Singularity analysis of a $p$-Ginzburg–Landau type minimizer

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Received 12 September 2007
Available online 20 April 2008

Abstract

This paper is concerned with the convergence of a $p$-Ginzburg–Landau type functional when the parameter goes to zero. By estimating the singularity of the energy and establishing the Pohozaev identity, we find the singularity of the energy concentrates on the domain near the singularities of a $p$-harmonic map.

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MSC: 35B25; 35J70; 49K20

Keywords: Singularity analysis; Ginzburg–Landau type functional; Regularized minimizer; $p$-harmonic map; Pohozaev’s identity

1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary $\partial G$. And let $g$ be a smooth map from $\partial G$ to $S^1$ and satisfy $\text{deg}(g, \partial G) = d \neq 0$. Without loss of generality, we may assume $d > 0$. We are concerned with the asymptotic behavior of the minimizer $u_\varepsilon$ of the $p$-Ginzburg–Landau type functional

$$E_\varepsilon(u, G) = \int_G e_\varepsilon(u)$$

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doi:10.1016/j.bulsci.2008.02.004
in the space \( W = \{ v \in W^{1,p}(G, \mathbb{R}^2); \, v|_{\partial G} = g \} \), where \( p > 2 \) and
\[
epsilon(u) = \frac{1}{p} |\nabla u|^p + \frac{1}{4\varepsilon^p} (1 - |u|^2)^2.
\]

When \( p = 2 \), lots of papers devote to the asymptotic behavior of minimizers \( u_\varepsilon \) of \( E_\varepsilon(u, G) \) in \( W \) as \( \varepsilon \to 0 \) (cf. [1–3,10], etc.). It was shown in those cases that \( u_\varepsilon \) converges strongly to a harmonic map \( u_0 \) on any compact subset away from the zeros (which are called the vortices in the theory of superconductivity, superfluids and XY-magnetism, etc.). In addition, Chapter VII in [2] shows the global convergence of \( |\log \varepsilon|^{-1} e_\varepsilon(u_\varepsilon) \) via investigating the properties near the singularities of \( u_0 \).

An analogous result in the case of \( p = n \) was obtained in [5], and a weaker form was obtained in [7], where \( n \geq 2 \) is the dimension. In the case of \( 1 < p < n \), the convergence of \( e_\varepsilon(u_\varepsilon) \) was also studied in [12]. The motivation for this paper is to generalize those ideas to the case \( p > n \).

We expect to show that, for the so-called regularized minimizer \( u_\varepsilon \), a subsequence of \( \varepsilon^{p-2} e_\varepsilon(u_\varepsilon) \) converges to a measure in the weak * topology of \( C(\overline{G}) \). It is helpful for understanding well the location of singularities of the \( p \)-harmonic maps.

When \( p > n = 2 \), [8] investigated the asymptotic behavior of the energy functional. We state the main results as follows. Assume \( u_\varepsilon \) is a minimizer of \( E_\varepsilon(u, G) \) in \( W \). Then the zeros of \( u_\varepsilon \) are included in finite disjoint bad discs \( B(x^j, h\varepsilon), \, j = 1,2,\ldots, N_1 \), where \( N_1 \) and \( h > 0 \) are independent of \( \varepsilon \in (0,1) \). As \( \varepsilon \to 0 \), there exist a subsequence \( x^j_{\varepsilon k} \) of the center \( x^j_\varepsilon \) and \( a_i \in \overline{G} \) such that \( x^j_{\varepsilon k} \to a_i, \, i = 1,2,\ldots, N_1 \). Since there may be at least two subsequences that converge to the same point, we denote by \( a_1, a_2,\ldots, a_N \) \( (N \leq N_1) \). It is convenient to enlarge \( G \) a little. Assume \( G' \subset \mathbb{R}^2 \) is a bounded, simply connected domain with smooth boundary such that \( \overline{G} \subset G' \). We can find a smooth map \( \tilde{g} : (G' \setminus G) \to S^1 \) such that \( \tilde{g} = g \) on \( \partial G \). Extend the definition domain of each element in \{ \( u : G \to \mathbb{R}^2; \, u|_{\partial G} = g \) \} to \( G' \) such that
\[
u = \tilde{g} \quad \text{on} \quad G' \setminus G. \quad (1.1)
\]

In particular, the minimizer \( u_\varepsilon \) can be defined on \( G' \). Suppose \( K \) is an arbitrary compact subset of \( G \setminus \bigcup_{i=1}^N \{ a_i \} \). Then there exists a subsequence \( u_{\varepsilon k} \) of \( u_\varepsilon \) such that as \( k \to \infty \), \( u_{\varepsilon k} \to u_p \) in \( W^{1,p}(K, \mathbb{R}^2) \), where \( u_p \) is a map of the least p-energy \( \int_K |\nabla u|^p \) in \( W^{1,p}(K, S^1) \). In addition, it is also shown that, there exists \( C > 0 \) \( (\text{independent of } \varepsilon) \), such that
\[
1/2 \leq |u_\varepsilon(x)| \leq 1, \quad \text{for } x \in K; \quad (1.2)
\]
\[
E_\varepsilon(u_\varepsilon, K) \leq C. \quad (1.3)
\]

When \( p > 2 \), the minimizer of \( E_\varepsilon(u, G) \) may be not \( C^2 \)-smooth, since its Euler–Lagrange system is degenerate. It is not convenient to deal with the singularities of \( u_\varepsilon \) by means of Euler–Lagrange system, such as applying Pohozaev’s identity or \( C^{1,\alpha} \)-regularity of \( u_\varepsilon \), etc. To overcome this difficulty, we will make research on one of the minimizers, the regularized minimizer, which was introduced in [7] by following Uhlenbeck’s idea. Clearly, a minimizer \( u_\varepsilon^\eta \) of the regularized functional
\[
E_\varepsilon^\eta(u, G) = \frac{1}{p} \int_G v^{p/2} + \frac{1}{4\varepsilon^p} \int_G (1 - |u|^2)^2
\]
in \( W \) is a classical solution to
\[
-\text{div}(v^{p-2}/2\nabla u) = \frac{1}{\varepsilon^p} u(1 - |u|^2), \quad \text{on } G, \quad (1.4)
\]
where \( v = |\nabla u|^2 + \eta \) with \( \eta \in (0, 1) \). In addition, \( |u_\varepsilon'| \leq 1 \) a.e. on \( G \) (cf. the proof of Theorem 2.2 in [7]). Letting \( \eta \to 0 \), we can find a subsequence \( \eta_k \) of \( \eta \), such that

\[
u^n_k \to u_\varepsilon, \quad \text{in} \ W^{1,p}(G, \mathbb{R}^2),
\]

where \( u_\varepsilon \) is also a minimizer of \( E_\varepsilon(u, G) \) in \( W \). It is called a regularized minimizer (cf. [7]). Clearly, (1.2) and (1.3) are still true if we replace \( u_\varepsilon \) by \( u_\varepsilon^n \). In addition, by an analogous argument in [9], from (1.2) and (1.3) we can deduce that there is a subsequence of regularized minimizer \( u_\varepsilon \) denoted by itself, such that as \( \varepsilon \to 0 \),

\[
u_\varepsilon \to u_p, \quad \text{in} \ C^{1,\alpha}_{\text{loc}}\left(G \setminus \bigcup_{i=1}^N [a_i]\right)
\]

for some \( \alpha \in (0, 1) \). By virtue of (1.6), we may call these points \( a_1, a_2, \ldots, a_N \) the singularities of the \( p \)-harmonic map \( u_p \). In this paper, we will investigate the limit of \( \varepsilon^{p-2}e_\varepsilon(u_\varepsilon) \) with the regularized minimizer \( u_\varepsilon \). From (1.3) we know that the main difficulty should be how to estimate

\[
\lim_{\varepsilon \to 0} \varepsilon^{p-2}E_\varepsilon(u_\varepsilon, B(a_j, \sigma)), \quad \text{i.e. the energy near the singularities} \ a_j \quad (j = 1, 2, \ldots, N).
\]

One of the main results, as follows, is proved in Section 3 based on the consequences listed in Section 2.

