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Alekseevskian spaces

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Abstract: D.V. Alekseevsky's classification of quaternionic Kählerian solvmanifolds (1975) is completed, confirming results which were obtained recently by the theoretical physicists B. de Wit and A. Van Proeyen in the context of supergravity. It is further shown that an Alekseevsky space is symmetric if and only if its sectional curvature is non-positive. This is an analogue of a well known theorem due to J.E. D'Atri and I. Dotti Miatello, which characterizes bounded symmetric domains by non-positive curvature.

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Introduction

In this paper we study (non-flat) quaternionic Kähler manifolds which admit a simply transitive and real solvable group of isometries. We call them Alekseevsky spaces after D.V. Alekseevsky, who classified them and hereby discovered the first non-symmetric examples of quaternionic Kählerian manifolds (see [3]). He conjectured that every homogeneous quaternionic Kähler manifold of negative scalar curvature should be Alekseevskian.

In the first chapter we give an introduction to the theory of Alekseevskian spaces established in [3].

As was kindly pointed out to me by A. Swann, the theoretical physicists B. de Wit and A. Van Proeyen have recently discovered, in the context of supergravity, that Alekseevsky's classification is incomplete.

In the second chapter we complete the classification (see Thm. 2.28) in line with Alekseevsky's original programme, confirming their discovery, without recourse to supergravity. Furthermore, some basic facts concerning the structure of Alekseevskian Lie algebras are presented.

In the third chapter we investigate the curvature of the Alekseevsky spaces. The main discovery is the fact that an Alekseevsky space is symmetric if and only if its sectional curvature is non-positive (see Thm. 3.10). This is an analogue of a well known theorem due to J.E. D'Atri and I. Dotti Miatello, which characterizes bounded symmetric domains by non-positive curvature.

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1. Preliminaries

In the first chapter we shall provide a frame of reference for the reader to fall back on in the course of this paper. We review [3], where one finds a wealth of technical details, which we omit here. After presenting the fundamental objects, namely Alekseevskian Lie algebras, in Sect. 1.1, we elucidate the conceptional framework of their classification in Sect. 1.2. In the process, we go into the notion of Q-representation in detail. Finally, in the third section, we define Alekseevsky's $\mathcal{W}$- and $\mathcal{V}$-spaces and explain the connection with isometric mappings.

1.1. Alekseevskian Lie algebras

Definition 1.1. A quaternionic Kähler manifold of non-trivial scalar curvature is said to be *Alekseevskian* or an *Alekseevsky space*, if it admits a simply transitive, real solvable group of isometries.

Lie algebras and Lie groups in this paper are always defined over the reals. Recall that a solvable Lie algebra s is said to be *real* (or *completely* [3, Sect. 1]) *solvable*, if the endomorphisms ad_X ($X \in s$) have only real eigenvalues. A Lie group is said to be real solvable, if its Lie algebra is real solvable.

In [3], the classification of Alekseevskian spaces is interpreted as classification problem for metric Lie algebras and, apart from the minor refinements contributed in Sect. 2, it is carried out.

Definition 1.2. A *metric Lie algebra* is a pair $(s, \langle\cdot\,,\cdot\rangle)$ consisting of a Lie algebra s and a scalar product $\langle\cdot\,,\cdot\rangle$ on s. An *isomorphism* of metric Lie algebras is a Lie algebra isomorphism which is, at the same time, an isometry of Euclidean vector spaces.

For every metric Lie algebra $(s, \langle\cdot\,,\cdot\rangle)$ there is a unique (connected) simply connected Lie group $\mathcal{S}$ with Lie algebra s and a unique left-invariant Riemannian metric which coincides with $\langle\cdot\,,\cdot\rangle$ on $T_e\mathcal{S} \cong s$. Thus, to every metric Lie algebra we have assigned a simply connected Riemannian manifold.

