Finite Subnormalizing Extensions of Rings

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The purpose of this paper is to provide proofs of or counterexamples to possible generalizations, to a larger class of ring extensions, of several basic results about finite normalizing extensions [1–7, 11, 13, 14, 16]. The situation we are concerned with involves rings $R \subset S$ (with common unity element 1) such that, for some finite set $a_1, \ldots, a_n$ of elements of $S$, we have

$$S = \sum_{i=1}^n R a_i = \sum_{i=1}^n a_i R \quad \text{and for each } j, 1 \leq j \leq n, \sum_{i=1} a_i R = \sum_{i=1}^j a_i R.$$ 

Such an extension we call a finite subnormalizing extension. We obtain a Cutting Down result for prime ideals (Theorem 4.1) and several results about chain conditions (Sect. 5). And we show that familiar results from the case of finite normalizing extensions concerning Integrality, Extension Ideals and Bounds do not extend to that of finite subnormalizing extensions (Sect. 6).

Our proofs rely heavily on the “primitivity machine” developed by Passman [11] with some further fine tuning to deal with nonminimal primes and mid-annihilators.

Clearly, any finite normalizing ring extension is a finite subnormalizing ring extension, and these latter have been much studied in recent years. But there exist familiar ring extensions which are finite subnormalizing but not finite normalizing (see Examples 1.7 and 1.8), and we have been unable to trace any discussion of finite subnormalizing extensions as such.

1. Preliminaries

In this section we define terms and notations and review results which will be used throughout this paper.

**Definition 1.1.** Let $R, S$ be rings and $M = R M_S$ be an $R$–$S$ bimodule. An element $x$ of $M$ is normalizing if $Rx = xS$. A nonempty subset $X$ of $M$ is self-conjugate if $RX = XS$. A finite or countably infinite sequence $x_1, x_2, \ldots$, is a subnormalizing sequence if each $X_j = \{x_1, \ldots, x_j\}$ is self-conjugate for
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\( j = 1, 2, \ldots \) (in cases where ambiguity is possible, we will say "\( R-S \) normalizing," "\( R-S \) self-conjugate," etc.).

A finite subnormalizing sequence has been discussed (in the bimodule \( _RRR \) for a ring \( R \)) by among others Walker [12] and McConnell [9, 10], the former using the label "normalizing set." We wish here to use a different label because in the present context confusion would be likely between "normalizing set" and "set of normalizing elements," and because we wish (at times) to emphasise the orderings of subnormalizing sequences, as in Proposition 1.2. However, by abuse of notation we shall write \( x_i \in X \) when \( X = \{x_1, ..., x_n, \ldots \} \) is a subnormalizing sequence, while if \( Y = \{y_1, ..., y_n\} \) is a family of normalizing elements we shall sometimes think about it as a subnormalizing sequence in the order \( y_1, y_2, ..., y_n \). In practice this will cause no confusion. All modules and bimodules are unital.

The following is straightforward.

**Proposition 1.2.** If \( R \subseteq S \) are rings and \( X = \{x_1, x_2, ..., x_n\} \) and \( Y = \{y_1, y_2, ..., y_m\} \) are \( R-R \) subnormalizing sequences in the \( R-R \) bimodule \( _RS_R \) then \( X*Y = \{x_i y_j : 1 \leq i \leq n, 1 \leq j \leq m\} \) is (with the lexicographic ordering) an \( R-R \) subnormalizing sequence of \( S \).

The notation \( X*Y \) of this proposition will be consistently used to denote \( \{xy : x \in X, y \in Y\} \) for non-empty subsets \( X \) and \( Y \) of a ring \( S \). We shall say a bimodule satisfies bi-a.c.c or is bi-Noetherian (or some similar expression) if it satisfies the ascending chain condition on sub-bimodules. Terms such as bi-d.c.c. and bi-Artinian are analogously defined. If \( A_{ij}, 1 \leq i \leq m, 1 \leq j \leq n \), are nonempty sets then \( \left(A_{ij}^{1}, \ldots, A_{ij}^{n}\right) \) denotes the set of \( m \times n \) matrices, each of which has \( i, j \) entry belonging to \( A_{ij} \). This notation will usually be used when \( n = m \) and the \( A_{ij} \) are rings or bimodules such that \( (A_{ij}) \) is a ring or an ideal under matrix multiplication. Similarly, \( e_{ij} \) will denote (for some finite \( n \)) the \( n \times n \) matrix whose \( kl \) entry is \( \delta_{ki} \delta_{jl} \), and \( n \) will be clear from the context.

**Definition 1.3.** A ring extension \( R \subseteq S \) is said to be a normalizing extension (with respect to \( X \)) if \( X \subseteq S \) is a nonempty set of \( R-R \) normalizing elements and \( S = RX = XR \). It is a finite normalizing extension (with respect to \( X \)) if in addition \( X \) is finite. The terms (finite) subnormalizing extension (with respect to \( Y \)) and (finite) self-conjugate extension (with respect to \( Z \)) are defined in the obvious analogous ways.

A normalizing extension with respect to \( X \) is a subnormalizing extension with respect to \( X \) and a subnormalizing extension with respect to \( Y \) is a self-conjugate extension with respect to \( Y \). The reverse statements are false, as is shown in Examples 1.7 and 1.8.
We have the following elementary results which depend only on certain given sets being self-conjugate (where \(|T|\) denotes the cardinality of a set \(T\)).

