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Parallel Multisplitting Explicit AOR Methods for Numerical Solutions of Semilinear Elliptic Boundary Value Problems

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Abstract—A class of parallel multisplitting explicit AOR methods for a large scale system of nonlinear algebraic equations, which is a finite difference approximation of a semilinear elliptic boundary value problem, are presented. This class of methods avoid the inner iteration and are shown to converge monotonically either from above or from below to a solution of the system without the monotonicity property of nonlinearity. Moreover, this class of methods are applicable to the pure Neumann boundary value problems. A sufficient condition for the uniqueness of the solutions is provided. The global convergence of the methods and the influence of the acceleration factor on the convergence rate are considered. The applications and numerical results are given.

Keywords—Semilinear elliptic boundary value problem, Finite difference, Multisplitting explicit relaxation method, Monotone convergence, Global convergence.

1. INTRODUCTION

In the study of the numerical solutions of semilinear elliptic boundary value problems by the finite difference method, the corresponding discrete problem is usually formulated as a large scale system of nonlinear algebraic equations. Consider the following semilinear elliptic boundary value problem

$$\begin{aligned} -\mathbf{L}u &= f(x,u), \qquad && x \in \Omega, \\ \mathbf{B}u &= g(x), && x \in \partial\Omega, \end{aligned} \tag{1.1}$$

with

$$\begin{aligned} \mathbf{L}u &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(x)\,\frac{\partial u}{\partial x_i}\right), \qquad && x \in \Omega, \\ \mathbf{B}u &= \alpha(x)\,\frac{\partial u}{\partial \nu} + \beta(x)u, && x \in \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a rectangular domain in $\mathbf{R}^n$ $(n = 1, 2, \dots)$, $x = (x_1, x_2, \dots, x_n)^\top$, $\partial\Omega$ is the boundary of Ω and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u on $\partial\Omega$. It is assumed that the

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functions $a_i(x)$ $(i = 1, 2, \dots, n)$ are positive on $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, $\alpha(x)$ and $\beta(x)$ are nonnegative on $\partial\Omega$ with $\alpha(x) + \beta(x) > 0$, and f and g are prescribed continuous functions in their respective domains. By using the standard finite difference approximations for the differential and boundary operators in (1.2) and arranging the mesh points in the usual fashion, the resulting discrete problem of (1.1) becomes a system of nonlinear algebraic equations which in matrix form is given by

$$AU = F(U) + G, \tag{1.3}$$

where if N denotes the total number of mesh points in $\overline{\Omega}$, then A is an $N \times N$ matrix associated with the coefficients $a_i(x)$ $(i = 1, 2, \dots, n)$, as well as, the boundary coefficients $\alpha(x)$ and $\beta(x)$, U and $F(U)$ are N-dimensional vectors with their respective elements U_i and $F_i(U_i)$, which are associated with the unknown function u and the nonlinearity $f(x, u)$, respectively, and G is an N-dimensional vector whose elements G_i $(i = 1, 2, \dots, N)$ are associated with the boundary function $g(x)$. It is well known that if we write $A = (a_{i,j})$, then

$$\begin{aligned} &a_{i,i} > 0, && i = 1, 2, \dots, N, \\ &a_{i,j} \le 0, && i, j = 1, 2, \dots, N, \quad i \ne j, \\ &a_{i,i} \ge \sum_{j=1, j\ne i}^{N} |a_{i,j}|, && i = 1, 2, \dots, N, \end{aligned} \tag{1.4}$$

(see [1]). Moreover, the connectedness of the set of mesh points ensures that A is irreducible (see [2]). When the boundary condition is not pure Neumann type (i.e., $\beta(x) \ne 0$ for some $x \in \partial\Omega$), strict inequality in the last relation of (1.4) holds for at least one i, and in this situation A is a diagonally dominant M-matrix (see [2]). Otherwise, if

$$a_{i,i} = \sum_{j=1, j\ne i}^{N} |a_{i,j}|, \qquad i = 1, 2, \dots, N, \tag{1.5}$$

which corresponds to the pure Neumann boundary condition (i.e., $\beta(x) \equiv 0$), the matrix A may be singular. But in this situation $A + D$ is still a diagonally dominant M-matrix, where D is a nonnegative matrix and is not identically zero.

For system (1.3), a major concern is to obtain reliable and efficient computational algorithms for computing the solution. Applying the matrix multisplitting concept introduced in [3], White [4] designed a class of parallel nonlinear Gauss-Seidel methods for solving this class of problems. As a two-parameter generalization of White's methods, a class of parallel nonlinear AOR methods also based on the matrix multisplitting concept was established in [5]. This method is also a generalization of the results in [6] to the nonlinear problems. However, the application of this method to system (1.3) requires that A is nonsingular and $F_i(U_i)$ is monotone decreasing in U_i $(i = 1, 2, \dots, N)$. These conditions limit the application of it since

(i) A may be singular when the boundary condition is the pure Neumann type;
(ii) in many practical problems $F_i(U_i)$ $(i = 1, 2, \dots, N)$ do not possess the monotonicity property.

Thus, a new method without these restrictions is needed. On the other hand, the method in [5] involves an implicit nonlinear equation in each iteration, which by itself often requires some iterative methods such as Newton method, Chord method, and Steffensen method, etc. (see [7]). This gives additional complications and brings inconvenience for the numerical computation. The purpose of this paper is to propose a class of parallel multisplitting explicit AOR methods for solving system (1.3). Besides its simple structure and strong parallel computation function, an advantage of this method is that it leads to a sequence which converges monotonically to the solution of system (1.3) with a low regularity requirement on the function $F(U)$. In particular, no monotone condition on $F(U)$ is required. On the other hand, we allow the possibility that condition (1.5) holds in the discussion of the convergence of the methods. This means that our method is applicable to system (1.3) where A may be singular. In addition, since our method is explicit it avoids the inner iteration in each iteration which is often needed in the method of [5]. This gives a practical advantage in the numerical computation.

