_0 = D_{i_1} z_0
\]

\[
= \kappa_n (n-2) \left( \frac{nx^2_1}{(1 + |x|^2)^{n/2}} - \frac{1}{(1 + |x|^2)^{n/2}} \right).\tag{53}
\]

We try to solve this equation with a function \( w_0 \) of the form \( w_0 = x_1^2 f(|x|^2) \); reasoning as before we see that (53) becomes

\[
4f''(t) + (2n + 8) f'(t) + \frac{m(n+2)}{(1 + t)^2} f(t) = m\kappa_n (n-2) \frac{n}{(1 + t)^{(n+2)/2}};
\]

\[
2f = -\kappa_n (n-2) \frac{1}{(1 + t)^{n/2}}.
\]

The solution is given by

\[
f(t) = -\frac{\kappa_n (n-2)}{2} \frac{1}{(1 + t)^{n/2}}.
\]
so $w_0$ becomes

$$w_0 = -\frac{\kappa_n (n - 2)}{2} \frac{x_i^2}{(1 + |x|^2)^{n/2}}.$$  

(A.2)

A generic $h$ is the sum of diagonal terms like Case 1, and non-diagonal ones like Case 2.

Summing the expressions (52) and (54) related to the coefficients $h_{ij}$ we obtain (29).

Proof of (A.2)

**Lemma A.3.** There holds

$$\sum_{i,j} h_{ij} \int D_i z_0 D_j w_0 = \frac{\kappa_n^2 (n - 2)^2}{2n} \left( \frac{\text{tr}(h^2)}{4} - \frac{1}{2} \left( \frac{\text{tr}(h^2)}{2} \right)^2 \right) I_0^2.$$

Proof. Integrating by parts we obtain

$$\sum_{i,j} h_{ij} \int D_i z_0 D_j w_0 = -\sum_{i,j} h_{ij} \int w_0 D_i^2 z_0.$$  

By writing the explicit expression of $z_0$ and $w_0$ we obtain

$$\sum_{i,j} h_{ij} \int D_i z_0 D_j w_0 = \frac{\kappa_n^2 (n - 2)^2}{2} \sum_{i,j,k,l} h_{ij} h_{kl} (\delta_{ik} I_{n+1}(x_i,x_j,x_k) - \delta_{jl} I_{n+1}(x_i,x_k)).$$  

Let us turn our attention to the term $I_{n+1}(x_i,x_j,x_k,x_l)$; it is different from zero only when $i = j$ and $l = k$, or when $i = k$ and $l = j$, or when $i = l$ and $j = k$. Hence, there holds

$$\sum_{i,j,k,l} h_{ij} h_{kl} I_{n+1}(x_i,x_j,x_k,x_l)$$

$$= \sum_{i,j,k,l} h_{ij} h_{kl} I_{n+1}(x_i,x_j,x_k,x_l) + \sum_{i,j,k,l} h_{ij} h_{kl} I_{n+1}(x_i,x_j,x_k,x_l)$$

$$+ \sum_{i,j,k,l} h_{ij} h_{kl} I_{n+1}(x_i,x_j,x_k,x_l) - 2 \sum_{i,j,k,l} h_{ij} h_{kl} I_{n+1}(x_i,x_j,x_k,x_l)$$

$$= \sum_{i,k} h_{ii} h_{kk} I_{n+1}(x_i^2 x_k^2) + 2 \sum_{i,j} (h_{ij})^2 I_{n+1}(x_i^2 x_j^2)$$