Conversely, a Riemannian manifold M which admits a simply transitive group $\mathcal{S}$ of isometries can be regarded as a Lie group with left-invariant metric and we can consider the corresponding metric Lie algebra $(s, \langle\cdot\,,\cdot\rangle)$. We express this fact in the following by saying that M admits the metric Lie algebra $(s, \langle\cdot\,,\cdot\rangle)$. According to Alekseevsky, a Riemannian manifold M admits *at most one real solvable* metric Lie algebra ([2, Thm. 1], cf. [13, Thm. 4.3, Thm. 5.2 and Cor. 5.3]). (In general, there may be more than one simply transitive group of motions.)

In the following, s denotes a metric Lie algebra and M the corresponding simply connected Riemannian manifold. We can transfer the notion of *Levi-Civita connection* and, for that reason, all derived notions from M to s or, equivalently, we can define it by the following *Koszul formula* ($X, Y, Z \in s$):

$$2\langle\nabla_X Y, Z\rangle = \langle[X, Y], Z\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle .$$

Canonical definitions (cf. [3, Def. 1.5–8]), which will be frequently used in the course of this paper, now follow.

Definition 1.3. Let s be a metric Lie algebra. The *Kostant algebra* kos is the Lie subalgebra of $so(s)$ generated by the endomorphisms ∇_X ($X \in s$).

The *holonomy algebra* hol is the Lie subalgebra of kos generated by the expressions of the type

$$[\nabla_{X_1}, \ldots [\nabla_{X_k}, R(X_{k+1}, X_{k+2})] \ldots] , \quad X_1, X_2, \ldots X_{k+2} \in s, k = 0, 1, 2, \ldots$$

Accordingly, hol is the Lie algebra generated by the covariant derivatives of arbitrary order of the curvature endomorphisms, i.e. the Lie algebra of the holonomy group of M (see [15, Ch. III, Thm. 9.2 and Ch. II, Thm. 10.8]).

Definition 1.4. Let V be a Euclidean vector space. A *complex structure* on V is a skew endomorphism J satisfying $J^2 = -\mathrm{Id}$.

A *quaternionic structure* on V is a Lie subalgebra q of $so(V)$ generated by two anticommuting complex structures J_1 and J_2.

We denote the centralizers of J and $\mathrm{q} = \mathrm{span}\{J_1, J_2, J_3\}$ ($J_3 := J_1 J_2$) in $so(V)$ by zent(J) and zent(q) respectively.

Definition 1.5. A metric Lie algebra s with complex structure J is said to be *Kählerian*, if

$$\mathrm{hol} \subset \mathrm{zent}(J) .$$

A metric Lie algebra s of dimension > 4 (resp. $= 4$) with quaternionic structure q is said to be *quaternionic Kählerian*, if

$$\mathrm{hol} \subset \mathrm{q} \oplus \mathrm{zent}(\mathrm{q})$$

(resp. if q annihilates the curvature tensor). Non-Abelian, real solvable, quaternionic Kählerian Lie algebras are said to be *Alekseevskian*.

In other words, s is Kählerian resp. quaternionic Kählerian resp. Alekseevskian, if M is. The condition "non-Abelian" in the definition of Alekseevskian Lie algebras guarantees that s has non-vanishing scalar curvature (see [3, Cor. 1.5 and Cor. 1.2], cf. [9, II.1.2, Satz 5]).

Alekseevskian spaces are Einsteinian with ric < 0 (see [3, p. 305], cf. [9, p. 11]) and, according to [3, Thm. 1.3], simply connected. In particular, the following theorem holds, which is the starting point for Alekseevsky's classification, that will be discussed in the next section.