The outline of this paper is as follows. In Section 2, we present the parallel multisplitting explicit AOR method based on the matrix multisplitting concept. In Section 3, by introducing the concept of ordered upper and lower solutions we discuss the monotone convergence of the method under condition (1.4) and a low regularity condition on $F(U)$. In addition, we give a sufficient condition ensuring the uniqueness of the solutions. At the end of this section, with an additional condition we establish the global convergence of the method. Section 4 gives a comparison of the monotone sequences, which shows that the convergence rate of the method can be improved by increasing the value of acceleration factor. Finally, the applications and numerical results are presented in Section 5. The numerical results coincide with the theoretical analysis in the previous sections.

2. PARALLEL MULTISPLITTING EXPLICIT AOR METHOD

In the development of parallel multisplitting explicit AOR methods for system (1.3), condition (1.4) is our basic hypothesis. Let $\mathcal{N} = \{1, 2, \ldots, N\}$. For $i \in \mathcal{N}$, U_i represents the i^{th} component of $U \in \mathbf{R}^N$. Given a positive integer $\alpha \leq N$ and for $k = 1, 2, \ldots, \alpha$, we take S_k to be a nonempty subset of $\mathcal{N}$ satisfying $\cup_{k=1}^{\alpha} S_k = \mathcal{N}$, $E_k = \operatorname{diag}(e_1^{(k)}, \ldots, e_N^{(k)})$ with

$$e_i^{(k)} = \begin{cases} e_i^{(k)} \geq 0, & i \in S_k, \\ 0, & i \notin S_k, \end{cases} \qquad i = 1, 2, \ldots, N,$$

and $\mathcal{L}_k = (l_{i,j}^{(k)})$, $\mathcal{U}_k = (u_{i,j}^{(k)})$ to be two $N \times N$ matrices with the elements

$$l_{i,j}^{(k)} = \begin{cases} l_{i,j}^{(k)}, & i, j \in S_k, \quad i > j, \\ 0, & \text{otherwise}, \end{cases} \qquad u_{i,j}^{(k)} = \begin{cases} u_{i,j}^{(k)}, & i \neq j, \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 1, 2, \ldots, N.$$

Let $\mathcal{D} = \operatorname{diag}(A)$. If

(i) $A = \mathcal{D} - \mathcal{L}_k - \mathcal{U}_k$, $k = 1, 2, \ldots, \alpha$,

(ii) $\sum_{k=1}^{\alpha} E_k = I$, where I denotes $N \times N$ identity matrix,

then the collection of triples $(\mathcal{D} - \mathcal{L}_k, \mathcal{U}_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ is called a multisplitting of the matrix A.

Based on the above concept, we set up the following parallel multisplitting explicit AOR method for solving system (1.3).

METHOD. Given initial value $U^0 \in \mathbf{R}^N$, we compute for $m = 0, 1, 2, \ldots,$

$$U_i^{m+1} = \omega \sum_{k=1}^{\alpha} e_i^{(k)} U_i^{m,k} + (1 - \omega) U_i^m, \qquad i = 1, 2, \ldots, N, \tag{2.1}$$

where for $k = 1, 2, \ldots, \alpha$, $U_i^{m,k}$ is concurrently computed by

$$U_i^{m,k} = \begin{cases} (a_{i,i} + M_i)^{-1} \left[r \sum_{j=1}^{i-1} l_{i,j}^{(k)} U_j^{m,k} + (1 - r) \sum_{j=1}^{i-1} l_{i,j}^{(k)} U_j^m \right. \\ \qquad \left. + \sum_{j \neq i} u_{i,j}^{(k)} U_j^m + M_i U_i^m + F_i(U_i^m) + G_i \right], \\ \qquad\qquad i \in S_k, \qquad i = 1, 2, \ldots, N, \\ U_i^m, \quad i \notin S_k, \qquad i = 1, 2, \ldots, N,. \end{cases} \tag{2.2}$$

Here, $r \in [0, \infty)$ is called the relaxation factor and $\omega \in (0, \infty)$ the acceleration factor, while M_i denotes some nonnegative constant specified later.

Evidently, this method is of strong parallel computation function because for $k = 1, 2, \dots, \alpha$, $U_i^{m,k}$ can be correspondingly calculated on the k^{th} processor of a multiprocessor system made up of α processors, and for different k the computations of $U_i^{m,k}$ $(k = 1, 2, \dots, \alpha)$ are thoroughly independent. Moreover, considerable savings are possible since for each $k = 1, 2, \dots, \alpha$, $U_i^{m,k}$ corresponding to $e_i^k = 0$ need not to be computed, and the α splittings and the weights e_i^k $(k = 1, 2, \dots, \alpha)$ can also be used to balance the distributions of the computational task among the α processors. Furthermore, with the different choices of the parameters r and ω many parallel multisplitting explicit relaxation methods can be generated from this method. For example, we get parallel multisplitting explicit Gauss-Seidel method if $r = \omega = 1$, and parallel multisplitting explicit SOR method if $r = \omega$. In addition, since this method is explicit it avoids the inner iterations and is easier to handle in the practical computations.

3. THE CONVERGENCE OF THE METHOD

First, we discuss the monotone convergence of the method. A crucial requirement for this possibility is the existence of a pair of upper and lower solutions which are defined as follows.

DEFINITION 3.1. *A vector $\overline{U} = (\overline{U}_1, \dots, \overline{U}_N)^\top \in \mathbf{R}^N$ is called an upper solution of (1.3) if*

$$A\overline{U} \geq F\left(\overline{U}\right) + G.$$

A vector $\underline{U} = (\underline{U}_1, \dots, \underline{U}_N)^\top \in \mathbf{R}^N$ is called a lower solution if

$$A\underline{U} \leq F\left(\underline{U}\right) + G.$$

It is obvious that every solution of (1.3) is an upper solution as well as a lower solution. We say that $\overline{U}$ and $\underline{U}$ are ordered if $\overline{U} \geq \underline{U}$. Given any ordered upper and lower solutions $\overline{U}$ and $\underline{U}$ we set

$$\mathbf{K}\left(\underline{U}, \overline{U}\right) = \left\{U \in \mathbf{R}^N \mid \underline{U} \leq U \leq \overline{U}\right\}.$$