**Lemma 1.4.** If \(M\) is an \(R\)-\(S\) bimodule and \(\omega\) is an infinite cardinal then there exists a self-conjugate set \(Z\) such that \(|Z| < \omega\) and \(M = RZ = ZS\) if and only if there exist sets \(X\) and \(Y\) such that \(|X| < \omega\), \(|Y| < \omega\) and \(M = RX = YS\).

**Proof.** Set \(Z = X \cup Y\).

**Lemma 1.5.** If \(R \subseteq S\) and \(S \subseteq T\) are self-conjugate extensions with respect to the sets \(X\) and \(Y\), respectively then \(R \subseteq T\) is a self-conjugate extension with respect to \(Z = X*Y \cup Y*X\), which is finite whenever \(X\) and \(Y\) are.

It is possible, if we drop the assumption that rings have unity elements and that bimodules are unital, to define different subbimodules "generated" (in various canonical ways) by a subset of a bimodule, and then to obtain corresponding trivial generalisations of Lemma 1.4.

The following examples indicate that the sets \(X\), \(Y\) and so on referred to in Definition 1.3 cannot lightly be left out of the discussion. Nevertheless, we shall drop the parenthetic phrase "with respect to \(T\)" wherever the identity of \(T\) is clear from the context.

**Example 1.6.** A ring extension \(R \subseteq S\) and sets (sequences) \(X\) and \(Y\) such that \(R \subseteq S\) is a subnormalizing but not a normalizing extension with respect to \(X\), and yet is a normalizing extension with respect to \(Y\).

Let \(D\) be the real quaternians and \(T\) be its centre (the real field), and let \(i, j, k\) have the usual meanings and \(K = T[i]\). Then \(K \subseteq D\) is a normalizing extension with respect to \(Y = \{1, j\}\) since \(Kj = Tj \oplus Tk = jK\), and is a subnormalizing extension with respect to \(X = \{1, i+j\}\) since \(K(i+j) \neq (i+j)K\).

The following examples illustrate that the concepts of (finite) normalizing, (finite) subnormalizing and (finite) self-conjugate extension are all distinct.

**Example 1.7.** A finite subnormalizing extension which is not a normalizing extension (finite or otherwise).

Let \(K\) be a commutative field, \(R\) the ring \((\begin{smallmatrix} \mathbb{K} & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{smallmatrix})\) and \(S = M_2(K)\). Then let \(x_1 = 1_S = 1_R\), and \(x_2 = (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})\); \(X = \{x_1, x_2\}\) is a subnormalizing sequence and \(S = RX = XR\). We need to show that \(S\) is not a normalizing extension: it is easy to check that if \(A \in S\) and \(RA = AR\) then \(A \in R\).
EXAMPLE 1.8. A finite self-conjugate extension which is not a subnormalizing extension (finite or otherwise).

Let $F \subset K$ be commutative fields, with $\infty > \text{dim}_F K = n > 1$. Let $R = \begin{pmatrix} e & 0 \\ 0 & K \end{pmatrix}$ and $S = \begin{pmatrix} 0 & K \\ 0 & \frac{e}{K} \end{pmatrix}$. It is straightforward to check that $x \in S$ is $R$-$R$ normalizing if and only if $x \in R$, and then that $Rx + Ry = yR + xR$ implies $y \in R$. But $R \subseteq S$ is finite self-conjugate by Lemma 1.4 since $F \subseteq R$ and $\text{dim}_F S = 2n + 1 < \infty$.

Suppose $R \subseteq S$ is a subnormalizing extension with respect to the not necessarily finite subnormalizing sequence $X = \{x_1, x_2, \ldots\}$, and that $S$ is finitely generated as a right or as a left $R$-module. Then clearly there is an integer $N$ such that $R \subseteq S$ is a finite subnormalizing extension with respect to the finite subnormalizing sequence $Y = \{x_1, x_2, \ldots, x_N\}$, and $S$ is finitely generated both as a left and as a right $R$-module. The analogous statements for a ring extension $R \subseteq S$ self-conjugate with respect to a countably infinite set $X$ are false, as is shown by

EXAMPLE 1.9. Let $F \subset K$ be an extension of commutative fields with $\text{dim}_F K$ countably infinite. Let $R$ be the ring $\begin{pmatrix} e & 0 \\ 0 & K \end{pmatrix}$ and $S$ be the ring $\begin{pmatrix} 0 & K \\ 0 & \frac{e}{K} \end{pmatrix}$, let $e_{12} = (i, A) \in S$ and let $\{y_j: j = 1, 2, \ldots\}$ be an $F$-basis of $K$. Let $Y_j = \{I_R, y_j e_{12}\}$ for some $j$, and $X = \{I_R, y_j e_{12}: j = 1, 2, \ldots\}$. Then $S = Y_j R$, so $S$ is finitely generated as right $R$-module, and $S = RX = XR$ so $R \subseteq S$ is a self-conjugate extension with respect to the countably infinite set $X$. But $R \subseteq S$ is not a self-conjugate extension with respect to any finite subset of $X$.

There are several natural classes of finite subnormalizing ring extension that deserve mention.