Theorem 1.1. (D.V. Alekseevsky [3, Sect. 1]) *There is a one-to-one correspondence between Alekseevskian spaces and Alekseevskian Lie algebras.*

1.2. Structure of Alekseevsky's classification

Let now $(s, \langle \cdot , \cdot \rangle, \mathrm{q})$ be an Alekseevskian Lie algebra and $\mathcal{S}$ the corresponding Alekseevsky space. Being a non-Ricci-flat quaternionic Kähler manifold, $\mathcal{S}$ is also locally de-Rham irreducible (see [6, Thm. 14.45b]). Homogeneous locally de-Rham irreducible spaces of non-vanishing Ricci curvature satisfy kos = hol, according to an old result of Kostant–Lichnerowicz [17]. (The notion of Kostant algebra can be easily defined for general homogeneous spaces, e.g. by an appropriate Koszul formula.) This justifies the following decomposition ($X \in s$):

$$\nabla_X = \sum_{\alpha=1}^{3} \omega_\alpha(X) J_\alpha + \bar{\nabla}_X , \tag{1}$$

where ω_α are linear forms on s and $\bar{\nabla}_X \in \mathrm{zent}(\mathrm{q})$. On the other hand, owing to the general structure theorem for algebraic curvature tensors of type $sp(1) \oplus sp(m)$ ($m = \dim_{\mathbb{H}} s = (\dim_{\mathbb{R}} s)/4$) we have:

$$R = \mathrm{sk}_r R_0 + \bar{R} , \tag{2}$$

where R_0 is the curvature tensor of the quaternionic projective space $P^m_{\mathbb{H}}$ and $\bar{R}$ is of type $sp(m)$ (see [1, Table 1] or in more detail [21, Prop. 9.3] and [20, Thm. 3.1]). sk_r denotes the reduced scalar curvature of R, which coincides with the scalar curvature sk, up to a dimension factor:

$$\mathrm{sk} = 4m(m+2)\mathrm{sk}_r . \tag{3}$$

Combining the equations (1) and (2) yields the fundamental structure and integrability identities [3, $(*)$ and $(**)$]. These make possible a first important step of the classification: the study of (quaternionic) one-dimensional quaternionic subalgebras.

Definition 1.6. ([3, Def. 1.9 and 2.1]) Let s be a metric Lie algebra. A linear subspace s' of s is said to be *totally geodesic*, if $\nabla_{s'} s' \subset s'$.

Suppose, in addition, that s is provided with a quaternionic structure q. A subalgebra s' of s is said to be a *quaternionic subalgebra*, if $\mathrm{q}s' \subset s'$.

Theorem 1.2. (D.V. Alekseevsky [3, Lemma 2.1, Remark 2.1, Prop. 3.1–3.2 and Def. 3.2]) *Let s be an Alekseevskian Lie algebra.*

(i) *Every quaternionic subalgebra of s is totally geodesic and has the same reduced scalar curvature as s.*

(ii) *The algebra s contains a quaternionic one-dimensional quaternionic subalgebra (this is canonically associated to the choice of a real one-dimensional ideal).*

(iii) *There are (up to scaling) exactly two quaternionic one-dimensional Alekseevskian Lie algebras; in Alekseevsky's notation: A_1^1 and C_1^1. The corresponding Alekseevsky spaces are the complex hyperbolic plane of reduced scalar curvature* $\mathrm{sk}_r = -\frac{1}{2}$ *and the quaternionic hyperbolic line* ($\mathrm{sk}_r = -1$).

(iv) *There is (up to scaling) a unique Alekseevskian Lie algebra of quaternionic dimension m with one-dimensional quaternionic subalgebra C_1^1. The corresponding Alekseevsky space is the quaternionic hyperbolic space $H^m_{\mathbb{H}}$.*

To describe Alekseevskian Lie algebras with one-dimensional quaternionic subalgebra A_1^1 we need further definitions. In the following we denote, in contrast to [3], metric Lie algebras and their linear subspaces generally by small letters; vectors by capital letters. Unless otherwise stated, we use the notation of [3]. The symbol "+" between Euclidean vector spaces *always* stands for the *orthogonal sum*.