THEOREM 3.1. *Let condition (1.4) hold and let $(\mathcal{D} - \mathcal{L}_k, \mathcal{U}_k, E_k)$ $(k = 1, 2, \dots, \alpha)$ be a multisplitting of A with*

$$\mathcal{L}_k = \left(l_{i,j}^{(k)}\right) \geq 0, \quad \mathcal{U}_k = \left(u_{i,j}^{(k)}\right) \geq 0, \qquad k = 1, 2, \dots, \alpha. \tag{3.1}$$

Assume that

(i) *$\overline{U}$, $\underline{U}$ is a pair of ordered upper and lower solutions of (1.3);*
(ii) *there exists constant γ_i such that*

$$F_i(s) - F_i(t) \geq -\gamma_i(s - t), \quad \underline{U}_i \leq t \leq s \leq \overline{U}_i, \qquad i = 1, 2, \dots, N. \tag{3.2}$$

Let $\gamma_i^+ = \max\{0, \gamma_i\}$. Then for $r \in [0, 1]$ and $\omega \in (0, 1]$, the sequences $\{\overline{U}^m\}$ and $\{\underline{U}^m\}$ generated by the parallel multisplitting explicit AOR method with $M_i = \gamma_i^+$ and the initial values $\overline{U}^0 = \overline{U}$ and $\underline{U}^0 = \underline{U}$, respectively, converge monotonically to the solutions $\overline{U}^*$ and $\underline{U}^*$ of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$. Moreover,

$$\underline{U} \leq \underline{U}^m \leq \underline{U}^{m+1} \leq \underline{U}^* \leq \overline{U}^* \leq \overline{U}^{m+1} \leq \overline{U}^m \leq \overline{U}, \qquad m = 0, 1, 2, \dots, \tag{3.3}$$

and if U^* is any solution of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$, then $\underline{U}^* \leq U^* \leq \overline{U}^*$.

PROOF. First, we prove that for all $m \geq 0$ and $k = 1, 2, \dots, \alpha$,

$$\underline{U} \leq \underline{U}^m \leq \underline{U}^{m+1} \leq \overline{U}^{m+1} \leq \overline{U}^m \leq \overline{U}, \tag{3.4a}$$

$$\underline{U} \leq \underline{U}^{m,k} \leq \underline{U}^{m+1,k} \leq \overline{U}^{m+1,k} \leq \overline{U}^{m,k} \leq \overline{U}. \tag{3.4b}$$

Assume that $1 \in S_k$. By (2.2), (3.1), and (3.2),

$$\overline{U}_1^{0,k} - \underline{U}_1^{0,k} = \left(a_{1,1} + \gamma_1^+\right)^{-1} \left[\sum_{j \neq 1} u_{1,j}^{(k)} \left(\overline{U}_j - \underline{U}_j\right) + \gamma_1^+ \left(\overline{U}_1 - \underline{U}_1\right) + F_1\left(\overline{U}_1\right) - F_1\left(\underline{U}_1\right)\right] \geq 0.$$

By again (2.2) and the definition of a upper solution,

$$\overline{U}_1 - \overline{U}_1^{0,k} = \left(a_{1,1} + \gamma_1^+\right)^{-1} \left[a_{1,1}\overline{U}_1 - \sum_{j \neq 1} u_{1,j}^{(k)} \overline{U}_j - F_1\left(\overline{U}_1\right) - G_1\right] \geq 0.$$

A similar argument using the definition of a lower solution gives $\underline{U}_1 - \underline{U}_1^{0,k} \leq 0$. Therefore,

$$\underline{U}_1 \leq \underline{U}_1^{0,k} \leq \overline{U}_1^{0,k} \leq \overline{U}_1, \qquad 1 \in S_k, \quad k = 1, 2, \dots, \alpha. \tag{3.5}$$

It is clear that (3.5) is also true for $1 \notin S_k$, $k = 1, 2, \dots, \alpha$. Assume that for some $i > 1$,

$$\underline{U}_j \leq \underline{U}_j^{0,k} \leq \overline{U}_j^{0,k} \leq \overline{U}_j, \qquad j = 1, 2, \dots, i-1, \quad k = 1, 2, \dots, \alpha. \tag{3.6}$$

Let $i \in S_k$. By (2.2), (3.1), (3.2), and (3.6),

$$\overline{U}_i^{0,k} - \underline{U}_i^{0,k} = \left(a_{i,i} + \gamma_i^+\right)^{-1} \left[r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \left(\overline{U}_j^{0,k} - \underline{U}_j^{0,k}\right) + (1-r) \sum_{j=1}^{i-1} l_{i,j}^{(k)} \left(\overline{U}_j - \underline{U}_j\right) \right.$$
$$\left. + \sum_{j \neq i} u_{i,j}^{(k)} \left(\overline{U}_j - \underline{U}_j\right) + \gamma_i^+ \left(\overline{U}_i - \underline{U}_i\right) + F_i\left(\overline{U}_i\right) - F_i\left(\underline{U}_i\right)\right] \geq 0.$$

By again (2.2), (3.6), and the definition of upper and lower solutions,

$$\overline{U}_i^{0,k} - \overline{U}_i \leq 0, \qquad \underline{U}_i^{0,k} - \underline{U}_i \geq 0.$$

This proves that (3.6) holds true for $j = i \in S_k$. When $j = i \notin S_k$, (3.6) is clear. Thus, by induction, (3.6) holds true for all $j = 1, 2, \dots, N$ and $k = 1, 2, \dots, \alpha$. This leads to

$$\underline{U} \leq \underline{U}^{0,k} \leq \overline{U}^{0,k} \leq \overline{U}, \qquad k = 1, 2, \dots, \alpha.$$

Furthermore by (2.1),

$$\underline{U} \leq \underline{U}^1 \leq \overline{U}^1 \leq \overline{U}.$$

Following the similar process as that for proving $\underline{U}^{0,k} \leq \overline{U}^{0,k}$ we can prove that

$$\underline{U}^{0,k} \leq \underline{U}^{1,k} \leq \overline{U}^{1,k} \leq \overline{U}^{0,k}, \qquad k = 1, 2, \dots, \alpha.$$