**Theorem 1.1.** Assume \( u_\varepsilon \) is a regularized minimizer of \( E_\varepsilon(u, G) \) on \( W \). Then \( \varepsilon^{p-2}e_\varepsilon(u_\varepsilon) \) is bounded in \( L^1(G) \). In addition, when \( \varepsilon \to 0 \), there exists a subsequence \( \varepsilon_k \) of \( \varepsilon \), such that

\[
\frac{1}{4\varepsilon_k^2}(|u_\varepsilon|^2 - 1)^2 \to \sum_{j=1}^N L_j \delta_{a_j}, \quad \text{weakly * in} \ C(\overline{G}),
\]

\[
\varepsilon_k^{p-2} |\nabla u_\varepsilon|^p \to \frac{2p}{p-2} \sum_{j=1}^N L_j \delta_{a_j}, \quad \text{weakly * in} \ C(\overline{G}).
\]

Here \( \delta_{a_j} \) is the Dirac mass at \( a_j \), and

\[
\sum_{j=1}^N L_j \in \left[ \pi \frac{\sum_{j=1}^N [k_j]H^{2-p}}{p} \left( 1 - \frac{2}{p} \right) I(1,1) d + \frac{2\pi d}{p^2} \right],
\]

where \( k_j = \text{deg}(u_\varepsilon, a_j) \), \( H \) is a positive constant which is independent of \( \varepsilon \), and

\[
I(1,1) = \text{Min} \left\{ \int_{B_1(0)} \left[ \frac{1}{p} |\nabla u|^p + \frac{1}{4} (1 - |u|^2)^2 \right] ; u \in W^{1,p}(B_1(0), \mathbb{R}^2), \ u(x)|_{\partial B_1} = x \right\}.
\]

Remark 1. When \( p = 2 \), (1.9) implies \( \frac{\pi}{2} \sum_{j=1}^N [k_j] \leq \sum_{j=1}^N L_j \leq \frac{\pi d}{2} \). In view of \( d = \sum_{j=1}^N k_j \), we can get \( k_j \geq 0 \). Therefore, \( \sum_{j=1}^N L_j = \frac{\pi d}{2} \). This is consistent with Theorem VII.2 in [2]. When \( p \) is not equal to the dimension 2, the energy \( \int |\nabla u|^p \) is not invariant under conformal transformations. Thus, unlike the results in [2] and [3], the upper and the lower bounds of \( \sum_{j=1}^N L_j \) may not be optimal. In addition, the relation between \( N \) and the degree is not clear. So, there is no estimate on each \( L_j \), but just on their sum. Now, the singularity of energy functional \( E_\varepsilon(u_\varepsilon, G) \) appears not only on the first term \( \int_G |\nabla u_\varepsilon|^p \), but also on the second term \( \frac{1}{p} \int_G (1 - |u_\varepsilon|^2)^2 \). It is much more complicated to estimate the lower bound of \( \sum_{j} L_j \) than to do it in the case of \( p = 2 \) in [3].
Remark 2. As in [2], we also use the Pohozaev identity to prove Theorem 1.1. However, we need not suppose that $G$ is star-shaped when $p > 2$, since the singularity of the energy does not come from the integrals on the boundary $\partial G$ (see the proof of (1.8)). The same reason also shows that the energy $\varepsilon^p E_\varepsilon(u_\varepsilon, G)$ can be expressed by the terms containing the integrals on the domain $G$. It is different from the case of $p = 2$ (cf. (35) and (36) in Chapter VII of [2, p. 73]).

Since the conformal transformation of $E_\varepsilon(u, G)$ is lost, it is very difficult to verify $a_j \in \partial G$ for each $j$. On the other hand, we will investigate the limit of $\varepsilon^p E_\varepsilon(u_\varepsilon, G)$. When $p = 2$, by applying $|\nabla u_\varepsilon| \leq C\varepsilon^{-1}$ (cf. [1, (33)]), we can see easily 

\[ \frac{1}{|\ln \varepsilon|} \int B(x_\varepsilon^j, h_\varepsilon) |\nabla u_\varepsilon|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

When $p > 2$, it is difficult to verify whether the limit is zero when $B(x_\varepsilon^j, h_\varepsilon)$ is a bad disc. However, we will prove in Section 4 the theorem:

**Theorem 1.2.** If $\lim_{\varepsilon \to 0} x_\varepsilon^j = a_j \in \partial G$, then

\[ \lim_{\varepsilon \to 0} \varepsilon^p - \frac{2}{\varepsilon} \int_{B(x_\varepsilon^j, h_\varepsilon)} |\nabla u_\varepsilon|^p = 0, \]

where $\varepsilon = \varepsilon_k$ is some subsequence.

**Remark 3.** According to the definition of the bad disc (cf. §2 in [8]),

\[ \frac{1}{4\varepsilon^p} \int_{B(x_\varepsilon^j, h_\varepsilon)} (1 - |u_\varepsilon|^2)^2 \geq \mu \varepsilon^{2-p} \to \infty, \]

when $\varepsilon \to 0$. Theorem 1.2 shows that the energy $\varepsilon^p E_\varepsilon(u_\varepsilon, B(x_\varepsilon^j, h_\varepsilon))$ concentrates on its second term $\frac{1}{4\varepsilon^2} \int_{B(x_\varepsilon^j, h_\varepsilon)} (1 - |u_\varepsilon|^2)^2$ when $\lim_{\varepsilon \to 0} x_\varepsilon^j = a_j \in \partial G$.

Far away from the singularities, we will investigate the convergence rate of the module of the minimizer converging to 1. Based on this result, we will prove in Section 5 the functional $E_\varepsilon(u_\varepsilon, K)$ converges to $p$-energy $\frac{1}{p} \int_K |\nabla u_p|^p$, where $K$ is an arbitrary compact subset of $G \setminus \{a_i\}_{i=1}^N$.

**Theorem 1.3.** Assume $u_\varepsilon$ is a regularized minimizer of $E_\varepsilon(u, G)$ in $W$. Then for any compact subset $K$ of $G \setminus (\bigcup_{j=1}^N \{a_j\})$, there exists a positive constant $C$, such that as $\varepsilon \in (0, \varepsilon_0)$,

\[ \int_K \left[ |\nabla u_\varepsilon|^p + \frac{1}{\varepsilon^p} (1 - |u_\varepsilon|^2)^2 \right] \leq C\varepsilon^p, \]

\[ \left| \int_K \left( \frac{1 - |u_\varepsilon|^2}{\varepsilon^p} - |\nabla u_\varepsilon|^p \right) \right| \leq C\varepsilon, \]

where $\varepsilon_0$ is sufficiently small. Furthermore, when $\varepsilon \to 0$,

\[ E_\varepsilon(u_\varepsilon, K) \to \frac{1}{p} \int_K |\nabla u_p|^p, \]

where $u_p$ is the $p$-harmonic map in (1.6).
Remark 4. Eq. (1.11) shows a convergence rate as in [1, 14]). Since $u_\varepsilon$ is not $C^2$-smooth, this consequence is weaker than the case of $p = 2$. In addition, if we notice that

$$E_\varepsilon(u, K) = \frac{1}{p} \int_K \left( |\nabla u|^2 + |u|^2 \right)^{p/2} + \frac{1}{4\varepsilon^p} \int_K (1 - |u|^2)^2,$$

the estimation (1.10) and the convergence (1.12) show that the energy functional $E_\varepsilon(u_\varepsilon, K)$ concentrates to the term $\frac{1}{p} \int_K |\nabla u_\varepsilon|^p$ when $\varepsilon$ is sufficiently small.

2. Preliminaries

Assume $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$ in $W$. Clearly, it satisfies the Euler–Lagrange system

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p} u\left(1 - |u|^2\right), \quad \text{on } G$$

in the weak sense. It is not difficult to prove $|u_\varepsilon| \leq 1$ a.e. in $\bar{G}$.

Proposition 2.1. Let $u_\varepsilon \in W$ be a weak solution to (2.1). Then there exists $\rho_0 > 0$, such that for any $\rho \in (0, \rho_0)$,

$$|u_\varepsilon(x)| \geq \frac{1}{2}, \quad \text{for } x \in G \setminus G^{2\rho_\varepsilon},$$

where $G^{\rho_\varepsilon} = \{ x \in G; \text{dist}(x, \partial G) > \rho_\varepsilon \}$. Moreover, there exists $C = C(\rho_0) > 0$ (independent of $\varepsilon$), such that

$$\|\nabla u_\varepsilon\|_{L^\infty(B(x, \frac{1}{4}\rho_\varepsilon))} \leq C \varepsilon^{-1}, \quad x \in G^{\rho_\varepsilon}.$$  \hspace{1cm} (2.2)

Proof. Let $y = x\varepsilon^{-1}$ in (2.1) and denote $v(y) = u(x), \ G_\varepsilon = \{ y = x\varepsilon^{-1}; \ x \in G \}$. Then

$$-\text{div}(|\nabla v|^{p-2}\nabla v) = \left(1 - |v|^2\right), \quad \text{on } G_\varepsilon.$$  \hspace{1cm} (2.3)

This implies that $v(y)$ is a weak solution to (2.3). By using the standard argument of the Hölder continuity of the weak solution to (2.3) near the boundary (for example, cf. Theorem 1.1 and lines 19–21 of p. 104 in [4]), we can see that for any $y_0 \in \partial G_\varepsilon$ and $y \in B(y_0, \rho_0)$ (where $\rho_0 > 0$ is a constant independent of $\varepsilon$), there exist positive constants $C = C(\rho_0)$ and $\alpha \in (0, 1)$ which are independent of $\varepsilon$, such that $|v(y) - v(y_0)| \leq C(\rho_0)|y - y_0|^{\alpha}$. Choose $\rho > 0$ sufficiently small to satisfy $y \in B(y_0, 2\rho) \subset B(y_0, \rho_0)$, and $C(\rho_0)|y - y_0|^{\alpha} \leq \frac{1}{4}$, then $|v(y)| \geq |v(y_0)| - C(\rho_0)|y - y_0|^{\alpha} = 1 - C(\rho_0)|y - y_0|^{\alpha} \geq \frac{3}{4}$. Let $x = x_\varepsilon$. Thus $|u_\varepsilon(x)| \geq \frac{1}{2}$ for $x \in B(x_0, 2\rho_\varepsilon) \cap G$, where $x_0 \in \partial G$.