Definition 1.7. ([3, Def. 6.1–5]) A metric Lie algebra $f = \operatorname{span}\{G, H\}$ with complex structure J is said to be a *key algebra with root* μ, if $G = JH$ and $[H, G] = \mu G$ with $\mu > 0$.

A metric Lie algebra $f + x$ with complex structure J is said to be an *elementary Kählerian Lie algebra*, if $f = \operatorname{span}\{G, H\}$ is a key subalgebra with root μ, $\operatorname{ad}_H|_x = \frac{1}{2}\mu \operatorname{Id}$, $\operatorname{ad}_G|_x = 0$ and $[X, Y] = \mu\langle JX, Y\rangle G$ for $X, Y \in x$ ([3, Def. 6.2] contains a misprint).

A representation $U \mapsto T_U$ of a Lie algebra u with complex structure J on a Euclidean vector space $(x, \langle\cdot, \cdot\rangle)$ with complex structure J_1 is said to be *symplectic*, if

(1) T (continued as derivation of the tensor algebra of x) annihilates the Kähler form $\rho_1 = \langle J_1\cdot, \cdot\rangle$ of J_1: $T_U\rho_1 = 0$ for all $U \in u$;

(2) $T_{JU}^{\mathrm{sym}} = J_1 T_U^{\mathrm{sym}}$ for all $U \in u$, where T_U^{sym} denotes the symmetric part of T_U. T is *non-degenerate*, if $T_u x = x$. The representation space x of a non-degenerate symplectic representation T of a key algebra $f = \operatorname{span}\{G, H\}$ with root μ admits a decomposition $x = x_+ + x_-$ with the following properties ([3, Prop. 6.1.3])):

$$x_- = J_1 x_+, \quad T_H|_{x_\pm} = \pm\tfrac{1}{2}\mu \operatorname{Id}, \quad T_G|_{x_+} = 0 \quad \text{and} \quad T_G|_{x_-} = -\mu J_1 .$$

This is called the *weight decomposition* of x with respect to the non-degenerate symplectic representation T.

In [3, Sect. 5] it is shown that an Alekseevskian Lie algebra s with one-dimensional quaternionic subalgebra A_1^1 is necessarily of the form $s = u + J_2 u$ (orthogonal sum), where $(u, J_1|_u)$ is a totally geodesic Kählerian subalgebra: the so-called *principal Kählerian subalgebra* of s (see [3, Def. 5.1]). u contains a key subalgebra $f_0 = \operatorname{span}\{G_0, H_0\}$ with root 1 such that $f_0 + J_2 f_0$ is the (canonical) one-dimensional quaternionic subalgebra of s and

$$u = f_0 + \{S \in f_0^\perp \mid \operatorname{ad}_{H_0} S = 0\},$$
$$u^\perp = \{S \in s \mid \operatorname{ad}_{H_0} S = \tfrac{1}{2} S\}.$$

The adjoint representation of s induces a representation of u on $u^\perp = J_2 u$ (addition of roots), which represents the model for the next definition.

Definition 1.8. ([3, Def. 5.2–3]) A Kählerian Lie algebra (u, J) is *admissible*, if $u = f_0 + u_0$ is a direct orthogonal sum of a key algebra $f_0 = \operatorname{span}\{G_0, H_0\}$ with root 1 and a real solvable Kählerian Lie algebra u_0.

A representation $U \mapsto T_U$ of such a Lie algebra u on a Euclidean vector space $\tilde{u}$ together with a vector space isometry $\phi : u \to \tilde{u}$ is said to be a *Q-representation*, if there is a linear form ω on u_0 such that, with the notation $\tilde{U} := \phi(U)$ $(U \in u)$, $J_1 = -\phi J \phi^{-1}$, $\hat{J}|_{\tilde{u}_0} = J_1$ and $\hat{J}|_{\tilde{f}_0} = -J_1$, the following is true:

(1) $T_{H_0} = \frac{1}{2}\operatorname{Id}$ and $T_{G_0} = 0$ (Q1).