The above conclusions show that (3.4a) and (3.4b) are true for $m = 0$ and $k = 1, 2, \dots, \alpha$. Assume, by induction, (3.4a) and (3.4b) are true for $m = m' \geq 0$ and $k = 1, 2, \dots, \alpha$. It is obvious by (2.1) and (3.4) with $m = m'$ that (3.4a) is also true for $m = m' + 1$ and $k = 1, 2, \dots, \alpha$. Imitating the process as that for proving $\underline{U}^{0,k} \leq \overline{U}^{0,k}$ we can show that

$$\underline{U}^{m'+1,k} \leq \underline{U}^{m'+2,k} \leq \overline{U}^{m'+2,k} \leq \overline{U}^{m'+1,k}, \qquad k = 1, 2, \dots, \alpha.$$

This implies that (3.4b) is also true for $m = m' + 1$ and $k = 1, 2, \dots, \alpha$. It follows again by an induction argument that (3.4a) and (3.4b) are true for all $m \geq 0$ and $k = 1, 2, \dots, \alpha$.

In view of (3.4a), there exist limits $\overline{U}^*$ and $\underline{U}^*$ such that

$$\lim_{m\to\infty} \overline{U}^m = \overline{U}^*, \qquad \lim_{m\to\infty} \underline{U}^m = \underline{U}^*,$$

and (3.3) holds. To prove that $\overline{U}^*$ is the solution of (1.3), we define

$$\mathcal{K}(i) = \{k \mid i \in S_k,\ k = 1,2,\dots,\alpha\}, \qquad i = 1,2,\dots,N.$$

By (3.4b), there exists the limit $\overline{U}^{*,k}$ such that $\lim_{m\to\infty} \overline{U}^{m,k} = \overline{U}^{*,k}$ and

$$\overline{U}_i^{*,k} = \begin{cases} \left(a_{i,i} + \gamma_i^+\right)^{-1} \left[r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^{*,k} + (1-r) \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^* \right. \\ \qquad \left. + \sum_{j\neq i} u_{i,j}^{(k)} \overline{U}_j^* + \gamma_i^+ \overline{U}_i^* + F_i\left(\overline{U}_i^*\right) + G_i \right], \\ \qquad\qquad i \in S_k, \quad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha, \\ \overline{U}_i^*, \qquad i \notin S_k, \quad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha. \end{cases} \tag{3.7}$$

So by (3.7), we need only prove that

$$\overline{U}_j^{*,k} = \overline{U}_j^*, \qquad j \in S_k, \quad j = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha. \tag{3.8}$$

By (2.1),

$$\overline{U}_j^* = \sum_{k=1}^{\alpha} e_j^{(k)} \overline{U}_j^{*,k}, \qquad j = 1,2,\dots,N. \tag{3.9}$$

Let $1 \in S_k$. Multiplying the first equality of (3.7) with $i = 1$ by e_1^k and summing the result over all $\mathcal{K}(1)$, we have from (3.9) and $\sum_{\mathcal{K}(1)} e_1^{(k)} = 1$ that

$$\left(a_{1,1} + \gamma_1^+\right) \overline{U}_1^* = \sum_{\mathcal{K}(1)} e_1^{(k)} \left(\sum_{j\neq 1} u_{1,j}^{(k)} \overline{U}_j^* \right) + \gamma_1^+ \overline{U}_1^* + F_1\left(\overline{U}_1^*\right) + G_1.$$

It follows from the above relation that

$$\sum_{\mathcal{K}(1)} e_1^{(k)} \left[a_{1,1} \overline{U}_1^* - \sum_{j\neq 1} u_{1,j}^{(k)} \overline{U}_j^* \right] = F_1\left(\overline{U}_1^*\right) + G_1,$$

or equivalently,

$$\left[A\overline{U}^*\right]_1 = F_1\left(\overline{U}_1^*\right) + G_1, \tag{3.10}$$

where $[U]_i$ denotes the i^{th} component of $U \in \mathbf{R}^N$. From (3.10) and (3.7) with $i = 1$, we easily obtain

$$\overline{U}_1^{*,k} = \overline{U}_1^*, \qquad 1 \in S_k, \quad k = 1,2,\dots,\alpha.$$

Assume that (3.8) is true for some $i > 1$ and $j = 1,2,\dots,i-1$. Then, (3.7) becomes

$$\overline{U}_i^{*,k} = \begin{cases} \left(a_{i,i} + \gamma_i^+\right)^{-1} \left[\sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^* + \sum_{j\neq i} u_{i,j}^{(k)} \overline{U}_j^* + \gamma_i^+ \overline{U}_i^* + F_i\left(\overline{U}_i^*\right) + G_i \right] \\ \qquad i \in S_k, \quad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha, \\ \overline{U}_i^*, \qquad i \notin S_k, \quad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha. \end{cases}$$

Repeating the above process for $j = 1$ leads to (3.8) is still true for $j = i$. Therefore, by induction, (3.8) holds. This proves that $\overline{U}^*$ is the solution of (1.3). In the same argument we can show that $\underline{U}^*$ is also the solution of (1.3). Now if U^* is a solution of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$, then $\overline{U}$ and U^* are ordered upper and lower solutions. Using $\overline{U}^0 = \overline{U}$ and $\underline{U}^0 = U^*$, (3.3) implies that $\overline{U}^* \geq U^*$. A similar argument using U^* and $\underline{U}$ as ordered upper and lower solutions yields $U^* \geq \underline{U}^*$. This proves the theorem.