EXAMPLE 1.10. Let $R$ be a ring and $n > 1$ an integer. If $S = M_n(R)$ and $T$ is the subring of $S$ comprising upper triangular matrices, then $T \subseteq S$ is a finite subnormalizing extension with respect to the sequence $x_1 = 1, x_2 = e_{21}, \ldots, x_j = e_{j1} + \cdots + e_{j(j-1)}, \ldots, x_n = e_{n1} + \cdots + e_{n(n-1)}$. Also $R \subseteq T$ is an intermediate normalizing extension in the sense of [5].

EXAMPLE 1.11. Let $R$ be a ring, $\alpha$ an automorphism of $R$ and $\delta$ a (left) $\alpha$-derivation of $R$, so that $(ab)\delta = a^\delta b^\delta + ab^\delta$ for all $a, b$ in $R$. Let $T$ be the skew polynomial ring $R[X; \alpha, \delta]$ where $X$ is an indeterminate and multiplication extends $aX = Xa^\alpha + a^\delta$ for $a$ in $R$. If $f$ is a monic polynomial of degree $n$, contained in an ideal $P$ of $T$ such that $P \cap R = 0$, if $S = T/P$ and if $x = X + P \in S$ then $R \subseteq S$ is a finite subnormalizing extension with respect to $1, x, x^2, \ldots, x^{n-1}$.

EXAMPLE 1.12. Let $L$ be a finite dimensional Lie algebra over a field $k$ and let $L$ act as derivations on $R$. Let $U(L)$ be the universal enveloping
algebra of $L$. Then the smash product $S = R \# U(L)$ is a subnormalizing extension of $R$. For suppose $x_1, \ldots, x_t$ is a basis of $L$. Then a "basis" of $S$ over $R$ is the set of ordered monomials $X = \{x_1^{i_1} \cdots x_t^{i_t} : i_j \geq 0\}$, and for $r$ in $R$ we have $x_i r = r x_i + d_i(r)$, where $d_i$ is the derivation associated to $x_i$.

Clearly $S$ is a subnormalizing extension of $R$ with respect to $X$ when $X$ is ordered by the total degree in the $x_i$; that is, $X$ is ordered as $\{1, x_1, \ldots, x_t, \text{degree 2 monomials, degree 3 monomials, etc.}\}$ and the ordering within the subset of degree $n$ monomials can be one of several possibilities.

**Example 1.13.** Let $L$ be a restricted Lie algebra acting as derivations on $R$, and let $R$ have characteristic $p \neq 0$. Then $u(L)$, the restricted enveloping algebra, is finite dimensional, spanned by $\{x_1^{i_1} \cdots x_t^{i_t} : 0 \leq i_j \leq p - 1\}$. Thus the smash product $S = R \# u(L)$ is a finite subnormalizing extension of $R$ (with respect to the obvious ordered set $X$).

**Example 1.14.** The "almost normalizing" extensions of J.C. McConnell [15] are easily seen to be subnormalizing extensions.

We now recall some crucial techniques due to Passman. Let $R$ be a ring, and $Y$, $Z$ be sets of variables, and $R(Y, Z) = (R \langle Y \rangle \langle Z \rangle)$ be a suitably defined (see [11]) power series-polynomial ring over $R$. Then Passman proved:

**Lemma 1.15.** If $R$ is a prime ring and $|Y|, |Z|$ are large enough (in terms of $|R|$) then $R(Y, Z)$ is (right) primitive.

In the remainder of this paper, when we deal with a ring extension $R \subseteq S$ we will choose $Y$ and $Z$ such that $|Y|$ and $|Z|$ are large enough in terms of $|S|$. They will then be large enough in terms of $|R|$ and of $|T|$ for any sub-ring or quotient ring $T$. With this convention we simply write $R^* = R(Y, Z)$. If $A$ is a subset of $R$ we let $A^* = A(Y, Z)$ denote the subset of $R^*$ comprising the power series-polynomials all of whose coefficients are in $A$. If $M$ is a left (right) $R$-module and $T \subseteq M$ is a non-empty set then $L_A(T) = L_A^R(T)$ and $R_A(T) = R_A^R(T)$ denote respectively the left and right annihilators of $T$ over $R$.

Passman also proved:

**Lemma 1.16.** With the above notation and conventions we have:

(i) if $I$ is an ideal of $R$ then $I^*$ is an ideal of $R^*$ with $I^* \cap R = I$ and $(R/I)^* \cong R^*/I^*$;

(ii) if $\{A_i\}$ is a collection of subsets of $R$ the $( \cap A_i)^* = \bigcap A_i^*$;

(iii) if $A$ and $B$ are ideals of $R$ then $A^* B^* \subseteq (AB)^*$, so $A^n = 0$ implies $(A^*)^n = 0$;
(iv) if $P$ is a prime ideal of $R$ then $P^*$ is a primitive ideal of $R^*$;
(v) if $P$ is a semiprime ideal of $R$ then $P^*$ is a semiprimitive ideal of $R^*$.