(2) The endomorphisms T_U $(U \in u)$ have only real eigenvalues (Q8).

(3) The endomorphisms T_U $(U \in u_0)$ annihilate the Kähler form $\hat{\rho} = \langle \hat{J}\cdot, \cdot\rangle$ of the complex structure $\hat{J}$ (Q7) and satisfy the following conditions (Q2–6), where (Q3) and (Q6) are only imposed on $U_0 \in u_0$. T_U^{sk} and T_U^{sym} denote the skew and symmetric parts of T_U respectively; ∇ the covariant derivative of u:

(Q2) $T_U \tilde{H}_0 = \frac{1}{2}\tilde{U} - 2\omega(U)\tilde{G}_0$, $T_U \tilde{G}_0 = \frac{1}{2}\widetilde{JU} + 2\omega(U)\tilde{H}_0$;

(Q3) $T_U\tilde{U}_0 \equiv \frac{1}{2}\langle U, U_0\rangle \tilde{H}_0 + \frac{1}{2}\langle JU, U_0\rangle \tilde{G}_0 \bmod \tilde{u}_0$;
(Q4) $T_U^{\mathrm{sk}} = \phi\nabla_U\phi^{-1} + 2\omega(U)J_1$;
(Q5) $T_{JU}^{\mathrm{sym}}|_{\tilde{u}_0} = J_1 T_U^{\mathrm{sym}}|_{\tilde{u}_0}$;
(Q6) $T_U^{\mathrm{sym}}\tilde{U}_0 - T_{U_0}^{\mathrm{sym}}\tilde{U} = \langle JU, U_0\rangle \tilde{G}_0$.
The numbering (Q1–8) refers to [3, Lemma 5.5].

We indicate how to construct an Alekseevskian Lie algebra out of a Q-representation.

Theorem 1.3. (D.V. Alekseevsky [3, Lemma 5.5 and Prop. 5.1]) *Q-representations of an admissible Kählerian Lie algebra (u, J) and Alekseevskian Lie algebras with principal Kählerian subalgebra (u, J) are related by the following natural correspondence.*

If s is an Alekseevskian Lie algebra with principal Kählerian subalgebra u, then the representation of u on J_2u induced by the adjoint representation of s is a Q-representation with $\phi = J_2|_u : u \xrightarrow{\sim} u^\perp$ ($\omega = \omega_1|_{u_0}$, see eq. 1).

Conversely let (T, ϕ) be a Q-representation of an admissible Kählerian Lie algebra (u, J) on the Euclidean vector space $\tilde{u} = \phi(u)$. A quaternionic structure $\mathrm{q} = \mathrm{span}\{J_\alpha \mid \alpha = 1, 2, 3\}$ *on the Euclidean vector space $s := u + \tilde{u}$ (orthogonal sum) is defined as follows:*

$$J_1|_u := J, \quad J_1|_{\tilde{u}} := -\phi J\phi^{-1},$$
$$J_2|_u := \phi, \quad J_2|_{\tilde{u}} := -\phi^{-1},$$
$$J_3 := J_1J_2.$$

The following conditions define the structure of a Lie algebra on s:
(1) *u is a subalgebra of s;*
(2) $\mathrm{ad}_U|_{\tilde{u}} := T_U$ *for all $U \in u$;*
(3) $[\tilde{U}, \tilde{V}] := \hat{\rho}(\tilde{U}, \tilde{V})G_0$ *for all $U, V \in u$.*

(Sketch of) proof. We only comment on the second part of the construction. That the above arrangements turn $s = u + \tilde{u}$ into a metric *Lie algebra* (Jacobi identity) with quaternionic structure q, follows from the fact that T is a representation and from the conditions (Q1) and (Q7) alone. s is real solvable, thanks to u_0 being real solvable and to (Q8). The remaining conditions (Q2–6) ensure that (s, q) is quaternionic Kählerian. □