In view of the relation $\underline{U}^* \leq U^* \leq \overline{U}^*$ for any solution U^* of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$, $\underline{U}^*$, and $\overline{U}^*$ are called minimal and maximal solutions of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$, respectively. If these two solutions coincide, then their common value is the unique solution of (1.3). In the following, we give a sufficient condition ensuring $\underline{U}^* = \overline{U}^*$. As being shown in the introduction, for any real number $\sigma > 0$, $A + \sigma I$ is a irreducible M-matrix. This property ensures that $(A + \sigma I)^{-1} \geq 0$ and $\rho((A + \sigma I)^{-1})$ is a real eigenvalue of $(A + \sigma I)^{-1}$ (see [2,8]), where $\rho(B)$ denotes the spectral radius of the matrix B. This implies that A has at least one real eigenvalue and the smallest real eigenvalue, denoted by $\underline{\mu}$, is nonnegative.

THEOREM 3.2. *Let the conditions in Theorem 3.1 hold. In addition, assume that there exists constant σ_i such that*

$$F_i(s) - F_i(t) \leq \sigma_i(s - t), \quad \underline{U}_i \leq t \leq s \leq \overline{U}_i, \qquad i = 1, 2, \ldots, N. \tag{3.11}$$

Let $\sigma = \max_i \sigma_i$. If $\sigma < \underline{\mu}$, then $\underline{U}^* = \overline{U}^*$ and is the unique solution of (1.3) in $\mathbf{K}(\underline{U}, \overline{U})$.

PROOF. Let $W^* = \overline{U}^* - \underline{U}^*$. Then $W^* \geq 0$ and by (1.3) and (3.11),

$$AW^* = F\left(\overline{U}^*\right) - F\left(\underline{U}^*\right) \leq \sigma\left(\overline{U}^* - \underline{U}^*\right) = \sigma W^*,$$

or equivalently,

$$(A - \sigma I)W^* \leq 0. \tag{3.12}$$

Let $\delta > \max\{0, -\sigma\}$. Then $\delta > 0$ and $\sigma + \delta > 0$. Define $B = A + \delta I$. We have $B^{-1} \geq 0$ and write

$$A - \sigma I = B\left(I - (\sigma + \delta)B^{-1}\right). \tag{3.13}$$

Let $0 < \varepsilon < 1/\sigma + \delta\,(1 - \sigma + \delta/\underline{\mu} + \delta)$. It is well known that there exists some matrix norm $\|\cdot\|$ such that

$$\left\|B^{-1}\right\| \leq \rho\left(B^{-1}\right) + \varepsilon, \tag{3.14}$$

(see [7]). Since B^{-1} is nonnegative, there exist a real eigenvalue μ of A such that $\mu + \delta > 0$ and

$$\rho\left(B^{-1}\right) = \frac{1}{\mu + \delta}, \tag{3.15}$$

(see [2,8]). Substituting (3.15) into (3.14) leads to

$$\left\|B^{-1}\right\| < \frac{1}{\mu + \delta} + \frac{1}{\sigma + \delta}\left(1 - \frac{\sigma + \delta}{\underline{\mu} + \delta}\right) = \frac{1}{\mu + \delta} + \frac{1}{\sigma + \delta} - \frac{1}{\underline{\mu} + \delta} \leq \frac{1}{\sigma + \delta},$$

which implies that

$$(\sigma + \delta)\left\|B^{-1}\right\| < 1.$$

Thus, we have from [9, Theorem 3, p. 298] that $(I - (\sigma + \delta)B^{-1})^{-1}$ exists and is nonnegative. Since $B^{-1} \geq 0$, we use (3.13) to deduce that the inverse $(A - \sigma I)^{-1}$ exists and is nonnegative. It follows from (3.12) that $W^* \leq 0$ which leads to $W^* = 0$. This proves $\overline{U}^* = \underline{U}^*$.

REMARK 3.1. Without the monotonicity property of $F(U)$ we obtain the monotone convergence of the method either from above or from below to a solution of system (1.3), depending on the choice of the initial value. Condition (3.2) means that F_i satisfies locally a uniform one-sided Lipschitz condition.

REMARK 3.2. In Theorems 3.1 and 3.2, no strict inequality in the last relation of (1.4) is required. This implies that all the conclusions in these theorems are applicable to the pure Neumann boundary value problems.

REMARK 3.3. In the special case of the linear function $f(x, u) = \sigma u + q(x)$, the corresponding finite difference system (1.3) becomes

$$AU = \sigma U + G + Q, \tag{3.16}$$

where Q is associated with $q(x)$. Since problem (3.16) has a unique solution when $\sigma < \underline{\mu}$ and it has either no or infinite number of solutions when $\sigma = \underline{\mu}$, we see that the uniqueness result in Theorem 3.2 can not be much improved without additional conditions on $F(U)$.

In the remainder of this section, we deal with the global convergence of the parallel multisplitting explicit AOR method.

THEOREM 3.3. *Let condition (1.4) hold and let $(\mathcal{D} - \mathcal{L}_k, \mathcal{U}_k, E_k)$ $(k = 1, 2, \ldots, \alpha)$ be a multisplitting of A, which satisfies (3.1). Assume that there exist constants γ_i and σ_i such that*

$$-\gamma_i(s-t) \le F_i(s) - F_i(t) \le \sigma_i(s-t), \qquad i = 1, 2, \ldots, N, \quad \forall s, t \in \mathbf{R}^1, \quad s \ge t. \tag{3.17}$$

Let $\gamma_i^+ = \max\{0, \gamma_i\}$ and $\sigma = \max_i \sigma_i$. If $\sigma < \underline{\mu}$ where $\underline{\mu}$ is the smallest real eigenvalue of A, then any sequence generated by the parallel multisplitting explicit AOR method with $M_i = \gamma_i^+$ and any initial value $U^0 \in \mathbf{R}^N$ converges to the unique solution U^* of system (1.3) provided that $r \in [0, 1]$ and $\omega \in (0, 1]$.