If $R$ is a ring, and $I, J$ are nonempty subsets and $C$ is an ideal, let $\text{Mid}(I, J; C) = \{x \in R: IxJ \subseteq C\}$. If $C = 0$, write $\text{Mid}(I, J)$; this is the mid-annihilator of $I$ and $J$. If $I$ and $J$ are ideals then so is $\text{Mid}(I, J; C)$, and if $\text{Mid}(I, J; C)$ is an ideal then $\text{Mid}(I, J; C) = \text{Mid}(RIR, RJR; C)$. An ideal $A$ of $R^*$ is closed if it is closed in the power-series topology. For example, $R^*$ and 0 are closed ideals.

Parts (ii) and (iii) of the following lemma were proved by Passman [11, Lemma 1.5] for the case of a left annihilator (prime) ideal.

**Lemma 1.17.** If $R, R^*$ are as usual and $A, B, C, I, J$ are ideals of $R^*$, then

(i) if $C$ is closed then so is $\text{Mid}(I, J; C)$;
(ii) if $A$ is closed then $(R \cap A)^* \subseteq A$;
(iii) if $A$ is closed and prime then $R \cap A$ is prime;
(iv) if $A$ is closed and a minimal prime of $R^*$ then $A = (R \cap A)^*$;
(v) if $A$ is closed and a minimal prime of $R^*$ and if $\sigma: R \to R^*/B \to R^*/A \simeq (R/R \cap A)^*$ is an epimorphism with kernel $B$ which takes each monomial of $R^*$ to the corresponding monomial of $(R/R \cap A)^*$ then $R \cap B$ is prime and $B = (R \cap B)^*$.

**Proof.** Parts (i) and (ii) are clear, and part (iii) follows just as in the corresponding result of Passman. Part (iv) then follows from Lemma 1.16 (i), and part (v) from part (iv) by some easy diagram chasing.

Most of the above result would hold with a slightly weaker condition on $A$ than closure, but that is the obvious "natural" property to assume. Passman gives an example of a ring $R$ and a prime ideal $Q$ of $R^*$ such that $Q \cap R$ is not a prime ideal of $R$. He also notes for finite normalizing extensions the obvious analogue of

**Lemma 1.18.** Let $R \subseteq S$ be an extension of rings.

(i) If $S/R$ is a finite self-conjugate, or a finite subnormalizing or a finite normalizing extension with respect to $X$ then $S^*/R^*$ has the corresponding property with respect to the same set $X$;
(ii) If in (i) $S$ is $R$-free on $X$ (on the right or left or both) then $S^*$ is $R^*$-free on $X$ (in the corresponding way or ways);
(iii) If $I$ is an ideal of $S$ with $I \cap R = 0$ then $I^* \cap R^* = 0$. 

2. MODULES AND BIMODULES

In this section we review the \( R \)-module structure of a (right) \( S \)-module \( M \) for ring extensions \( R \subseteq S \) subject to various conditions; then we comment on the situation for bimodules. Much of our material is generalized from [1, 2].

First we note that if \( R \subseteq S \) is a ring extension there always exists a self-conjugate set \( X \) such that \( S = RX = XR \); for example, \( X = S \). Let \( M \) be an \( S \)-module, \( N \) an abelian additive subgroup, and for \( x \in S \) define \( Nx^{-1} = \{ m \in M : mx \in N \} \). In general \( Nx^{-1} \) is also an abelian additive subgroup, but even if \( N \) is an \( R \)-submodule \( Nx^{-1} \) need not be an \( R \)-submodule. However, we clearly have

**Lemma 2.1.** If \( R \subseteq S \) is a self-conjugate extension with respect to \( X \) and if \( M \) is a right \( S \)-module with \( R \)-submodule \( N \) then \( b(N) = \bigcap_{x \in X} Nx^{-1} \) is the largest \( S \)-submodule of \( N \).

We call this module \( b(N) \) the *bound* of \( N \); it is of course independent of the choice of \( X \).

**Lemma 2.2.** If \( R \subseteq S \) is a finite self-conjugate extension with respect to \( X \) and \( M \) is a (right) \( S \)-module then \( M \) contains an \( R \)-submodule \( N \) maximal with respect to \( b(N) = 0 \).

*Proof.* Zorn's Lemma.

Now suppose \( R \subseteq S \) is a subnormalizing extension with respect to \( X = \{ x_1, x_2, \ldots \} \). If \( M \) is a right \( S \)-module and \( N \) is an \( R \)-submodule define \( V_j(N) = \bigcap_{x \in X} Nx^{-1} \), for \( j = 1, 2, \ldots \), and \( V_0(N) = M \). We will repeatedly use this notation.

**Lemma 2.3.** With the above notation, \( V_j(N) \) is an \( R \)-submodule of \( M \) for each \( j \geq 0 \), and \( b(N) = \bigcap_{j=1}^{\infty} V_j(N) \). Furthermore, for \( j \geq 0 \) the natural group monomorphism \( V_j/V_{j+1} \to M/N \) defined by \( m + V_{j+1} \to mx_{j+1} + N \) induces a lattice embedding \( \mathcal{L}(V_j/V_{j+1}) \to \mathcal{L}(M/N) \) of the lattices of \( R \)-submodules.

*Proof.* These assertions are straightforwardly verified.