Theorem 1.3 makes it possible to apply the structure theory of real solvable Kählerian Lie algebras and their symplectic representations, developed by S.G. Gindikin, I.I. Pjatecky–Šapiro and E.B. Vinberg, to the classification problem in question. Admissible Kählerian Lie algebras $u = f_0 + u_0$ which admit a Q-representation are non-degenerate (see [3, Def. 1.8d, Def. 5.5 and Cor. 5.1]) and, hence (see [3, Prop. 6.2], which contains two misprints in 4), decompose as semidirect sum of elementary Kählerian Lie algebras $u_0 = \sum_{i\geqslant 1}(f_i + x_i)$, i.e. $[f_i + x_i, f_j + x_j] \subset f_j + x_j$ for $i \geqslant j$, with symplectic representations $\mathrm{ad}_{f_i}|_{x_j}$ for $i > j$ and commuting key algebras: $[f_i, f_j] = 0$ $(i \neq j)$.

Definition 1.9. ([4, Def. 3.1]) The *rank* of a solvable Lie algebra s is the dimension of a Cartan subalgebra of s. The rank of a Alekseevsky space $\mathcal{S}$ is the rank of its Alekseevskian Lie algebra s.

A Cartan subalgebra is a nilpotent subalgebra which coincides with its normalizer. According to [22, Lemma 4.1.1], Cartan subalgebras are necessarily maximally nilpotent. The rank is well defined, as, according to [7, Cor. 2.3], the Cartan subalgebras of a solvable Lie algebra are conjugate.

Remark 1.1. *The rank of* $u = f_0 + \sum_{i\geqslant 1}(f_i + x_i)$ *coincides with the number of key algebras of* u, *because* $a := \mathrm{span}\{H_i \mid i = 0, 1, \dots\}$ *is obviously an (Abelian) Cartan subalgebra of* u. *Due to* $T_{H_0} = \frac{1}{2}\mathrm{Id}$ $(Q1)$, *it is even true that* $a = \mathrm{norm}_s(a)$ *such that* a *is also a Cartan subalgebra of* s, *whence* u *and* s *are of the same rank.*

Alekseevsky shows that the existence of a Q-representation of u imposes severe restrictions on the rank of u and on the allowed roots of the key algebras occurring in the decomposition of u_0. For instance, the roots μ_i can only be 1, $1/\sqrt{2}$ or $1/\sqrt{3}$. Correspondingly Alekseevsky distinguishes between three types of admissible Kählerian Lie algebras u: u is of *type* α ($\alpha = 1, 2, 3$), if $1/\sqrt{\alpha}$ is the smallest root.

Type 3: There is a unique Lie algebra of type 3 which admits a Q-representation: $u = f_0 + f_1$. The Q-representation is unique and yields the symmetric Alekseevsky space $G_2^{(2)}/SO(4)$ (see [3, Prop. 8.1], in which the the formulas for T_H are incorrectly printed).

Type 2: Lie algebras of type 2 which admit Q-representations must have rank 3. We shall see in Sect. 2.1 that there is a countably infinite family $u(p)$ $(p = 0, 1, \dots)$ of such Lie algebras and that any of them admits a unique Q-representation. The corresponding Alekseevsky spaces will be denoted by $\mathcal{T}(p)$ and will be described in Sect. 2.1. In the original classification ([3, Sect. 8, Prop. on p. 329]) only $\mathcal{T}(0) \cong SO_0(3, 4)/(SO(3) \times SO(4))$ occurs, due to an error in the calculations.

Type 1: Apart from the hermitian symmetric Alekseevsky spaces ([3, Prop. 9.1]), this class contains the Alekeevskian spaces of rank 4: the $\mathcal{W}$- and $\mathcal{V}$-spaces which will be reviewed in the next section.

1.3. $\mathcal{W}$-Spaces, $\mathcal{V}$-Spaces and isometric mappings

The following concept (going back to Pjatecky-Šapiro) from the theory of real solvable Kählerian Lie algebras is all-important for the classification of Kählerian Lie algebras of type 1 and rank > 2 which admit Q-representations.