$$- 2 \sum_{i} (h_{ii})^2 I_{n+1}(x_i^4).$$
Since $I_{n+1}(x_i^4) = 3I_{n+1}(x_i^2 x_j^2)$, \( i \neq j \), then
\[
\sum_{i,k} h_i h_k I_{n+1}(x_i^2 x_k^2) = \sum_{i,k} h_i h_k I_{n+1}(x_i^2 x_k^2) + \frac{2}{3} \sum_i (h_i)^2 I_{n+1}(x_i^4),
\]
and
\[
\sum_{i,j} (h_i)^2 I_{n+1}(x_i^2 x_j^2) = \sum_{i,j} h_i h_j I_{n+1}(x_i^2 x_j^2) + \frac{2}{3} \sum_i (h_i)^2 I_{n+1}(x_i^4).
\]
The last two equalities imply
\[
\sum_{i,j,k,l} h_i h_j h_k h_l I_{n+1}(x_i x_j x_k x_l) = \sum_{i,k} h_i h_k I_{n+1}(x_i^2 x_k^2) + 2 \sum_{i,j} (h_i)^2 I_{n+1}(x_i^4),
\]
from which we deduce
\[
\sum_{i,j} h_i h_j \iint D_x z_0 D_j w_0 = \frac{\kappa^2 (n-2)^2}{2n} \left[ \frac{n}{n+2} ((tr h)^2 + 2 tr(h^2)) I^4_{n+1} - (tr h)^2 I^2_n \right].
\]
If we use (43), we get
\[
\sum_{i,j} h_i h_j \iint D_i z_0 D_j w_0 = \frac{\kappa^2 (n-2)^2}{2n} \left( tr(h^2) - \frac{1}{2} (tr h)^2 \right) I^2_n.
\]
This concludes the proof.

4. Proof of Proposition 4.5.

We want to expand near $\mu = 0$ the function $\Gamma$ as
\[
\Gamma(\mu, \xi) = G_2(\mu, \xi) + \frac{1}{2} V G_1(\mu, \xi) \tilde{w}_{\mu, \xi} = \sum_{i=1}^4 \gamma_i(\xi) \mu^i + o(\mu^4).
\]
In order to do this we find a similar expansion for $G_2(\mu, \xi)$ and $V G_1(\mu, \xi) \tilde{w}_{\mu, \xi}$. By using the change of variables $x = \mu y + \xi$ we have
$$G_2(z_{\mu, \xi}) = \int \left[ \frac{1}{2} c_n a^2 (\mu y + \xi) |D_1 z_0|^2 
olimits - \frac{1}{8} a^2 (\mu y + \xi) \left( c_n |\nabla z_0|^2 - \frac{1}{2} |z_0|^2 \right) \right. $$

$$\left. + \mu^2 \left( \frac{1}{2} R_2 + \frac{1}{4} a R_1 \right) (\mu y + \xi) z_0^2 \right] dy. \quad (57)$$

Setting $a_0 = a(\xi)$, it is $a(\mu y + \xi) = a_0 + \mu \sum_j y_j D_j a(\xi) + \frac{1}{2} \mu^2 \sum_{i,j} D_i^2 a(\xi) y_i y_j + o(\mu^3)$, and hence

$$a^2 (\mu y + \xi) = a_0^2 + 2 \mu \sum_j a D_j a(\xi) y_j + \mu^2 \sum_{i,j} (a D_i^2 a(\xi)) y_i y_j + o(\mu^3). \quad (58)$$

Terms of order $\mu$ in $G_2$ have coefficients like

$$\int y_i |D_1 z_0|^2, \quad \int y_i \varphi(|y|^2),$$

for a suitable radial function $\varphi$, which are all zero. Similarly, the first order expansion in $\mu$ of the term $V G_1(z_{\mu, \xi}) \psi_{\mu, \xi}$ involves only integrals of odd functions and thus $\gamma_1(\xi) \equiv 0$. For the same reason, $\gamma_j(\xi) \equiv 0$.