PROOF. Condition (3.17) implies that $F_i(U_i) - \sigma U_i$ is monotone decreasing in U_i. In the proof of Theorem 3.2, we know that $A - \sigma I$ is an M-matrix provided $\sigma < \underline{\mu}$. Thus, we conclude that system (1.3) has a unique solution U^* (see [7, Theorem 13.1.5]). For given $U^0 \in \mathbf{R}^N$, let

$$\overline{U} = U^0 + U^+, \qquad \underline{U} = U^0 + U^-,$$

with

$$U^+ = (A - \sigma I)^{-1} \left| AU^0 - F\left(U^0\right) - G \right|, \qquad U^- = -U^+,$$

where $|U| = (|U_1|, |U_2|, \ldots, |U_N|)^\top \in \mathbf{R}^N$ provided $U = (U_1, U_2, \ldots, U_N)^\top \in \mathbf{R}^N$. It is obvious that $\underline{U} \le U^0 \le \overline{U}$. By (3.7) and an elementary calculation,

$$A\overline{U} \ge F\left(\overline{U}\right) + G, \qquad A\underline{U} \le F\left(\underline{U}\right) + G.$$

This fact indicates that $\overline{U}$, $\underline{U}$ is a pair of ordered upper and lower solutions of (1.3). Let $\{\overline{U}^m\}$, $\{\underline{U}^m\}$, and $\{U^m\}$ be sequences generated by the parallel multisplitting explicit AOR method starting from the initial values $\overline{U}$, $\underline{U}$, and U^0, respectively, with the same parameters r, ω, and $M_i = \gamma_i^+$. By Theorem 3.1 and the uniqueness of the solutions,

$$\lim_{m \to \infty} \underline{U}^m = U^* = \lim_{m \to \infty} \overline{U}^m. \tag{3.18}$$

Since $\underline{U} \le U^0 \le \overline{U}$, along the same line of Theorem 3.1 we can show that for all $m \ge 0$,

$$\underline{U}^m \le U^m \le \overline{U}^m.$$

In light of (3.18),

$$\lim_{m \to \infty} U^m = U^*.$$

This proves the theorem.

4. A COMPARISON OF THE MONOTONE SEQUENCES

In this section, we give a comparison result for the monotone sequences obtained by parallel multisplitting explicit AOR method. This result demonstrates that the convergence rate of the method can be improved by increasing the value of acceleration factor. It is described in the following theorem.

THEOREM 4.1. *Let the conditions of Theorem 3.1 hold, and let ω, $\omega' \in (0,1]$, $r \in [0,1]$ be given for which $\omega \le \omega'$. Starting the same initial values $\overline{U}^0 = \overline{U} = \overline{U}'^0$, let $\{\overline{U}^m\}$ and $\{\overline{U}'^m\}$ be sequences generated by the parallel multisplitting explicit AOR method with $M_i = \gamma_i^+$ and parameter pairs $\{r,\omega\}$ and $\{r,\omega'\}$, respectively. Also, denote the corresponding sequences by $\{\underline{U}^m\}$ and $\{\underline{U}'^m\}$ when the initial values are taken as $\underline{U}^0 = \underline{U} = \underline{U}'^0$. Then*

$$\overline{U}^m \ge \overline{U}'^m, \quad \underline{U}^m \le \underline{U}'^m, \qquad m = 0,1,2,\dots. \tag{4.1}$$

PROOF. We by induction prove that for $m = 0,1,2,\dots,$

$$\overline{U}^m \ge \overline{U}'^m. \tag{4.2}$$

When $m = 0$, (4.2) is trivial. Assume that (4.2) holds for some $m \ge 0$. Based on (2.2), we see that when $i \in S_k$, $k = 1,2,\dots,\alpha$,

$$\begin{aligned}
\left(a_{i,i} + \gamma_i^+\right) \overline{U}_i^{m,k} &= r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^{m,k} + (1-r) \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^m + \sum_{j \ne i} u_{i,j}^{(k)} \overline{U}_j^m \\
&\quad + \gamma_i^+ \overline{U}_i^m + F_i\left(\overline{U}_i^m\right) + G_i \\
&\ge r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j^{m,k} + (1-r) \sum_{j=1}^{i-1} l_{i,j}^{(k)} \overline{U}_j'^m + \sum_{j \ne i} u_{i,j}^{(k)} \overline{U}_j'^m \\
&\quad + \gamma_i^+ \overline{U}_i'^m + F_i\left(\overline{U}_i'^m\right) + G_i \\
&= \left(a_{i,i} + \gamma_i^+\right) \overline{U}_i'^{m,k} + r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \left(\overline{U}_j^{m,k} - \overline{U}_j'^{m,k}\right).
\end{aligned}$$

This gives

$$\begin{gathered}
\overline{U}_i^{m,k} - \overline{U}_i'^{m,k} \ge \left(a_{i,i} + \gamma_i^+\right)^{-1} r \sum_{j=1}^{i-1} l_{i,j}^{(k)} \left(\overline{U}_j^{m,k} - \overline{U}_j'^{m,k}\right), \\
i \in S_k, \quad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha.
\end{gathered}$$

Since $\overline{U}_i^{m,k} = \overline{U}_i^m \ge \overline{U}_i'^m = \overline{U}_i'^{m,k}$ for $i \notin S_k$ ($i = 1,2,\dots,N$; $k = 1,2,\dots,\alpha$), by induction we can easily conclude from the above relation that

$$\overline{U}_i^{m,k} \ge \overline{U}_i'^{m,k}, \qquad i = 1,2,\dots,N, \quad k = 1,2,\dots,\alpha.$$

By (2.1), we have that for $i = 1,2,\dots,N$,

$$\begin{aligned}
\overline{U}_i^{m+1} - \overline{U}_i'^{m+1} &= (\omega - \omega') \sum_{k=1}^{\alpha} e_i^{(k)} \overline{U}_i^{m,k} + \omega' \sum_{k=1}^{\alpha} e_i^{(k)} \left(\overline{U}_i^{m,k} - \overline{U}_i'^{m,k}\right) \\
&\quad + (1-\omega')\left(\overline{U}_i^m - \overline{U}_i'^m\right) - (\omega - \omega') \overline{U}_i^m \\
&\ge (\omega - \omega') \left[\sum_{k=1}^{\alpha} e_i^{(k)} \overline{U}_i^{m,k} - \overline{U}_i^m\right] \\
&= \frac{\omega - \omega'}{\omega} \left(\overline{U}_i^{m+1} - \overline{U}_i^m\right).
\end{aligned}$$

Since $\omega \le \omega'$ and $\overline{U}_i^m \ge \overline{U}_i^{m+1}$, the above inequality implies that $\overline{U}_i^{m+1} \ge \overline{U}_i'^{m+1}$ $(i = 1, 2, \dots, N)$ which show that (4.2) is true for $m+1$. The monotonicity property (4.2) for all m follows from the induction principle. The proof for $\underline{U}^m \le \underline{U}'^m$ $(m = 0, 1, 2, \dots)$ is similar.