Definition 1.10. ([3, Def. 6.7 and 9.1]) Let x, y and z be Euclidean vector spaces. A bilinear mapping $\psi : x \times z \to y$ is said to be *isometric*, if

$$\langle \psi(X, Z), \psi(X, Z)\rangle = \langle X, X\rangle\langle Z, Z\rangle$$

for all $X \in x$ and $Z \in z$. Isometric mappings $\psi : x \times z \to y$ and $\psi' : x' \times z' \to y'$ are said to be *equivalent*, if there are isomorphisms $\sigma : x \to x'$, $\tau : z \to z'$ and $\upsilon : y \to y'$ of Euclidean vector spaces such that the following diagram commutes:

$$\begin{array}{ccc} x \times z & \xrightarrow{\psi} & y \\ {\scriptstyle \sigma\times\tau}\downarrow & & \downarrow {\scriptstyle \upsilon} \\ x' \times z' & \xrightarrow{\psi'} & y' \end{array}$$

An isometric mapping $\psi : x \times z \to y$ with $k = \dim x \neq 0$ is said to be *special*, if $\dim y = \dim z \neq 0$. k is called the *order* of the special isometric mapping.

As Alekseevsky notices, there is an intimate relation between special isometric mappings and $\mathbb{Z}_2$-graded Clifford modules. This will be discussed in great detail in Sect. 2.2.

Let x_-, z_- and y_- be Euclidean vector spaces. Every isometric mapping $\psi : x_- \times z_- \to y_-$ defines a Kählerian Lie algebra $u(\psi) = (f_0 + u_0, J)$ of type 1 and rank 4 by means of the following recipe ([3, Prop. 9.3]):

1. u_0 is a semi-direct sum $u_0 = (f_1 + x_1) + (f_2 + x_2) + f_3$ of elementary Kählerian Lie algebras with commuting key algebras with root 1.

2. x_1 admits a (J-invariant) decomposition $x_1 = y + z$ such that the following is true for $x := x_2$, y and z: $\mathrm{ad}_{f_3}|_y$, $\mathrm{ad}_{f_2}|_z$ and $\mathrm{ad}_{f_3}|_x$ are non-degenerate symplectic representations with weight decompositions $y = y_+ + y_-$, $z = z_+ + z_-$ and $x = x_+ + x_-$, where $y_+ = Jy_-$, $z_+ = Jz_-$ and $x_+ = Jx_-$. Furthermore:

$$[f_1, x] = [f_2, y] = [f_3, z] = [y, z] = [x, z_+] = [x_+, y_+] = [x_-, y_-] = 0\,.$$

3. The remaining Lie brackets are computed according to the rules ($X \in x$, $X_\pm \in x_\pm$, $Y_\pm \in y_\pm$ and $Z_\pm \in z_\pm$):

$$[X_-, Z_-] = \tfrac{1}{\sqrt{2}}\psi(X_-, Z_-)\,, \quad [JX, Z_-] = J[X, Z_-]\,, \qquad [x_-, y_+] \subset z_+\,,$$
$$[X_+, Y_-] = [JX_+, JY_-]\,, \qquad \langle [X_-, Y_+], Z_+\rangle = -\tfrac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,.$$

(The factor $1/\sqrt{2}$ is apparently missing in [3, Prop. 9.3].)

Remark 1.2. Let V and W be vectors of a Euclidean vector space. $V \otimes W$ denotes, for the remainder of this paper, the linear mapping $S \mapsto V\langle W, S\rangle$. Its domain should be clear from the corresponding context. In the formulas below, e.g., it is $\tilde{u}$.