To prove (2.2), we test (2.3) by $v^p \xi, \ \xi \in C_0^\infty(G_\varepsilon, R)$. Then

$$\int_{G_\varepsilon} |\nabla v|^p \xi \ dy \leq p \int_{G_\varepsilon} |\nabla v|^{p-1} |\nabla \xi| |v| \ dy + \int_{G_\varepsilon} |v|^2 (1 - |v|^2) \xi^p \ dy.$$  \hspace{1cm} (2.4)

For any $\rho \in (0, \rho_0)$, take $y \in G_\varepsilon \setminus G^{\rho_\varepsilon}, \ B(y, \rho/2) \subset G_\varepsilon$. Setting $\zeta = 1$ in $B(y, \rho/4), \ \zeta = 0$ in $G_\varepsilon \setminus B(y, \rho/2)$ and $|\nabla \zeta| \leq C(\rho)$, we have

$$\int_{B(y, \rho/2)} |\nabla v|^p \xi \ dy \leq C(\rho) \int_{B(y, \rho/2)} |\nabla v|^{p-1} \xi^{p-1} \ dy + C(\rho).$$  \hspace{1cm} (2.5)
Using Hölder’s inequality we can derive \( \int_{B(y,\rho/4)} |\nabla v|^p \, dy \leq C(\rho) \). Combining this with the Tolksdorf’s theorem in [11] (p. 244, lines 19–23) yields
\[
\|\nabla v\|_{L^\infty(B(y,\rho/8))} \leq C(\rho) \int_{B(y,\rho/4)} (1 + |\nabla v|)^p \, dy \leq C(\rho)
\]
which implies (2.2) if we let \( x = y\varepsilon \). Proposition 2.1 is proved. \(\square\)

**Theorem 2.2.** Assume \( u_\varepsilon \) is a regularized minimizer. For any compact subset \( K \subset G \setminus \{a_i\}_{i=1}^N \), there exists a constant \( C > 0 \) such that
\[
\left\| \frac{1}{\varepsilon^2} (1 - |u_\varepsilon|)^2 \right\|_{C(K)} \leq C \varepsilon^{2p-2}.
\]

**Proof.** Multiplying (1.4) by \( u = u_\eta^\varepsilon \) and using (1.2), we have
\[
\frac{1}{\varepsilon^p} (1 - |u|^2) \leq C \left[ v^{(p-2)/2} |\nabla u|^2 + \varepsilon^p \text{div}(v^{(p-2)/2} \nabla \psi) \right],
\]
where \( \psi = \psi_\eta^\varepsilon = \frac{1}{\varepsilon^p} (1 - |u|^2) \). At the point \( x_0 \) where \( \psi \) achieves its maximum in \( K \), we have \( \nabla \psi(x_0) = 0 \) and \( \Delta \psi(x_0) \leq 0 \). Hence,
\[
\psi(x_0) \leq C \left( v(x_0) \right)^{(p-2)/2} |\nabla u(x_0)|^2.
\]\n(2.4)

According to Proposition 5.2 in [9], the right-hand side of (2.4) is bounded. Namely, \( \psi(x) \leq \psi(x_0) \leq C \), where \( C > 0 \) is independent of \( \varepsilon \) and \( \eta \). Based on this result, applying Tolksdorf’s theorem in [11] to (1.4) yields \( |u_\eta^\varepsilon|_{C^{1,\alpha}(K)} \leq C \) with \( \alpha \in (0, 1) \), where \( C > 0 \) is independent of \( \varepsilon \) and \( \eta \). Inserting it into (2.4), we can deduce that
\[
\frac{1}{\varepsilon^p} (1 - |u_\eta^\varepsilon(x)|) \leq \psi(x) \leq \psi(x_0) \leq C
\]
with \( C > 0 \) independent of \( \varepsilon \) and \( \eta \). Letting \( \eta \to 0 \) and using \( |u_\eta^\varepsilon|_{C^{1,\alpha}(K)} \leq C \) and (1.5), we have \( 1 - |u_\varepsilon| \leq C \varepsilon^p \). This result implies the consequence of Theorem 2.2. \(\square\)

**Proposition 2.3.** Assume \( u_\varepsilon \) is a solution to (1.4). Then for any disc \( B \subset G \),
\[
- \int_{\partial B} |x| v^{(p-2)/2} |\partial_v u|^2 \, ds + \int_B v^{(p-2)/2} |\nabla u|^2 - \frac{2}{p} \int_B v^{p/2} + \frac{1}{p} \int_{\partial B} |x| v^{p/2} \, ds
\]
\[
= - \frac{1}{4\varepsilon^p} \int_{\partial B} |x| (1 - |u|^2)^2 \, ds + \frac{1}{2\varepsilon^p} \int_B (1 - |u|^2)^2,
\]\n(2.5)
where \( v = |\nabla u|^2 + \eta \) with \( \eta \in (0, 1) \).

**Proof.** As in the proof of the Pohozaev identity, multiplying the regularized system (1.4) by \( x \cdot \nabla u \) yields
\[
- \int_B \nabla \cdot (v^{(p-2)/2} \nabla u)(x \cdot \nabla u) = \frac{1}{\varepsilon^p} \int_B (1 - |u|^2) u(x \cdot \nabla u).
\]
Integrating by parts, we obtain
\[-\int_{\partial B} |x|^2 |\partial_v v|^2 \, ds + \int_B v^{(p-2)/2} \nabla u \nabla (x \cdot \nabla u) \]
\[= -\frac{1}{4 \varepsilon p} \int_{\partial B} (1 - |u|^2)^2 |x| \, ds + \frac{1}{2 \varepsilon p} \int_B (1 - |u|^2)^2. \]

Substituting
\[
\int_B v^{(p-2)/2} \nabla u \nabla (x \cdot \nabla u) = \int_B v^{(p-2)/2} \left( |\nabla u|^2 + \frac{1}{2} x \cdot \nabla v \right)
\]
\[= \int_B v^{(p-2)/2} |\nabla u|^2 + \frac{1}{p} \int_B x \cdot v^{p/2}
\]
\[= \int_B v^{(p-2)/2} |\nabla u|^2 + \frac{1}{p} \int_{\partial B} |x| v^{p/2} \, ds - \frac{2}{p} \int_B v^{p/2}
\]
into the equality above, we can obtain (2.5). Proposition 2.3 is complete. \(\Box\)

For each disc \(B \subset \mathbb{R}^2\), we write \(D = B \cap G\). Replacing \(B\) in the proof of Proposition 2.3 by \(D\), we also deduce another Pohozaev identity:
\[
-\int_{\partial D} (x \cdot \nu) v^{(p-2)/2} |\partial_v v|^2 \, ds + \int_D v^{(p-2)/2} |\nabla u|^2 - \frac{2}{p} \int_D v^{p/2} + \frac{1}{p} \int_{\partial D} x \cdot v^{p/2} \, ds
\]
\[= -\frac{1}{4 \varepsilon p} \int_{\partial D} (1 - |u|^2)^2 |x| \, ds + \frac{1}{2 \varepsilon p} \int_D (1 - |u|^2)^2, \tag{2.6} \]
where \(v\) is the unit outward norm vector on \(\partial D\).