5. APPLICATIONS AND NUMERICAL RESULTS

Consider system (1.3) with

$$A = \begin{pmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix}.$$

This problem comes from the standard finite difference approximation of the following two-point boundary value problem:

$$\begin{aligned} -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} &= f(x, u), \qquad x \in (0,1), \\ \left.\frac{\partial u}{\partial \nu}\right|_{x=0} &= 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{x=1} = 0. \end{aligned}$$

Clearly, A is a singular matrix and so the method in [5] is not applicable to this problem. However, we see that A satisfies our basic hypothesis (1.4) of this paper.

To give an application of algorithms (2.1),(2.2) we consider a logistic model problem which is given by

$$\begin{aligned} -(u_{xx} + u_{yy}) &= \rho u(1-u) + q(x,y), \qquad && 0 < x < 1, \quad 0 < y < 1, \\ u(0,y) &= u(1,y) = 0, && 0 \le y \le 1, \\ u(x,0) &= u(x,1) = 0, && 0 \le x \le 1, \end{aligned} \tag{5.1}$$

where $\rho = \pi^2$ and

$$q(x,y) = 2\sin(\pi y) + \pi^2 x^2 (1-x)^2 \sin^2(\pi y).$$

It is easy to verify that the analytical solution of (5.1) is given by $u(x,y) = x(1-x)\sin(\pi y)$. Let $h = 1/M$ and let $x_i = ih$, $y_j = jh$ for $i, j = 0, 1, 2, \dots, M$. The set of points (x_i, y_j) in $\Omega = (0,1) \times (0,1)$ is denoted by Ω_h. We write a point (x_i, y_j) in Ω_h by $(i,j) \in \Omega_h$ and define

$$u_{i,j} = u(x_i, y_j), \qquad q_{i,j} = q(x_i, y_j), \qquad f_{i,j}(u_{i,j}) = \rho u_{i,j}(1 - u_{i,j}) + q_{i,j}.$$

Then by using the central finite difference approximation for the second-order differential operator in (5.1), a finite difference system is given in the form

$$\begin{aligned} -u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} &= h^2 f_{i,j}(u_{i,j}), \qquad && (i,j) \in \Omega_h, \\ u_{0,j} = u_{M,j} &= 0, && j = 0, 1, \dots, M, \\ u_{i,0} = u_{i,M} &= 0, && i = 0, 1, \dots, M. \end{aligned} \tag{5.2}$$

Define vectors by

$$\begin{aligned} U &= (u_{1,1}, u_{2,1}, \dots, u_{M-1,1}, u_{1,2}, u_{2,2}, \dots, u_{M-1,2}, \dots, u_{1,M-1}, u_{2,M-1}, \dots, u_{M-1,M-1})^{\mathrm{T}}, \\ \overline{F}(U) &= (f_{1,1}(u_{1,1}), f_{2,1}(u_{2,1}), \dots, f_{M-1,1}(u_{M-1,1}), f_{1,2}(u_{1,2}), f_{2,2}(u_{2,2}), \dots, f_{M-1,2}(u_{M-1,2}), \\ &\qquad \dots, f_{1,M-1}(u_{1,M-1}), f_{2,M-1}(u_{2,M-1}), \dots, f_{M-1,M-1}(u_{M-1,M-1}))^{\mathrm{T}}, \\ F(U) &= h^2 \overline{F}(U), \\ G &= (0, 0, \dots, 0)^{\mathrm{T}}. \end{aligned}$$

Then problem (5.2) may be written in the form (1.3) where A is an $(M-1)^2 \times (M-1)^2$ matrix. If we write $A = (a_{i,j})$, then A satisfies condition (1.4) and $\text{diag}(A) = \text{diag}(4, 4, \ldots, 4)$.

We take $\alpha = 2$ and

$$S_1 = \{1, 2, \ldots, N_1\}, \qquad S_2 = \{N_2, N_2 + 1, \ldots, N\}$$

with N_1, N_2 being positive integers satisfying $1 \le N_2 \le N_1 \le N = (M-1)^2$. The multisplitting $(\mathcal{D} - \mathcal{L}_k, \mathcal{U}_k, E_k)$ $(k = 1, 2)$ of A is taken as follows:

$$\mathcal{D} = \text{diag}(A) = \text{diag}(4, 4, \ldots, 4),$$

$$\mathcal{L}_k = \left(l_{i,j}^{(k)}\right), \qquad l_{i,j}^{(k)} = \begin{cases} 1, & i > j, \quad i, j \in S_k, \quad a_{i,j} \ne 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{U}_k = \left(u_{i,j}^{(k)}\right), \qquad u_{i,j}^{(k)} = \begin{cases} -\left(a_{i,j} + l_{i,j}^{(k)}\right), & i \ne j, \\ 0, & \text{otherwise}, \end{cases}$$

$$e_i^{(1)} = \begin{cases} 1, & 1 \le i < N_2, \\ \dfrac{1}{2}, & N_2 \le i \le N_1, \\ 0, & N_1 < i \le N, \end{cases} \qquad e_i^{(2)} = \begin{cases} 0, & 1 \le i < N_2, \\ \dfrac{1}{2}, & N_2 \le i \le N_1, \\ 1, & N_1 < i \le N. \end{cases}$$

Since $0 \le q(x, y) \le 2 + \pi^2/16$ for $(x, y) \in \overline{\Omega}$, the nonnegative constant pair $\overline{U} = C$ and $\underline{U} = 0$ are ordered upper and lower solutions whenever $\pi^2 C(C-1) \ge 2 + \pi^2/16$.