Let us now show that $\gamma_2(\xi) \equiv 0$. First, the coefficients coming from $G_2$ are of the type

$$A_{ij} \int y_i y_j |r|^2, \quad \text{and} \quad A_{ij} \int y_i y_j |\tilde{r}|^2,$$

and such integrals, which are non-zero only for $i = j$, can be explicitly computed in terms of $I_4^a$, by means of relations of the same type as (43). For $h$ of the form (6) there holds

$$\frac{1}{2} R_2 + \frac{1}{4} a R_1 = -\frac{1}{4} (a A a - a D_1^2 a + |\nabla a|^2 - |D_1 a|^2). \quad (59)$$

So, using (57) and taking into account (58) and (59) we see that the coefficient of $\mu^3$ in $G_2$ contains terms of the form $a_0 D_2^2 a$ and $D_i a D_j a$. For the sake of brevity we will restrict ourselves to the very specific case that

$$Da(\xi) = 0, \quad D_2^2 a(\xi) = A \neq 0, \quad D_i^2 a(\xi) = 0 \quad \text{for} \quad i, j \neq 2. \quad (60)$$
In this case the coefficient of \( \mu^2 \) contains a factor of the form \( a_0 A \), and precisely there results

\[
G_2(\mu, \bar{\zeta}) = G_2(0, \bar{\zeta}) + \mu^2 \frac{1}{2} \frac{(n-1)(n-2)}{m(n+2)} a_0 A I^4 + o(\mu^2). \tag{61}
\]

It remains to evaluate \( \frac{1}{2} VG_1 \bar{\phi} = -\frac{1}{4} \bar{\eta}, f_0^*(z) \bar{\phi} \). After a change of variable, one finds

\[
(w_{\mu, \zeta}, f_0^*(z_{\mu, \zeta}) \bar{w}_{\mu, \zeta}) = (w_{\mu, \zeta}^*, f_0^*(z_0) w_{\mu, \zeta}^*).
\]

According to the arguments used in Lemma A.1 we can develop \( w \) in powers of \( \mu \). Actually, if

\[
k_{\mu, \zeta} = k_0 + \mu k_1 + \cdots,
\]

then

\[
w_0 = w_0 + \mu w_1 + \cdots,
\]

where \( w_i \) satisfies

\[
2 c_n A w_i + \frac{n+2}{n-2} \bar{c}_0^{2s-1} w_i = k_i.
\]

If (60) holds, then \( w_1 = 0, w_0 \) contains the coefficient \( a_0 \), and \( w_2 \) contains the coefficient \( A \). We then deduce that \( \mu^2 \) appears as a multiple of

\[
\frac{1}{2}((w_2, f_0^*(z_0) w_0) + (w_0, f_0^*(z_0) w_2)) \text{, and more precisely, since even } w_2 \text{ can be evaluated explicitly, a straight calculation yields}
\]

\[
\frac{1}{2} \left( w, f_0^*(z) \bar{w} \right) = \frac{1}{2} (w_0, f_0^*(z_0) w_0) + \mu^2 \frac{1}{2} \frac{(n-1)(n-2)}{m(n+2)} a_0 A I^4 + o(\mu^2).
\]

This equals (61) because \( G_2(0, \bar{\zeta}) - \frac{1}{2} (w_0, f_0^*(z_0) w_0) = \Gamma(0, \bar{\zeta}) = 0 \) and hence \( \bar{\gamma}_2 = 0 \).

We now describe the procedure to evaluate \( \partial^4 \Gamma \). We assume \( n > 6, h \) of the form (6) and write

\[
G_2(z) = \beta_1 + \beta_2,
\]

where

\[
\beta_1 = \int a^2 (\mu y + \bar{\zeta}) \left[ \frac{c_n}{2} |D_z z_0(y)|^2 - \frac{1}{8} (c_n |V z_0(y)|^2) - \frac{1}{2} |z_0(y)|^2 \right] dy \tag{62}
\]

\[
\beta_2 = \mu^2 \int \left( z_0(y) \cdot \left( R_2(\mu y + \bar{\zeta}) + \frac{1}{2} a(\mu y + \bar{\zeta}) R_1(\mu y + \bar{\zeta}) \right) \right) dy. \tag{63}
\]