Fix a small constant \(\sigma > 0\) such that \(B(a_j, \sigma) \subset G', \ j = 1, 2, \ldots, N; \ 4\sigma < |a_j - a_i|, \ i \neq j; \ 4\sigma < \text{dist}(G, \partial G')\). Set \(J = \{i; \ B(x_{\varepsilon_k}^i)\ \text{is a bad disc}\}, \ \text{and} \ \Lambda_j = \{i \in J : x_{\varepsilon_k}^i \rightarrow a_j\}, \ j = 1, 2, \ldots, N\). We have
\[
\bigcup_{i \in \Lambda_j} B(x_{\varepsilon_k}^i, h_{\varepsilon_k}) \subset B(a_j, \sigma), \quad j = 1, 2, \ldots, N,
\]
\[
\bigcup_{j \in J} B(x_{\varepsilon_k}^j, h_{\varepsilon_k}) \subset \bigcup_{j=1}^N B(a_j, \sigma/4),
\]
\[
B(x_{\varepsilon_k}^i, h_{\varepsilon_k}) \cap B(x_{\varepsilon_k}^j, h_{\varepsilon_k}) = \emptyset, \quad i, j \in J, \ i \neq j,
\]
as long as \(\varepsilon_k\) is sufficiently small. Let \(u_\varepsilon\) be a minimizer of \(E_\varepsilon(u, G)\) and denote \(d_{i}^\varepsilon = \deg(u_\varepsilon, \partial B(x_{\varepsilon_k}^i, h_{\varepsilon_k})), k_j^\varepsilon = \deg(u_\varepsilon, \partial B(a_j, \sigma))\), thus
\[
k_j^\varepsilon = \sum_{i \in \Lambda_j} d_{i}^\varepsilon, \quad d = \sum_{j=1}^N k_j^\varepsilon. \tag{2.7}
\]

**Proposition 2.4.** There exists a constant \(C > 0\) which is independent of \(\varepsilon_k\) such that
\[
|d_{i}^\varepsilon| \leq C, \quad i \in J; \quad |k_j^\varepsilon| \leq C, \quad j = 1, 2, \ldots, N. \tag{2.8}
\]
Furthermore, \( d_i^\epsilon \equiv d_i \) and \( k_j^\epsilon \equiv k_j \) for each \( i \) and \( j \), when \( \epsilon \) is sufficiently small. Namely, they are independent of \( \epsilon \).

**Proof.** Since \( u = u_\epsilon \) is a weak solution to (2.1), according to [4, Theorem 1.1 and Line 19–21 of page 104], we know \( u \in C(\partial B(x_i^\epsilon, h_\epsilon), \mathbb{R}^2) \). In addition, [8, (2.11)] implies \( |u| \geq 1/2 \) on \( \partial B(x_i^\epsilon, h_\epsilon) \), such that \( \frac{u}{|u|} \in C(\partial B(x_i^\epsilon, h_\epsilon), S^1) \). According to the definition of degree, we have

\[
\left| d_i^\epsilon \right| \leq \frac{1}{2\pi} \int_{\partial B(x_i^\epsilon, h_\epsilon)} \left| \partial_t \left( \frac{u}{|u|} \right) \right| ds.
\]

In view of \( B(x_i^\epsilon, h_\epsilon) \cap B(x_j^\epsilon, h_\epsilon) = \emptyset \) if \( i \neq j \), there exists a small constant \( \delta \in (0, 1) \) such that \( |u| \geq 1/2 \) on \( [B(x_i^\epsilon, (h + \delta)\epsilon)] \cap G^\epsilon \). Hence, by (2.2), for \( \xi \in (h, h + \delta) \), there exists a constant \( C > 0 \) which is independent of \( \epsilon_k \), such that

\[
\left| d_i^\epsilon \right| \leq \frac{1}{2\pi} \int_{\partial B(x_i^\epsilon, \xi\epsilon) \cap G^\epsilon} \left| \partial_t \left( \frac{u}{|u|} \right) \right| \leq C.
\]

Combining this with (2.7) we can complete the proof of (2.8).

By virtue of (2.8), for each \( j \), there exist \( k_j^\epsilon \) which is independent of \( \epsilon_k \), and a subsequence of \( k_\epsilon j \) denoted by itself, such that \( k_\epsilon j \rightarrow k_j \) as \( \epsilon_k \rightarrow 0 \). Since \( k_\epsilon j \) and \( k_j \) are positive integer, \( \{k_\epsilon j\} \) must be constant sequence when \( \epsilon_k \) is sufficiently small, namely \( k_\epsilon j \equiv k_j \). The same reason shows \( d_\epsilon i \) can be written as \( d_i \) (which is also a number independent of \( \epsilon_k \)). Proposition 2.4 is proved.

Write \( \Omega' = G' \setminus \bigcup_{j=1}^{N} B(a_j, \sigma) \). Fixing \( j \in \{1, 2, \ldots, N\} \) and taking \( i_0 \in \Lambda_j \), we have

\[
\left. x_i \rightarrow a_j \right|_{\epsilon \rightarrow 0} \text{ holds with } \epsilon \text{ small enough.}
\]

Denote \( \Omega_j = B(a_j, \sigma) \setminus \bigcup_{i \in \Lambda_j} B(x_i^\epsilon, h_\epsilon), \bigcup \Omega_j = \bigcup_{i \in \Lambda_j} B(x_i^\epsilon, h_\epsilon) \). To estimate the lower bound of \( \|\nabla u_\epsilon\|_{L^p(\Omega_j)} \), the following proposition is necessary.

**Proposition 2.5.** Let \( A_{s,t}(x_i) = (B(x_i, s) \setminus B(x_i, t)) \cap G \) with \( s \leq t < R \). Assume that \( u \in W \). If

\[
\frac{1}{2} \leq |u| \leq 1 \quad \text{on } A_{s,t}(x_i),
\]

then

\[
\int_{A_{s,t}(x_i)} |\nabla u|^p \geq \frac{2\pi}{2-p} |d_i|^p (s^{2-p} - t^{2-p}) - C \int_{A_{s,t}(x_i)} (1 - |u|^p) |\nabla \frac{u}{|u|}|^p,
\]

where \( C \) is a constant which is independent of \( \epsilon \) and \( d_i \) is the degree of \( u \) on each \( \partial (B(x_i, r) \cap G) \), \( t \leq r \leq s \).

**Proof.** By virtue of (2.9), we can write \( w(x) = w(r, \theta) = u(r, \theta) / |u(r, \theta)| \). Clearly,

\[
|\nabla u|^2 = \left| \nabla |u| \right|^2 + |u|^2 \left| \nabla \frac{u}{|u|} \right|^2 \geq |u|^2 r^{-2} |\partial_r w|^2.
\]

Hence, for \( A = A_{s,t}(x_i) \),
\[
\int_A |\nabla u|^p \geq \int_t \int_{S^1} |\partial_\tau w|^p \, dS \, dr - \int_A \left(1 - |u|^p\right)|\nabla w|^p. \tag{2.10}
\]

According to the definition of degree and by Hölder’s inequality, we obtain

\[
 di = \frac{1}{2\pi} \int_{S^1} (w \wedge \partial_\tau w) \, ds \leq (2\pi)^{-1/p} \left(\int_{S^1} |\partial_\tau w|^p \, ds\right)^{1/p}.
\]

Substituting this into (2.10) yields

\[
\int_A |\nabla u|^p \geq \frac{2\pi}{p-2} (d_i)^p \left(\frac{1}{2^p} - s^2\right) - \int_A \left(1 - |u|^p\right)|\nabla w|^p.
\]

Proposition 2.5 is proved. \(\square\)

**Proposition 2.6.** Assume \(u_\varepsilon\) is a minimizer. Then

\[
\int_{\Omega_j} |\nabla u_\varepsilon|^p \geq \int_{\Omega_{j,\sigma}} |\nabla u_\varepsilon|^p \geq \frac{2\pi}{p-2} |k_j|(H_\varepsilon)^{2-p} - C, \tag{2.11}
\]

where \(C = C(\sigma, h, N)\) and \(H = H(h, N)\) are positive constants which are independent of \(\varepsilon\).

**Proof.** For \(R > 0\), set \(A = B(x_i, R) \setminus B(x_i, h\varepsilon)\). (1.2) implies (2.9) is true. By Proposition 2.5 and Hölder’s inequality, there holds

\[
\int_A |\nabla u_\varepsilon|^p \geq \frac{2\pi}{p-2} (d_i)^p (H_\varepsilon)^{2-p} - \left(\int_{G} \left(1 - |u_\varepsilon|^2\right)^2\right)^{(q-p)/q} \left(\int_A |\nabla w|^q\right)^{p/q} - C(R), \tag{2.12}
\]

for any \(q > p\). Using (1.2) and [8, (2.4)], we obtain

\[
\left(\int_{G} \left(1 - |u_\varepsilon|^2\right)^2\right)^{(q-p)/q} \left(\int_A |\nabla w|^q\right)^{p/q} \leq C\varepsilon^{2(q-p)/q} \left(\int_A |\nabla u_\varepsilon|^q\right)^{p/q}.
\]

According to [8, Propositions 3.1, 2.1], we can deduce that, from the inequality above,

\[
\left(\int_{G} \left(1 - |u_\varepsilon|^2\right)^2\right)^{(q-p)/q} \left(\int_A |\nabla w|^q\right)^{p/q} \leq C\varepsilon^{2(q-p)/q} \left(\int_{G} |\nabla u_\varepsilon|^q\right)^{p/q} \leq C\varepsilon^{2(q-p)/q + 2-p}
\]

with some \(q > p\). Inserting it into (2.12), and noting

\[
\frac{2\pi d_i^p}{p-2} h^{2-p} - C\varepsilon^{2(q-p)/q} \geq \frac{2\pi d_i^p}{p-2} (2h)^{2-p}
\]

when \(\varepsilon\) is sufficiently small, we get

\[
\int_A |\nabla u|^p \geq \frac{2\pi}{p-2} (d_i)^p (2h)^{2-p} - C(R). \tag{2.13}
\]