Finally, we apply these results to bimodules. Suppose \( R \subseteq S \) is a finite subnormalizing extension with respect to \( X = \{ x_1, \ldots, x_n \} \). Taking tensor products over the ring of rational integers there are natural ring homomorphisms \( R^{op} \otimes R \to S^{op} \otimes S \to S^{op} \otimes S \). Now, \( S^{op} \otimes S \) is a subnormalizing extension of \( im \beta \) with respect to \( \{ x_i \otimes 1 : 1 \leq i \leq n \} \) and is also a subnormalizing extension of \( im \alpha \beta \) with respect to \( \{ x_i \otimes x_j : 1 \leq i, j \leq n \} \) under the lexicographic ordering. Clearly an \( R-S \) subbimodule \( N \) or an
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**R–R subbimodule** $L$ of an $S$–$S$ bimodule $M$ is simply a right submodule of $M$ when viewed over $\text{im} \beta$ or $\text{im} \alpha \beta$, respectively. And then $b(N) = \bigcap_i N(x_i \otimes 1)^{-1}$ and $b(L) = \bigcap_{i,j} L(x_i \otimes x_j)^{-1}$ are the largest $S$–$S$ subbimodules of $M$ contained in $N$ and $L$, respectively (where we regard $M$ as a right $S^{\text{op}} \otimes S$ module). All this, apart from minor notational differences, closely resembles [1, p. 3].

3. Modules with Composition Series

Throughout this section (except where stated to the contrary) we assume that $R \subseteq S$ is a finite subnormalizing extension with respect to $X = \{x_1, \ldots, x_n\}$. We develop relations between right $S$-modules with composition series and right $R$-modules with composition series.

We also fix the following notation. $W_0 = 0 \subseteq S$, and for $1 \leq j \leq n$, $W_j = Rx_1 + \cdots + Rx_j$ and $\tilde{x}_j = x_j + W_{j-1} \subseteq W_j/W_{j-1}$. Finally, $L_j = L_{A_R} (W_j/W_{j-1}) = L_{A_R} (\tilde{x}_j)$ and $N_j = R_{A_R} (W_j/W_{j-1}) = R_{A_R} (\tilde{x}_j)$; note that each $\tilde{x}_j$ is $R$–$R$ normalizing, that $(\bigcap_i L_j)^n = (\bigcap_i N_j)^n = 0$, and that $S$ is left (right) free on $x_1, \ldots, x_n$ if and only if $L_j = 0$ ($N_j = 0$) for each $j$. If $R$ is binoetherian, then, for each $j$, $L_j = 0$ if and only if $N_j = 0$.

**Lemma 3.1.** An $S$-module has an $S$-composition series if and only if it has an $R$-composition series. If $M$ has an $S$-composition series of length $m$ then it has an $R$-composition series of length $\leq mn$. In particular if $W$ is a simple $S$-module then it has a composition series as $R$-module of length $k \leq n$. Furthermore there exist primitive ideals $P_1, \ldots, P_k$ of $R$ (not necessarily distinct) such that

(i) each $P_i$ is the annihilator of an $R$-composition factor of $W$, and each such annihilator is one of $P_1, \ldots, P_k$;

(ii) $WP_1, \ldots, P_k = 0$;

(iii) $R/P_i \simeq R/P_j$ for $1 \leq i, j \leq k$.

If, in addition, $W$ is $S$-faithful, then

(iv) $P_1 \cdots P_k = 0 = (P_1 \cap \cdots \cap P_k)^k$, and

(v) each minimal prime of $R$ (which is among $P_1, \ldots, P_k$) is the mid-annihilator $\text{Mid}(A, B)$ for ideals $A, B$ of $R$ such that $AB \neq 0$.

**Proof.** It clearly suffices to consider the case of a simple $S$-module $W$. Since $W$ is $S$-cyclic on, say, $w$ it is clear that $W$ is finitely generated over $R$ by $wx_1, \ldots, wx_n$. Hence $W$ has a maximal $R$-submodule, say $N$, and in the notation of Lemma 2.3 we have a chain $0 = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 (N) = W$ of $R$-submodules of $W$. By Lemma 2.3 each $V_j/V_{j+1}$ can be
embedded in $W/N$, in a way which preserves submodules, by the map $m + V_{j+1} \rightarrow mx_{j+1} + N$. Hence each $V_j/V_{j+1}$ is simple or zero, so $W$ has a composition series of length $k \leq n$. Let the successive annihilators of simple composition factors, going down the series from $W$, be $P_1, \ldots, P_k$; (i) and (ii) follow at once. Now let $P = RA_R(W/N)$; since $W/N$ is a composition factor of $W$, $P = P_i$ for some $i$. Suppose $1 \leq a \leq k$ and $P_a$ is the annihilator of $V_b/V_{b+1}$, a simple module. It is routine to check that the map $r \rightarrow r'$, where $r^2 = x_r r'$ for $r, r' \in R$, induces a ring isomorphism $s: R/L \rightarrow R/N$, and that $L = \text{ann}(P)$ and $N = \text{ann}(P)$, so that $s$ induces an isomorphism $R/P \cong R/P_i$ giving (iii). If $W$ is $S$-faithful then (iv) is clear, and it remains to prove (v).