The leading terms in \( \beta_1 \) to be considered are those containing the coefficients of \( \mu^4 \) in the Taylor expansion of \( a^2 \), which are of the form

\[
\mu^4 \sum A_{\alpha \beta \delta \epsilon} (\bar{\zeta}) y_\alpha y_\beta y_\delta y_\epsilon.
\]
Taking into account that $|D_1 z_0(y)|^2 = y_1^2 \psi(|y|^2)$ for some smooth $\psi(r)$, it turns out that the non-vanishing integrals are only those like

$$
\frac{\kappa}{2} A_{abk} \int y_1^2 y_k^2 y_1^2 \psi(|y|^2) \, dy - \frac{1}{8} A_{abk} \int y_1^2 y_k^2 z_0^2 \, dy
$$

and similar terms containing $A_{abk}$ and $A_{abk}$. All these integrals can be evaluated explicitly by means of $I_n^x$. We again assume (60) and also that $D^4 a(x) = 0$. One finds under the hypotheses

$$
\beta_1 = -\mu^4 \cdot A^2 \cdot \frac{9 \kappa^2}{8} \frac{(n-1)(n-2)}{8(n+2)(n+4)} I_n^x + o(\mu^4).
$$

As for $\beta_2$, there results

$$
\frac{1}{2} R_2 + \frac{1}{4} R_1 a = \frac{1}{4} (a A a - a D_1^2 a + |\nabla a|^2 - |D_1 a|^2) = \frac{3}{8} A^2 x_1^2 + o(|x|^2).
$$

Hence it follows

$$
\beta_2 = \mu^4 \cdot A^2 \cdot \frac{3}{8} \int y_1^2 z_0^2 (|y|) \, dy \cdot A^2 + o(\mu^4).
$$

Let us point out that the above integral is finite whenever $n > 6$. Putting together $\beta_1$ and $\beta_2$ we deduce

$$
G_2(z_0, \xi) = G_2(0, \xi) + \mu^4 \cdot \left( \frac{3 \kappa^2}{8} \frac{(n-1)(n-2)}{8(n+2)(n+4)} I_n^x \right) A^2 + o(\mu^4).
$$

Expanding again $w^*$ in powers of $\mu$ we get

$$(w_{\mu, \xi}, f_0^\mu(z_0) w_{\mu, \xi}) = (w_0, f_0^\mu(z_0) w_0) + \mu^4 (w_2, f_0^\mu(z_0) w_2) + o(\mu^4).$$

Since $w_2$ can be evaluated explicitly, a straight calculation yields for $n > 6$

$$(w, f_0^\mu(z) \, \bar{w}) = (w_0, f_0^\mu(z_0) \, w_0) + \mu^4 \cdot \frac{\kappa^2}{12} \frac{13n^2 - 47n + 42}{12(n+2)(n+4)} I_n^x \cdot A^2 + o(\mu^4).$$

In conclusion we find

$$
\Gamma(\mu, \xi) = G_2(z_0, \xi) - \frac{1}{2} (w, f_0^\mu(z) \, \bar{w}) = -\mu^4 \frac{(n-2)(n-3)}{6(n+2)(n+4)} I_n^x \cdot A^2 + o(\mu^4).
$$
Hence
\[ \frac{\partial^4 \Gamma}{\partial t^4}(0, \xi) = -c^* A^2 = -c^* |D^2_{\omega a}(\xi)|^2, \]
where \( c^* \) is a positive constant.

The general case is proved similarly.

If \( n = 6 \) the integral in the preceding formula of \( \beta_2 \) becomes infinite. This causes that \( G_2 \) is no more of order \( \mu^4 \). Similarly, also \( (\bar{\omega}, f_0'^{(z)}(\bar{\omega})) \) is not of order \( \mu^4 \). However, repeating the preceding calculation one finds
\[ \mu^{-4}(G_2(z_{\mu}, \xi) - \frac{1}{2}(\bar{\omega}, f_0'(z)(\bar{\omega}))) \to -\infty, \quad \text{as} \quad \mu \to 0+. \]
This proves Remark 4.6.

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