Here \(C(R)\) is independent of \(\varepsilon\).
Based on (2.13), by applying the technique in [10] and arguing as in the proof of Theorem 3.10 in [7], we can also deduce a lower bound. Indeed, set \( N_2 = \text{Card}\ A_j \). Suppose \( x_1, x_2, \ldots, x_{N_2} \) converge to \( a_j \), and \( d_{i,R} (i = 1, 2, \ldots, N_2) \) is the degree of \( u_\varepsilon \) around \( \partial B(x_i, R) \). Let \( R_\sigma^0 \) denote the set of all numbers \( R \in [\varepsilon, \sigma] \) such that \( \partial B(x_i, R) \cap B(x_j, \varepsilon) = \emptyset \) for all \( i \neq j \) and such that for some collection \( J_R \subset \{1, 2, \ldots, N_2\} \), satisfying \( J_R \subset J_R' \) if \( R' \leq R \), the family \( \{B(x_i, R)\}_{i \in J_R} \) is disjoint and

\[
\bigcup_{i=1}^{N} B(x_i, \varepsilon) \subset \bigcup_{i \in J_{R'}} B(x_i, R') \subset \bigcup_{i \in J_R} B(x_i, R), \quad R' \leq R.
\]

Note that \( R_\sigma^0 \) is the union of closed intervals \([R_0^l, R_l^l] \), \( 1 \leq l \leq L \), whose right endpoints correspond to a number \( R = R_l^l \) such that \( \partial B(x_i, R) \cap B(x_j, \varepsilon) \neq \emptyset \) for some pair \( i \neq j \in J_R \) and whose left endpoints correspond to a number \( R_0^l \) such that \( B(x_i, R_l^l) \setminus \bigcup_{j \in J_0} B(x_j, R_0^l) \neq \emptyset \) for \( i \in J_{R_l^l} \). \( J_R = J_l^l \) is a constant for \( R \in [R_0^l, R_l^l] \) and \( J_l^{l+1} \subset J_l^l, J_l^{l+1} \neq J_l^l \). Thus \( L \leq N \). Moreover, there exists a constant \( M = M(h) > 0 \) such that

\[
R_0^l \leq M \varepsilon, \quad R_l^l \geq \sigma / M, \quad R_0^{l+1} \leq M R_l^l
\]

for all \( l = 1, 2, \ldots, L - 1 \). Finally, observe that for all \( R \in R_\sigma^0 \) and \( J \in J_R \),

\[
|k_j| = \left| \sum_{i \in J_R} d_{i,R} \right| \leq \sum_{i \in J_R} |d_{i,R}|^p.
\]

Applying (2.15) and Proposition 2.5, we have

\[
\int_{\Omega_{J,\sigma}} |\nabla u_\varepsilon|^p \geq \sum_{l=1}^{L} \sum_{i \in J_{A_l^l \sigma}} \int_{A_l^l \sigma} |\nabla u_\varepsilon|^p
\]

\[
\geq \sum_{l=1}^{L} \sum_{i \in J_{A_l^l \sigma}} \frac{2\pi}{p - 2} |d_{i,R}| \left[ (R_0^l)^{2-p} - (R_l^l)^{2-p} \right] - C \int_{G} (1 - |u_\varepsilon|^p) |\nabla w|^p
\]

\[
\geq \frac{2\pi}{p - 2} |k_j| \sum_{l=1}^{L} \left[ (R_0^l)^{2-p} - (R_l^l)^{2-p} \right] - C \int_{G} (1 - |u_\varepsilon|^p) |\nabla w|^p.
\]

The second term of the right-hand side can be handled as in the corresponding term in (2.12). Similar to the derivation of (2.13), by (2.14) we can find a properly large constant \( H > 0 \) which only depends on \( M, N, h \), such that (2.11) holds. \( \square \)

3. Proof of Theorem 1.1

**Proof of (1.7).** In view of Proposition 2.3 in [8], \( V_\varepsilon := \frac{(1 - |u_\varepsilon|^2)^{\frac{1}{2}}}{4e \varepsilon^2} \) is bounded in \( L^1(G) \). Moreover, according to Theorem 2.2, there holds

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left( 1 - |u_\varepsilon| \right)^2 = 0, \quad \text{in } C(K)
\]

for any compact subset \( K \) of \( G \setminus \bigcup_{i=1}^{N} \{a_i\} \). In addition, (1.1) means \( 1 - |u_\varepsilon| = 0 \) in \( G' \setminus G \). Thus, we can find a subsequence \( \varepsilon_k \) of \( \varepsilon \), such that as \( k \to \infty \), \( V_{\varepsilon_k} \) converges weakly star in
C(\overline{G'}) to a measure \(\omega_1\) with \(\text{supp}(\omega_1) \subset \bigcup_j \{a_j\}\). Then, there exist \(L_j\) \((j = 1, 2, \ldots, N)\) such that \(\omega_1 = \sum_{j=1}^{N} L_j \delta_{a_j}\). Thus, (1.7) holds. According to the definition of the bad disc \(B(x_i^\varepsilon, \varepsilon)\), we have
\[
\frac{1}{\varepsilon^2} \int_{B(x_i^\varepsilon, \varepsilon)} \left(1 - |u_\varepsilon|^2\right)^2 > \mu > 0, \tag{3.1}
\]
which implies \(L_j > 0\). □

**Proof of (1.8).** By virtue of [8, Proposition 2.1] and (1.1), we see that \(\varepsilon^{p-2} |\nabla u_\varepsilon|^p\) is bounded in \(L^1(G')\). When \(\varepsilon \to 0\), there exists a subsequence \(\varepsilon_k\) of \(\varepsilon\) such that \(\varepsilon_k^{p-2} |\nabla u_{\varepsilon_k}|^p\) converges weakly star in \(C(\overline{G'})\) to a Radon measure \(\omega_2\). (1.3) and (1.1) lead to \(\text{supp}(\omega_2) \subset \bigcup_j \{a_j\}\).

Take \(\sigma\) sufficiently small such that \(B(a_j, 2\sigma) \subset G\) for each \(a_j \in G\). Since (1.3) is still true for \(u_\varepsilon\), by the mean value theorem we can also find \(r \in (\sigma, 2\sigma)\) such that
\[
\int_{\partial B(x,r)} v_{p/2}^2 / \varepsilon + \int_{\partial B(x,r)} \left(1 - |u_\varepsilon|^2\right)^2 \, ds \leq C. \tag{3.2}
\]
Let \(B = B(a_j, r)\) in (2.5). Multiply it with \(\varepsilon^{p-2}\) and let \(\varepsilon \to 0\). In view of (3.2), we see that the terms containing the integral on the boundary \(\partial B\) are vanishing. Thus, as \(\varepsilon \to 0\),
\[
\varepsilon^{p-2} \int_B v^{(p-2)/2} |\nabla u|^2 - \frac{2}{p} \varepsilon^{p-2} \int_B v^{p/2} - \frac{1}{2\varepsilon^2} \int_B (1 - |u|^2)^2 \to 0.
\]
Letting \(\eta \to 0\) and using (1.5), we have
\[
\lim_{\varepsilon \to 0} \left[ \left(1 - \frac{2}{p}\right) \varepsilon^{p-2} \int_B |\nabla u_\varepsilon|^p - \frac{1}{2\varepsilon^2} \int_B (1 - |u_\varepsilon|^2)^2 \right] = 0. \tag{3.3}
\]
Combining this with (1.7), we can see (1.8) for \(a_j \in G\).