We may assume that the distinct minimal primes of $R$ are $Q_1 = P_{i_1}$, $Q_2 = P_{i_2}, \ldots, Q_g = P_{i_g}$ for some $g$, $1 \leq g \leq k$ and some $i_1, \ldots, i_g$. Clearly, $\bigcap \mathfrak{p} Q_i = \bigcap \mathfrak{p} P_i$, so $\bigcap Q_i$ is nilpotent. Thus the zero ideal may be written as a product of the $Q_i$'s, and hence as a produce with the minimal number $r \geq g$ of (not necessarily distinct) factors. Suppose one such shortest product is $0 = Q_{c(1)}Q_{c(2)}\cdots Q_{c(r)}$, each $Q_{c(d)}$ is one of the $Q_i$'s, and we consider the descending chain $W \supseteq WQ_{c(1)} \supseteq WQ_{c(1)}Q_{c(2)} \supseteq \cdots \supseteq WQ_{c(1)}Q_{c(2)}\cdots Q_{c(r-1)} = 0$. By the minimality of $r$, each inclusion is strict. Since each $Q_{c(d)}$ is the annihilator of a composition factor of $W$, for each $d$ there is an integer $s < r$ such that $Q_{d} = \text{ann}(Q_{c(d)})$ of the nonzero module $(WQ_{c(1)}Q_{c(2)}\cdots Q_{c(s)})/(WQ_{c(1)}Q_{c(2)}\cdots Q_{c(s+1)})$. If $s = 1$ set $A = R$ and $B = Q_{c(2)}\cdots Q_{c(r)}$. If $s = r - 1$ set $A = Q_{c(1)}\cdots Q_{c(r-1)}$ and $B = R$. Otherwise, set $A = Q_{c(1)}\cdots Q_{c(s)}$ and $B = Q_{c(s+2)}Q_{c(s+3)}\cdots Q_{c(r)}$. In each case $Q_{d} = \text{ann}(Q_{c(d)}) = \text{Mid}(A, B)$, since if $Q_{d} \subset J$ for an ideal $J$ then $WAB \neq 0$.

**Example 3.2.** The bounds described in the previous lemma can be attained. Let $R \subseteq S$ be the finite subnormalizing extension with respect to $X = \{1, x_2\}$ given in Example 1.7. Then $S$ is a simple ring, and in the standard notation $M = (e_1 + e_2)$ is a maximal right ideal of $S$. It is easy to check that $V = S/M$ is a faithful $R$-module. There are two prime ideals of $R$, namely $Q = (x_0 x_0)$ and $P = (x_0 x_0)$, and $QP = Q \cap P = (0 x_0)$ say, but $PQ = 0$. A composition series for $V_R$ has length 2, and $Q$ annihilates one simple factor while $P$ annihilates the other, but $0 = J^2 \neq J = Q \cap P$. Many features of this example are typical of the case in which $S$ is simple, as Theorem 4.1 shows.

**Lemma 3.3.** If $M$ is a maximal right ideal of $R$ and $V$ is the $S$ module $V = S/MS$ then $V$ has a composition series over $R$ of length $\leq n$.

**Proof.** As in the corresponding result of Passman [11, Lemma 2.2] $V = \Sigma_i^1((Rx_i + MS)/MS)$. If, for $1 \leq i \leq n$ we define $U_i = \Sigma_i^1((Rx_i + MS)/MS)$
and $U_0 = 0$ then it is easily checked that each factor $U_{j+1}/U_j$, for $0 < j < n$, is an $R$-homomorphic image of $((R + MS)/MS) \cong R/(R \cap MS)$, which is zero or simple.

Our next result closely resembles one in [11], and the proof is accordingly brief.

**Lemma 3.4.** Suppose $S$ is left free over $R$ on $X = \{x_1 \cdots x_n\}$, and $M$ is a right ideal of $R$ with $RA_R(R/M) = J$. Then $I = RA_S(S/MS) \subseteq \bigoplus Jx_i = JS$. Thus if $R$ acts faithfully on the direct sum $\bigoplus_j R/M_j$ then $S$ acts faithfully on $\bigoplus_j S/M_j S$.

**Proof.** This is almost the same as Passman's proof [11, p. 561] in the case of a finite normalizing extension. The only difference is that if (for each $k$) we set $I_k = \{r \in R : \sum_i r_ix_i \in I, r = r_k\}$ then $I_k$ is not necessarily a two-sided ideal of $R$. However, $I_n$ is such an ideal, contained in $J$, and then an easy reverse induction establishes that $I_k + J$ is an ideal for $1 \leq k \leq n$. But $I_k + J \subseteq M$ by left freeness, and $J$ is the largest two-sided $R$-ideal contained in $M$, so each $I_k \subseteq J$.

**Lemma 3.5.** Suppose $R$, $T$ are rings and $M$ is a simple $R$-$T$ bimodule such that $M = Rx = xT$ for some $x \in M$. Then $LA(x)$ and $RA(x)$ are maximal two-sided ideals of $R$ and $T$, respectively.

**Proof.** If $A$ is any ideal of $R$ then $Ax$ is a subbimodule of $M$, so $Ax = 0$ or $Ax = M$. Obviously $LA(x)$ is a prime two-sided ideal of $R$. Suppose $y \in R$ and $y \notin LA(x)$. Then $M = RyRx$ so $x = (\sum \alpha_i y \beta_i) x$ for some $\alpha_i, \beta_i \in R$, so $(1 - \sum \alpha_i y \beta_i) \in LA(x)$, hence $LA(x)$ is a maximal ideal. Similarly, $RA(x)$ is a maximal ideal of $T$.