Notice that $F(U)$ is not monotone decreasing in this example. Hence, the method given in [5] is not used. But nevertheless we do have a parallel monotone iterative procedure of the form given in (2.1) and (2.2) of this paper, because all the assumptions of the Theorem 3.1 are satisfied.

Using $C = 5/4$ for $\overline{U}$ and $\underline{U} = 0$, we compute the maximal and minimal sequences $\{\overline{U}^m\}$, $\{\underline{U}^m\}$ from the parallel multisplitting explicit AOR method. The computations are proceeded with $N_1 = [4N/5]$, $N_2 = [N/5]$, $M = 20$, and $M_i = 15h^2$, where $[a]$ denotes the integer part of a positive real number a. All numerical computations show that the maximal sequence $\{\overline{U}^m\}$ is monotone decreasing and the minimal sequence $\{\underline{U}^m\}$ is monotone increasing for various values of parameters $r \in [0, 1)$ and $\omega \in (0, 1)$. This coincides with the monotonicity described in Theorem 3.1. Numerical results at point $(x_i, y_j) = (1/4, 1/4)$ with $r = 0.8$ and $\omega = 0.9$ are given in Figure 5.1, where $\overline{u}_{i,j}^m$ and $\underline{u}_{i,j}^m$ are the respective components of $\overline{U}^m$ and $\underline{U}^m$. We determine the iteration criterion for the maximal solution $\overline{U}^*$ and minimal solution $\underline{U}^*$ from the conditions

$$\left\|A\overline{U}^m - F\left(\overline{U}^m\right) - G\right\|_\infty < \varepsilon, \qquad \left\|A\underline{U}^m - F\left(\underline{U}^m\right) - G\right\|_\infty < \varepsilon$$

for various values of ε, respectively. Table 5.1 gives the number of iteration for the maximal and minimal sequences $\{\overline{U}^m\}$, $\{\underline{U}^m\}$ with $\varepsilon = 10^{-5}$ and different parameters r and ω. It is seen from this table that for fixed r, the number of iteration is reduced when the value of acceleration factor ω is increased. This agrees with the theoretical analysis in Theorem 4.1. In addition, we see that the number of iteration is also reducing with r increasing. Numerical results with $(r, w) = (0.8, 0.9)$ and $\varepsilon = 10^{-5}$ for the approximate solutions $\overline{u}_{i,j}^*$, $\underline{u}_{i,j}^*$, and the analytic solution $u_{i,j}^* = x_i(1 - x_i)\sin(\pi y_j)$ at the quarter points along the x-axis and y-axis are given in Table 5.2, where $\overline{u}_{i,j}^*$ and $\underline{u}_{i,j}^*$ are the respective components of $\overline{U}^*$ and $\underline{U}^*$. In all computations, we also find that the sequences $\{\overline{U}^m\}$ and $\{\underline{U}^m\}$ have the same limit and so it is the unique solution of (5.2) in $\mathbf{K}(\underline{U}, \overline{U})$.

Table 5.1. The number of iteration for $\{\overline{U}^m\}$ and $\{\underline{U}^m\}$ $(\varepsilon = 10^{-5})$.

r		0.8	0.8	0.8	0.8	0.8	0.7	0.9	1.0	1.05	1.1	0.9
ω		0.7	0.8	0.9	1.0	1.05	0.8	0.8	0.8	0.8	0.9	1.1
Number	$\overline{U}^m$	839	733	652	586	558	792	674	616	587	497	490
of iteration	$\underline{U}^m$	720	630	559	503	479	680	579	529	504	427	421

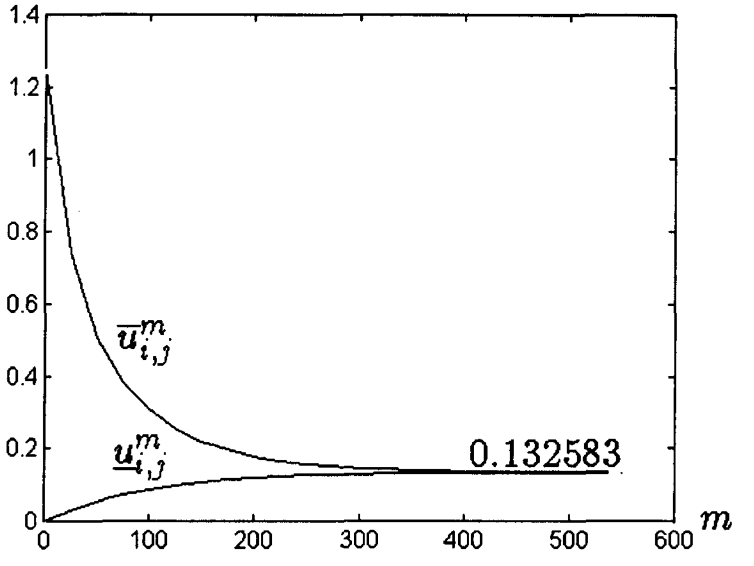

Figure 5.1. The monotonicity of $\{\overline{U}^m\}$ and $\{\underline{U}^m\}$ at $(x_i, y_j) = (1/4, 1/4)$ $(r = 0.8,\ \omega = 0.9)$

Table 5.2. The approximate solutions and the analytic solution $(\varepsilon = 10^{-5}, (r, \omega) = (0.8, 0.9))$.

Solution	$x_i \backslash y_j$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\overline{u}^*_{i,j}$	$\frac{1}{4}$	0.132943	0.187994	0.132932
$\underline{u}^*_{i,j}$		0.132617	0.187557	0.132628
$u^*_{i,j}$		0.132583	0.187500	0.132583
$\overline{u}^*_{i,j}$	$\frac{1}{2}$	0.177271	0.250675	0.177256
$\underline{u}^*_{i,j}$		0.176828	0.250084	0.176843
$u^*_{i,j}$		0.176777	0.250000	0.176777
$\overline{u}^*_{i,j}$	$\frac{3}{4}$	0.132934	0.187982	0.132924
$\underline{u}^*_{i,j}$		0.132625	0.187569	0.132628
$u^*_{i,j}$		0.132583	0.187500	0.132583

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