When \(a_j \in \partial G\), \(B(a_j, 2\sigma) \setminus B(a_j, \sigma) \subset G'\) as \(\sigma\) is sufficiently small. In view of (1.1) and (1.3), one has \(E_\varepsilon(u_\varepsilon, B(a_j, 2\sigma) \setminus B(a_j, \sigma)) \leq C\). By the mean value theorem, we can find \(r \in (\sigma, 2\sigma)\) such that \(E_\varepsilon(u_\varepsilon, \partial B) \leq C\), where \(B = B(a_j, r)\). Taking \(D = B \cap G\) in (2.6), we also have
\[
- \int_{\partial (B \cap G)} x \cdot v u^{(p-2)/2} |\partial_s u|^2 / \varepsilon^{p/2} + \int_{B \cap G} v^{(p-2)/2} |\nabla u|^2 - \frac{2}{p} \int_{B \cap G} v^{p/2} \, ds + \frac{1}{p} \int_{\partial (B \cap G)} x \cdot v u^{p/2} / \varepsilon^{p/2} \, ds - \frac{1}{4\varepsilon^p} \int_{\partial (B \cap G)} x \cdot v (1 - |u|^2)^2 / \varepsilon^{p/2} \, ds + \frac{1}{2\varepsilon^p} \int_{B \cap G} (1 - |u|^2)^2. \tag{3.4}
\]
Thus, when we multiply (3.4) with \(\varepsilon^{p-2}\) and let \(\varepsilon \to 0\),
\[
\text{the terms containing the integral on } G \cap \partial B \text{ must be vanishing}. \tag{3.5}
\]
Next, we will estimate the integrals on \(B \cap \partial B\). For this purpose, we borrow the ideas of “blow-up” in [6, §6] or [3, §3]. Suppose \(0 \in G \cap B\). Otherwise, we can move the coordinate. Let \(y = \varepsilon^{-1} x \in G\) and write \(U(y) = u(x)\) on \(G_{\varepsilon^{-1}} = \{y = \varepsilon^{-1} x; \ x \in G\}\). According to Proposition 2.1 in [8], we know \(E_\varepsilon(u_\varepsilon, \partial B) \leq C \varepsilon^{2-p}\). Thus,
\[
\int_{G_{-1}} \left[ \frac{1}{p} (|\nabla U|^2 + \eta \varepsilon^2)^{p/2} + \frac{1}{4} (1 - |U|^2)^2 \right] \, dy \leq C.
\]

The following argument benefits from the idea in [6, §6]. Letting \( \varepsilon \to 0 \) in the consequence above yields \( \lim_{\varepsilon \to 0} \int_{G_{-1}} |\nabla U|^p \, dy \leq C \). In view of (1.1), we can see that

\[
\frac{(p - 2) \ln R}{2} \inf_{r \in [R, R^{p/2}]} \int_{\partial G_r} |z| |\nabla U|^p \, d\zeta \leq \int_{G_{R^{p/2}} \setminus G_R} |\nabla U|^p \, dy \leq C,
\]

where \( z \in \partial G_r \) and \( R = \varepsilon^{-1} \). Therefore, there exists a subsequence \( r_k \), such that \( r_k \) converges to \( \infty \) as \( R \to \infty \), and

\[
\lim_{k \to \infty} \int_{\partial G_{r_k}} |z| |\nabla U|^p \, d\zeta = \lim_{R \to \infty} \inf_{r \in [R, R^p]} \int_{\partial G_r} |z| |\nabla U|^p \, d\zeta \leq \lim_{R \to \infty} \frac{C}{\ln R} = 0.
\]

Since \( R' \) is continuous for \( t \in [1, p/2] \), we can find \( t_0 \in [1, p/2] \) such that \( r_k = R^{t_0} \) by the intermediate value theorem. Let \( z = y \varepsilon^{1-t_0} = x \varepsilon^{-t_0} \), and write \( \tilde{u}(z) = U(y) = u(x) \) and \( s = \zeta \varepsilon^{t_0} \).

When \( \varepsilon \to 0 \), from the result above we deduce that

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2t_0} \int_{B \cap \partial G} |x| |\nabla_y \tilde{u}(z)|^p \, ds = 0.
\]

Going back from \( \nabla_y \tilde{u}(z) \) to \( \nabla_x u(x) \) in the integral, we have

\[
\varepsilon^{p-2t_0} \int_{B \cap \partial G} |x| |\nabla u|^p \, ds \leq \varepsilon^{-2t_0} \int_{B \cap \partial G} |x| |\nabla_y \tilde{u}(z)|^p \, ds.
\]

Hence, we obtain from \( 0 \leq p - 2t_0 \leq p - 2 \) that, as \( \varepsilon \to 0 \),

\[
\lim_{\varepsilon \to 0} \varepsilon^{p-2} \left| \int_{B \cap \partial G} (x \cdot v)^{p/2} \, ds \right| = 0. \quad (3.6)
\]

Obviously, in view of \(|u| = |g| = 1\) on \( \partial G \), it follows that

\[
\int_{\partial G \cap B} x \cdot v (1 - |u|^2)^2 \, ds = 0. \quad (3.7)
\]

Multiplying (3.4) with \( \varepsilon^{p-2} \), letting \( \varepsilon \to 0 \), and applying (3.5)–(3.7), we can see that

\[
\varepsilon^{p-2} \int_{G \cap B} v^{(p-2)/2} |\nabla u|^2 - \frac{2}{p} \varepsilon^{p-2} \int_{G \cap B} v^{p/2} - \frac{1}{2\varepsilon^2} \int_{G \cap B} (1 - |u|^2)^2 \to 0.
\]

Letting \( \eta \to 0 \) and using (1.5), we have

\[
\lim_{\varepsilon \to 0} \left[ \left( 1 - \frac{2}{p} \right) \varepsilon^{p-2} \int_{G \cap B} |\nabla u_{\varepsilon}|^p - \frac{1}{2\varepsilon^2} \int_{G \cap B} (1 - |u_{\varepsilon}|^2)^2 \right] = 0. \quad (3.8)
\]

This result, together with (1.7), implies (1.8) for \( a_j \in \partial G \). □
Proof of (1.9). For $R > 0$, set
\[
I(\varepsilon, R) = \operatorname{Min} \left\{ \int_{B(0, R)} \left[ \frac{1}{p} |\nabla u|^p + \frac{1}{4\varepsilon^p} \left( 1 - |u|^2 \right)^2 \right] ; \ u \in W_R \right\},
\]
where $W_R = \{ u(x) \in W^{1,p}(B(0, R), \mathbb{R}^2) ; \ u(x)|_{\partial B(0, R)} = \frac{1}{R} \}$ from the proof of Proposition 2.1 in [8], and $u_\varepsilon = \overline{g}$ on $G' \setminus G$, we can deduce that
\[
\varepsilon^{p-2} E_\varepsilon(u_\varepsilon, G') \leq d \left( I(1, 1) + \frac{2\pi}{p(p-2)} \right) + C_\varepsilon^{p-2}. \tag{3.9}
\]
Here $I(1, 1) < \infty$ is a constant independent of $\varepsilon$, since $I(1, 1) \leq \int_{B(1)} \left[ \frac{1}{p} |\nabla x|^p + \frac{1}{4} (1 - |x|^2)^2 \right]$. Using (1.7) and (1.8), we get
\[
\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon) = \frac{2}{p-2} \sum_j L_j \delta_{a_j} + \sum_j L_j \delta_{a_j} = \frac{p}{p-2} \sum_j L_j \delta_{a_j}. \tag{3.10}
\]
Combining with (3.9), we obtain easily the upper bound of $\sum_{j=1}^N L_j$ in (1.9).

According to the results in §2 of [8], the bad discs $\{ B(x^\varepsilon_i, h^\varepsilon) ; \ i \in J_\varepsilon \}$ are disjoint. In view of (3.1), we have
\[
\frac{1}{\varepsilon^2} \int_{\bigcup_{j \in J_\varepsilon} B(x^\varepsilon_i, h^\varepsilon)} (1 - |u_\varepsilon|^2)^2 > N_1 \mu,
\]
where $N_1$ is the integer introduced in Section 1. Letting $\varepsilon \to 0$ and using (1.7), we see a lower bound of $\sum_{j=1}^N L_j > N_1 \mu$.

In order to obtain a more accurate lower bound, we will give an estimation of lower bound for the energy $\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, G)$. From (1.3) it follows $\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, G \setminus \bigcup_j B(a_j, \sigma)) = 0$. Therefore, the energy concentrates on $\varepsilon^{p-2} E_\varepsilon(u_\varepsilon, \bigcup_j B(a_j, \sigma) \cap G)$. Combining (3.3) and (3.8) with
\[
\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon \left( u_\varepsilon, \bigcup_j B(a_j, \sigma) \cap G \right) = \lim_{\varepsilon \to 0} \frac{\varepsilon^{p-2}}{p} \int_{\bigcup_j B(a_j, \sigma) \cap G} |\nabla u_\varepsilon|^p + \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon^2} \int_{\bigcup_j B(a_j, \sigma) \cap G} (1 - |u|^2)^2,
\]
yields
\[
\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon \left( u_\varepsilon, \bigcup_j B(a_j, \sigma) \cap G \right) = \lim_{\varepsilon \to 0} \varepsilon^{p-2} \left[ \frac{1}{p} \int_{\bigcup_j B(a_j, \sigma) \cap G} |\nabla u_\varepsilon|^p + \frac{p-2}{2p} \int_{\bigcup_j B(a_j, \sigma) \cap G} |\nabla u_\varepsilon|^p \right] = \lim_{\varepsilon \to 0} \frac{\varepsilon^{p-2}}{2} \int_{\bigcup_j B(a_j, \sigma) \cap G} |\nabla u_\varepsilon|^p
\]
\[= \lim_{\varepsilon \to 0} \left[ \frac{\varepsilon^{p-2}}{2} \int_{(\bigcup_j B(a_j, \sigma) \cap G) \setminus \bigcup_i B(x_i, h\varepsilon)} |\nabla u_\varepsilon|^p + \frac{\varepsilon^{p-2}}{2} \int_{\bigcup_i B(x_i, h\varepsilon)} |\nabla u_\varepsilon|^p \right] \]
\[:= I_1 + I_2 \geq I_1. \tag{3.11} \]

By Proposition 2.6 we have
\[\int_{\bigcup_{i=1}^N (B(a_j, \sigma) \setminus \bigcup_i B(x_i, h\varepsilon))} |\nabla u_\varepsilon|^p \geq \frac{2\pi}{p - 2} \sum_{j=1}^N |k_j|H^{2-p} \varepsilon^2 - p - C. \]

Substituting this into (3.11), we have
\[\lim_{\varepsilon \to 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, \bigcup_j B(a_j, \sigma) \cap G) \geq \frac{\pi}{p - 2} \sum_{j} |k_j|H^{2-p}. \]

This result, together with (3.10), implies \(\sum_j L_j \geq \frac{\pi}{p} \sum_j |k_j|H^{2-p}\). Thus, (1.9) is proved. \(\square\)

4. Proof of Theorem 1.2

In this section, we will investigate whether \(I_2\) in (3.11) is equal to zero. When \(\lim_{\varepsilon \to 0} x_\varepsilon^j = a_j \in \partial G\), we will give the positive answer.