This result easily fails without the assumption that $M$ is generated by a normalizing element, for example if $M$ is the maximal ideal of the ring $R = T$ of endomorphisms of a countably infinite-dimensional vector space.

4. Cutting Down

The following result resembles the Cutting Down results of [1, 6] with the difference that $P \cap R$ need not (as examples show) be semiprime, but instead must contain a power of its prime radical, which is (as in the normalizing case) the intersection of finitely many primes.

**Theorem 4.1.** If $P$ is a prime ideal of $S$ then there exist integers $f, k$ with $1 \leq f \leq k \leq n$, and distinct prime ideals $Q_1, \ldots, Q_f$ of $R$, such that

(i) every prime ideal of $R$ minimal over $P \cap R$ occurs among $Q_1, \ldots, Q_f$;
(ii) \((Q_1 \cap \cdots \cap Q_f)^k \subseteq P \cap R \subseteq (Q_1 \cap \cdots \cap Q_f)\);

(iii) \(R/Q_i \cong R/Q_j\) for \(1 \leq i, j \leq f\);

(iv) if \(P\) is right (left) primitive then so are each of \(Q_1, \ldots, Q_f\);

(v) if \(P\) is maximal then so are each of \(Q_1, \ldots, Q_f\).

Proof. We can pass to the extension \((R/R \cap P) \subseteq (S/P)\), and hence assume \(P = 0\), so \(S\) is prime.

Case 1. \(S\) primitive.
In this case assertions (i)–(iv) are immediate consequences of Lemma 3.1.

Case 2. \(S\) prime.
As in Section 1, we have \(R^* \subseteq S^*\) and \(S^*\) is primitive by Lemma 1.16. Assertions (i)–(iv) are easy consequences of Lemmas 3.1 and 1.17.

Case 3. \(S\) simple.
In this case, there exists (in the notation of Sect. 3) a least integer \(j\) such that \(S = W_j = Rx_1 + \cdots + Rx_j\). If \(j = 1\) then it follows easily that \(S = R\) and our assertions are trivial. Otherwise, we can find an \(R\)-\(R\) sub-bimodule \(N\) of \(S\) such that \(W_{j-1} \subseteq N \subseteq S\) and \(S/N\) is a simple \(R\)-\(R\) bimodule, clearly generated by the normalizing element \(x_j + N\), so Lemma 3.5 applies to \(S/N\). An easy adaptation of the proof of Lemma 3.1 to the simple \(S\)-\(S\) bimodule \(S\) then yields assertion (v).

Corollary 4.2. If \(P\) is a semiprime ideal of \(S\) and \(N\) is the prime radical in \(R\) of \(P \cap R\) then there exists a least integer \(k \leq n\) such that \(N^k \subseteq P \cap R\). If \(P\) is a semiprimitive ideal of \(S\) and \(M\) is the intersection of all primitive ideals of \(R\) containing \(P \cap R\) then there exists a least integer \(h \leq n\) such that \(M^h \subseteq P \cap R\).

Proof. This is clear from Theorem 4.1 and Lemma 2.1.

Recall that Example 1.7 was a case in which every prime ideal concerned was primitive, and in which the bound \(k = n\) \((= 2)\) was attained. If \(S\) is left (or right) free over \(R\) on \(x_1, \ldots, x_n\) then, as in the case of a finite normalizing extension, more can be said about \(S\). The corresponding result for normalizing extensions is in [8].

Theorem 4.6. (Nilpotent ideals). Let \(R\) be a semiprime ring and suppose that \(S\) is left (or right) free over \(R\) on \(x_1, \ldots, x_n\). Then \(S\) has a unique maximal nilpotent ideal \(T\) and \(T^n = 0\).

Proof. This is exactly the same as the proof of [11, Theorem 3.4] except that Passman's Lemmas 2.2, 1.6, and 1.4 are replaced, respectively, by our Lemmas 3.3, 1.18, and 1.16.
The following result parallels a structure theorem due to Lorenz [6] for finite normalizing extensions.

**Theorem 4.7.** Let $R$ be a prime ring and assume that $S$ is left free over $R$ on $x_1, \ldots, x_n$. Then $S$ has finitely many minimal primes $P_1, \ldots, P_k$ with $k \leq n$, and $T = P_1 \cap \cdots \cap P_k$ is its unique largest nilpotent ideal, and $T^n = 0$.

**Proof.** This is just like the proof of [11, Theorem 3.5] except that Passman's Lemmas 2.2, 2.3, 1.6, 1.4, and 1.5 are replaced, respectively, by our Lemmas 3.3, 3.4, 1.18, 1.16, and 1.17.

5. Chain Conditions

In this brief section we collect together several results about chain conditions and finite subnormalizing extensions. We assume except where otherwise stated that $R \subseteq S$ is a finite subnormalizing extension with respect to $X = \{x_1 \cdots x_n\}$.

**Lemma 5.1.** If $M$ is a right $S$-module and $N \subseteq M$ is an $R$-module such that $b(N) = 0$ then $M$ is a noetherian (artinian) $R$-module if and only if $M/N$ is a noetherian (artinian) $R$-module.

**Proof.** If $V_0(N), V_1(N), \ldots, V_n(N)$ have the same meanings as in Sect. 2, we have $0 = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0 = M$, and each factor $V_i/V_{i+1}$ is $R$-noetherian ($R$-artinian) provided that $M/N$ is, by Lemma 2.3. If $M/N$ is not $R$-noetherian ($R$-artinian) then $M$ certainly is not.

**Theorem 5.2.** (i) If $M$ is a right $S$-module then $M$ is $S$-noetherian if and only if $M$ is $R$-noetherian; in particular $S$ is a right noetherian ring if and only if $R$ is right noetherian.

(ii) If $M$ is a right $S$-module then $M$ has a composition series as $S$-module if and only if $M$ has a composition series as $R$-module; in particular $S$ is a right artinian ring if and only if $R$ is a right artinian ring;

(iii) $S$ is bi-noetherian if and only if $R$ is bi-noetherian;

(iv) $S$ is bi-noetherian and bi-artinian if and only if $R$ is bi-noetherian and bi-artinian.

**Proof.** Assertion (i) follows as in [2] and assertion (ii) from Lemma 3.1 combined with the observation that $S$ is right artinian if and only if it has a composition series as $S$-module. Assertions (iii) and (iv) then follow from applying (i) and (ii) to the $S^{\text{op}} \otimes S$ module $SS_S$.

The corresponding results to those of this section for finite normalizing extensions were originally developed in [13], and alternative proofs are
given in [2]. In [14] it was shown using Krull dimension techniques that for a finite normalizing extension $R \subseteq S$ a right $S$-module $M$ is $S$-artinian if and only if it is $R$-artinian, and consequently $S$ is a bi-artinian ring if and only if $R$ is a bi-artinian ring. Modified proofs were given in [2]. The analogous problems for finite subnormalizing extensions do not seem to be susceptible to the methods of this paper.

6. COUNTER EXAMPLES

If $R \subseteq S$ is a finite normalizing extension, the following results are due to Bit-David and Robson:

**Proposition 6.1.** [1] If $M$ is an $S$-module and $N$ is an $R$-essential $R$-submodule of $M$ then $b(N)$ is $R$-essential in $M$.

**Proposition 6.2.** [1] If $Q$ is an ideal of $R$ essential as $R$-$R$ subbimodule of $R$ then there exists an ideal $I$ of $S$ such that $0 \neq I \cap R \subseteq Q$.

The proof of Proposition 6.2 depends fundamentally on (the bimodule version of) Proposition 6.1. Both propositions are false for finite subnormalizing extensions, as is shown by Example 6.5.

For a finite normalizing extension $R \subseteq S$ (with respect to $\{x_1, \ldots, x_n\}$), the following results are due to Lorenz [6], extending earlier work in [7].

**Proposition 6.3.** There exists an integer $t \geq 1$, depending only upon $n$, such for any right ideal $A$ of $R$ and any $s_1, s_2, \ldots, s_t \in AS$ we have $s_1 s_2 \cdots s_t = F(s_1, s_2, \ldots, s_t)$ where $F(s_i)$ is a sum of elements of $S$ of form $h_1 g_1 h_2 g_2 \cdots h_r g_r h_{r+1}$, where $1 \leq r < t$, each $h_i \in \{1\} \cup A$ and each $g_j \in \{s_1, s_2, \ldots, s_t\}$.

**Proposition 6.4.** With the above notation, if $A$ is a proper right ideal of $R$ then $AS \neq S$.

The second of these results follows easily from the first. Both propositions are false for finite subnormalizing extensions, as is also shown by

**Example 6.5.** Let $J, K, R, S, P, Q$ be as in Example 3.2. Then $S$ is a simple ring, and its only $R$-$R$ subbimodules are $S, R, P, Q, J, 0$. Recalling from Section 2 that we may treat bimodules rather like one-sided modules, we note that any nonzero $R$-$R$ subbimodule of $S$, other than $S$ itself, is an essential $R$-$R$ subbimodule of $S = SS_S$, but each has bound 0, which is not essential. Furthermore, each nonzero ideal of $R$ is essential as $R$-$R$ sub-
bimodule of \( R \), but there is no ideal \( I \) of \( S \) such that \( 0 \neq I \cap R \subseteq Q \) (or \( 0 \neq I \cap R \subseteq P \) or \( 0 \neq I \cap R \subseteq J \)). Finally, \( PS = S \), so Proposition 6.3 and 6.4 clearly do not extend to the case of finite subnormalizing extensions.

Although we have not studied a finite self-conjugate extension \( R \subseteq S \) in this paper, it is known that they can be very "badly" behaved. In [16] Resco gives an example of a ring extension \( R \subseteq S \) which is finite self-conjugate with respect to a certain set \( X \), but with \( S \) a left–right artinian simple ring and \( R \) not even noetherian on either side. Hence most of the results of Section 5 cannot be extended to the case of a finite self-conjugate extension. More recently [17, p. 34] Stafford gives an example of a prime left–right noetherian ring \( R \) and a ring embedding of \( R \) in \( S = M_2(R) \) such that \( S \) is free and finitely generated both as a left and a right \( R \)-module (so \( R \subseteq S \) is a finite self-conjugate extension), but \( S \) has a simple right \( S \)-module \( B \) such that \( B \) has infinite length as an \( R \)-module. This example makes explicit what was implicit in Resco's example, that the key results of Section 3 do not extent to the finite self-conjugate case.

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References