Taking \(B = B_i = B(x_\varepsilon^i, h\varepsilon)\) in (2.5) with \(u = u_\eta\), and multiplying with \(\varepsilon^{p-2}\), we obtain
\[-\varepsilon^{p-2} \int_{\partial B_i} |x|v^{(p-2)/2} |\partial_x u|^2 \, ds + \varepsilon^{p-2} \int_B v^{(p-2)/2} |\nabla u|^2 \]
\[-\frac{2\varepsilon^{p-2}}{p} \int_B v^{p/2} + \frac{\varepsilon^{p-2}}{p} \int_{\partial B} |x|v^{p/2} \, ds \]
\[= -\frac{1}{4\varepsilon^2} \int_{\partial B_i} |x|(1 - |u|^2)^2 \, ds + \frac{1}{2\varepsilon^2} \int_B (1 - |u|^2)^2. \tag{4.1} \]

By applying the mean value theorem, we can see that

\(\text{(R1)}\) \(\int_{\partial B_i} |x|(1 - |u|^2)^2 \, ds = 2\pi (h\varepsilon)^2 (1 - |u_\varepsilon(\xi_1)|^2)^2, \quad \xi_1 \in \partial B_i;\)

\(\text{(R2)}\) \(\int_{B_i} (1 - |u|^2)^2 = \pi (h\varepsilon)^2 (1 - |u_\varepsilon(\xi_2)|^2)^2, \quad \xi_2 \in B_i;\)

\(\text{(R3)}\) \(\int_{B_i} v^{p/2} = \pi (h\varepsilon)^2 [v(\xi_3)]^{p/2}, \quad \xi_3 \in B_i;\)

\(\text{(R4)}\) \(\int_{\partial B_i} |x|v^{p/2} \, ds = 2\pi (h\varepsilon)^2 [v(\xi_5)]^{p/2}, \quad \xi_5 \in \partial B_i;\)

\(\text{(R5)}\) \(\int_{\partial B_i} v^{(p-2)/2} |\nabla u|^2 = \pi (h\varepsilon)^2 [v(\xi_4)]^{(p-2)/2} |\nabla u(\xi_4)|^2, \quad \xi_4 \in B_i;\)
By the same argument of [8, Theorem 1.1], we can also prove that the zeros of $u_\varepsilon$ since the boundary data $g$ is smooth. Hence, the results (R1)–(R4) above lead to

\[\frac{1}{4\varepsilon^2} \int_{\partial B_i} |x|(1-|u|^2)^2 \, ds \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon^2} \int_{B_i} (1-|u|^2)^2 \, ds \quad \text{(4.2)}\]

We may apply the inner estimate to get

\[\lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{B_i} |v|^2 / 2 \, ds \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\varepsilon^{-2}}{2} \int_{\partial B_i} |x|^2 |\nabla u|^2 \, ds \quad \text{(4.3)}\]

Substituting (4.2) and (4.3) into (4.1), we can deduce that

\[\lim_{\varepsilon \to 0} \frac{\varepsilon^{-2}}{2} \int_{B_i} |v|^{(p-2)/2} |\nabla u|^2 - \frac{\varepsilon^{-2}}{2} \int_{\partial B_i} |x| |v|^{(p-2)/2} |\partial_\nu u|^2 \, ds = 0. \quad \text{(4.4)}\]

Combining (R5) with (R6), we have

\[\int_{B_i} |v|^{(p-2)/2} |\nabla u|^2 - \int_{\partial B_i} |x| |v|^{(p-2)/2} |\partial_\nu u|^2 \, ds = \pi (h\varepsilon)^2 \left[ |v(x_\varepsilon)|^{(p-2)/2} \left( |\partial_\nu u(x_\varepsilon)|^2 + |\partial_\nu u(x_\varepsilon)|^2 \right) - 2 |v(x_\varepsilon)|^{(p-2)/2} |\partial_\nu u(x_\varepsilon)|^2 \right].\]

Inserting it into (4.4) we obtain

\[\lim_{\varepsilon \to 0} \varepsilon^{p} \left[ 2 |v(x_\varepsilon)|^{(p-2)/2} |\partial_\nu u(x_\varepsilon)|^2 - \left( |v(x_\varepsilon)|^{(p-2)/2} |\partial_\nu u(x_\varepsilon)|^2 \right) \right] = \lim_{\varepsilon \to 0} \varepsilon^{p} \left[ |v(x_\varepsilon)|^{(p-2)/2} |\partial_\nu u(x_\varepsilon)|^2 \right]. \quad \text{(4.5)}\]

Similar to the proof of Proposition 2.1, for the solution to (1.4), we also have $|u_\varepsilon(x)| \geq 1/2$ on $G \setminus G^{2\rho_\varepsilon}$, and the inner estimate

\[\| \nabla u_\varepsilon \|_{L^\infty(B(x_\varepsilon, 2h\varepsilon))} \leq C\varepsilon^{-1}.\]

By the same argument of [8, Theorem 1.1], we can also prove that the zeros of $u_\varepsilon$ are contained in the bad discs. Thus, dist$(x_\varepsilon, \partial G) \geq 2h\varepsilon$ as long as we take $\rho = h$ in $|u_\varepsilon| \geq 1/2$ on $G \setminus G^{2\rho_\varepsilon}$.

We may apply the inner estimate to get

\[\pi h^2 \varepsilon^{p} \left[ |v(x_\varepsilon)|^{(p-2)/2} |\partial_\nu u(x_\varepsilon)|^2 \right] \leq C\varepsilon^{-2} |\partial_\nu u(x_\varepsilon)|^2. \quad \text{(4.6)}\]

When $a_j \in \partial G$, one has $|\partial_\nu u(a_j)|^2 = |\partial_\nu g(a_j)|^2 \leq C$, where $C > 0$ is independent of $\varepsilon$ and $\eta$, since the boundary data $g$ is smooth. Let $\varepsilon \to 0$ in (4.5). Noting $u_\varepsilon \in C^1(G \cap B(a_j, \sigma))$ and (4.6), we can deduce
\[\lim_{\varepsilon \to 0} \pi h^2 \varepsilon^p [v(aj)]^{(p-2)/2} \pi h^2 |\partial_\tau u(aj)|^2 \leq C \lim_{\varepsilon \to 0} \varepsilon^2 |\partial_\tau u(\xi_4)|^2 = C \lim_{\varepsilon \to 0} \varepsilon^2 |\partial_\tau u(aj)|^2 = 0. \quad (4.7)\]

Combining (4.7) with (4.5), and noting \(u_\eta \varepsilon \in C^1(G \cap B(a_j, \sigma))\), we obtain \[
\lim_{\varepsilon \to 0} \pi h^2 \varepsilon^p [v(aj)]^{(p-2)/2} |\partial_\nu u(aj)|^2 = 0.
\]

Substituting this result and (4.7) into \[
\varepsilon^p [v(aj)]^{(p-2)/2} \left|\nabla u(aj)\right|^2 = \varepsilon^p [v(aj)]^{(p-2)/2} \left|\partial_\tau u(aj)\right|^2 + \left|\partial_\nu u(aj)\right|^2 = 0.
\]

By virtue of (R5) and \(u_\eta \varepsilon \in C^1(G \cap B(a_j, \sigma))\), the consequence above implies \[
\lim_{\varepsilon \to 0} \varepsilon^{p-2} \int_{B_i} v^{(p-2)/2} |\nabla u|^2 = 0.
\]

Letting \(\eta \to 0\) and using (1.5), we can complete Theorem 1.2 at last.

5. Proof of Theorem 1.3

Theorem 2.2 shows a convergence rate of \(|u_\varepsilon|\) in the \(C(K)\) sense. We will prove (1.10), which is a convergence rate in the \(W^{1,p}(K)\) sense.

**Proof of (1.10).** Let \(R > 0\) be a small constant such that \(B(x, 2R) \subseteq G \setminus \bigcup_{i=1}^N \{a_i\}\). Applying (1.2), we have \(\frac{1}{2} \leq |u_\varepsilon(y)| \leq 1\) as \(y \in B(x, 2R)\). By the integral mean value theorem and (1.3), there is \(r \in [R, 2R]\) such that
\[
\int_{\partial B(x,r)} |\nabla u_\varepsilon|^p \, ds + \frac{1}{\varepsilon^p} \int_{\partial B(x,r)} (1 - |u_\varepsilon|^2)^2 \, ds \leq C
\]
with \(C = C(r) > 0\) independent of \(\varepsilon\). Denote \(B(x, r)\) by \(B\). If \(\rho_1\) is a minimizer of the functional
\[
E(\rho, B) = \frac{1}{p} \int_B \left(|\nabla \rho|^2 + |\rho|\right)^{p/2} \, ds + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2,
\]
in \(W^{1,p}_{|u_\varepsilon|}(B, \mathbb{R}^+ \cup \{0\})\), then it solves
\[
-\text{div} \left[ (|\nabla \rho|^2 + 1)^{(p-2)/2} \nabla \rho \right] = \frac{1}{\varepsilon^p} (1 - \rho). \quad (5.2)
\]

Multiplying (5.2) with \(\partial_\nu \rho\). By the same derivation of [8, (3.13)], from (5.1), we can also deduce that
\[
\int_{\partial B} (|\nabla \rho_1|^2 + 1)^{p/2} \, ds \leq C. \quad (5.3)
\]

Multiplying (5.2) with \((1 - \rho)\), and applying Theorem 2.2 and (5.3), we get
\[
\int_B \left[ (|\nabla \rho_1|^2 + 1)^{(p-2)/2} |\nabla \rho_1|^2 + \frac{1}{\epsilon^p} (1 - \rho_1)^2 \right] \leq C \epsilon^p. \tag{5.4}
\]

Set \( U = \rho_1 w \) on \( B \); \( U = u_\epsilon \) on \( G \setminus B \), where \( w = \frac{u_\epsilon}{|u_\epsilon|} \). Then \( U \in W \). Since \( u_\epsilon \) is a minimizer of \( E_\epsilon(u, G) \), we have

\[
E_\epsilon(u_\epsilon, G) \leq E_\epsilon(U, G) = E_\epsilon(\rho_1 w, B) + E_\epsilon(u_\epsilon, G \setminus B). \tag{5.5}
\]

Eq. (1.6) implies

\[
\sup_{B_{2R}} |\nabla u_\epsilon| \leq C, \tag{5.6}
\]

when \( \epsilon \) is sufficiently small. In view of (1.2) and (5.6), we see that

\[
\|\nabla w\|_{L^\infty(B)} \leq C \tag{5.7}
\]

with \( C > 0 \) independent of \( \epsilon \). Thus

\[
\int_B (|\nabla \rho_1|^2 + \rho_1^2 |\nabla w|^2)^{p/2} - \int_B (\rho_1^2 |\nabla w|^2)^{p/2} = \frac{p}{2} \int_B \int_0^1 (s|\nabla \rho_1|^2 + (1 - s)\rho_1^2 |\nabla w|^2)^{(p-2)/2} |\nabla \rho_1|^2 ds dx
\]

\[
\leq C \int_B (|\nabla \rho_1|^2 + 1)^{(p-2)/2} |\nabla \rho_1|^2.
\]

Combining this with (5.5) and (5.4) yields

\[
E_\epsilon(u_\epsilon, B) \leq E_\epsilon(\rho_1 w, B) \leq \frac{1}{p} \int_B (\rho_1^2 |\nabla w|^2)^{p/2} + C \epsilon^p.
\]

Since \( \rho_1 \) solves (5.2) and \( \rho_1|_{\partial B} = |u_\epsilon| \), from (1.2) we can see \( \rho_1 \leq 1 \) on \( B \) easily. By the inequality above, we can deduce that

\[
E_\epsilon(u_\epsilon, B) \leq \frac{1}{p} \int_B |\nabla w|^p + C \epsilon^p. \tag{5.8}
\]

By Jensen’s inequality and (5.8), we have

\[
\frac{1}{p} \int_B |\nabla h_\epsilon|^p + \frac{1}{p} \int_B (h_\epsilon^p - 1)|\nabla w|^p + \frac{1}{4\epsilon^p} \int_B (1 - h_\epsilon^2)^2 \leq E_\epsilon(u_\epsilon, B) - \frac{1}{p} \int_B |\nabla w|^p \leq C \epsilon^p. \tag{5.9}
\]

Here \( h_\epsilon = |u_\epsilon| \). In view of (5.7) and Theorem 2.2, we can obtain

\[
\frac{1}{p} \int_B (1 - h_\epsilon^p)|\nabla w|^p \leq C \epsilon^p.
\]

Combining this with (5.9), we can complete Proposition 5.1. \( \square \)
Proof of (1.11). Taking the inner product of both sides of (1.4) with \( u = u_\eta \), we have
\[
- \text{div}(v^{(p-2)/2} \nabla u) = \frac{1}{\varepsilon^p} |u|^2 (1 - |u|^2).
\]
Noting \( \nabla(|u|^2) = 2u \cdot \nabla u \), and \(- \text{div}(v^{(p-2)/2} \nabla u) = - \text{div}(v^{(p-2)/2} u \cdot \nabla u) + v^{(p-2)/2} |\nabla u|^2\), we get
\[
\frac{1}{\varepsilon^p} |u|^2 (1 - |u|^2) = v^{(p-2)/2} |\nabla u|^2 - \frac{1}{2} \text{div}(v^{(p-2)/2} \nabla(|u|^2)).
\]
Adding \( \frac{1}{\varepsilon^p} (1 - |u|^2)^2 \) to both sides of the equality above, we obtain
\[
\frac{1}{\varepsilon^p} (1 - |u|^2) - v^{(p-2)/2} |\nabla u|^2 = \frac{1}{\varepsilon^p} (1 - |u|^2)^2 - \frac{1}{2} \text{div}(v^{(p-2)/2} \nabla(|u|^2)).
\]
(5.10)

Similar to the derivation of (5.1), from (1.10), we can deduce that
\[
\int_{\partial B} |\nabla u_\eta| |\partial B| \leq \frac{1}{\varepsilon} C \varepsilon,
\]
(5.11)

Thus (1.11) is deduced by an argument of the finite covering. \( \square \)

Proof of (1.12). At first, (5.9) implies
\[
0 \leq E_\varepsilon(u_\varepsilon, B) - \frac{1}{p} \int_B |\nabla w|^p + \frac{1}{p} \int_B (1 - h_\varepsilon^p) |\nabla w|^p
\]
\[
\leq C \varepsilon^p + \frac{1}{p} \int_B (1 - h_\varepsilon^p) |\nabla w|^p.
\]
Combining this with Theorem 2.2 and (5.7), and letting \( \varepsilon \to 0 \), we have
\[
E_\varepsilon(u_\varepsilon, B) - \frac{1}{p} \int_B |\nabla w|^p \to 0.
\]
(5.12)

Next, we observe that
\[
\left| \int_B \left( |\nabla u_\varepsilon|^p - |\nabla w|^p \right) \right| \\
\leq \left| \int_B \left( |\nabla u_\varepsilon|^p - h_\varepsilon^p |\nabla w|^p \right) \right| + \left| \int_B |\nabla w|^p (1 - h_\varepsilon^p) \right| \\
= J_1 + J_2. \tag{5.13}
\]

In view of Theorem 2.2 and (5.7), we have \( \lim_{\varepsilon \to 0} J_2 = 0 \). In addition, the mean value theorem implies

\[
J_1 \leq C \int_B \left( \int_0^1 \left[ s |\nabla h|^2 + (1 - s) h_\varepsilon^2 |\nabla w|^2 \right]^{(p-2)/2} ds \right) |\nabla h_\varepsilon|^2 dx
\]

\[
\leq C \left( \int_B |\nabla u_\varepsilon|^p \right)^{(p-2)/p} \left( \int_B |\nabla h_\varepsilon|^p \right)^{2/p}.
\]

This result, together with (5.6) and (1.10), implies \( \lim_{\varepsilon \to 0} J_1 = 0 \). Substituting these results into (5.13), and using (1.6) we deduce that \( \lim_{\varepsilon \to 0} \int_B |\nabla w|^p = \int_B |\nabla u_p|^p \). Combining this with (5.12) yields (1.12). \( \square \)

**Acknowledgements**

The author is grateful to the referees for their helpful suggestions. The research was supported partly by NSF of China (No. 10571087), Natural Science Foundation of Jiangsu (No. BK2006523), Specialized Research Fund for the Doctoral Program of Higher Education (No. 20050319001) and Natural Science Foundation of Jiangsu Higher Education Institutions (No. 06KJB110056).

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