Theorem 1.4. (D.V. Alekseevsky [3, Prop. 9.2–4]) *The Kählerian Lie algebras of type* 1 *and rank* > 2 *which admit a Q-representation are given by the Lie algebras $u = u(\psi)$ for which either*

(i) $x_- = 0$ *(hence $\psi = 0$) and $u = u(p, q) \cong u(q, p)$ is completely determined by the parameters* $p = \dim y_-$ *and* $q = \dim z_-$ *or*

(ii) ψ *is a special isometric mapping.*

Every such Lie algebra u has a unique Q-representation T. The corresponding Alekseevsky spaces are denoted by $\mathcal{W}(p, q)$ and $\mathcal{V}(\psi)$ in the cases (i) and (ii) respectively.

The definition of the Q-representation T of $u(\psi)$ now follows. Set $f := \sum_{i=0}^{3} f_i$. The operators $T_{H_\alpha}|_{\tilde{f}}$ and $T_{G_\alpha}|_{\tilde{f}}$ are given, with respect to the orthonormal basis

$$\tilde{P}_0 := \tfrac{1}{2}(\tilde{H}_0 + \tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3)\,,$$
$$\tilde{P}_\alpha := \tfrac{1}{2}(-\tilde{H}_0 - \tilde{H}_\alpha + \tilde{H}_\beta + \tilde{H}_\gamma)\,, \quad \{\alpha, \beta, \gamma\} = \{1, 2, 3\},$$
$$\tilde{Q}_i := \hat{J}\tilde{P}_i\,, \quad i \in \{0, 1, 2, 3\}$$

of $\tilde{f}$, by the following formulas:

$$
\begin{aligned}
T_{H_\alpha}|_{\operatorname{span}\{\tilde P_0,\tilde P_\alpha,\tilde Q_\beta,\tilde Q_\gamma\}} &= \tfrac{1}{2}\mathrm{Id}\,,\\
T_{H_\alpha}|_{\operatorname{span}\{\tilde Q_0,\tilde Q_\alpha,\tilde P_\beta,\tilde P_\gamma\}} &= -\tfrac{1}{2}\mathrm{Id}\,,
\end{aligned}
$$

$$
T_{G_\alpha}:\begin{cases} \tilde Q_0 \mapsto \tilde P_\alpha \mapsto 0\,,\\ \tilde Q_\alpha \mapsto \tilde P_0 \mapsto 0\,,\\ \tilde P_\beta \mapsto \tilde Q_\gamma \mapsto 0\,. \end{cases}
$$

$T_{f_1}|_{\tilde x}$, $T_{f_2}|_{\tilde y}$ and $T_{f_3}|_{\tilde z}$ are non-degenerate symplectic representations with weight decompositions $\tilde x = \tilde x_+ + \tilde x_-$, $\tilde y = \tilde y_+ + \tilde y_-$ and $\tilde z = \tilde z_+ + \tilde z_-$ and

$$T_{f_1}(\tilde y + \tilde z) = T_{f_2}(\tilde x + \tilde z) = T_{f_3}(\tilde x + \tilde y) = 0\,.$$

In the following, let $X_\pm$, $Y_\pm$ and $Z_\pm$ denote arbitrary elements and $(X^i_\pm)_i$, $(Y^i_\pm)_i$ and $(Z^i_\pm)_i$ arbitrary orthonormal bases of $x_\pm$, $y_\pm$ and $z_\pm$ respectively. $U \circ V := 2\nabla_U V$ defines a bilinear mapping from the product of two of the spaces x, y and z to the third. With these conventions the expressions for the remaining operators T_U $(U \in y + z + x)$ read

$$
\begin{aligned}
T_{X_+} &= \tilde P_0 \otimes \tilde X_+ - \tilde X_+ \otimes \tilde P_1 + \widetilde{JX_+} \otimes \tilde Q_0 - \tilde Q_1 \otimes \widetilde{JX_+}\\
&\quad + \sum_i \widetilde{X_+ \circ Y^i_-} \otimes \tilde Y^i_- + \sum_i \widetilde{X_+ \circ Z^i_-} \otimes \tilde Z^i_-\,,\\
T_{Y_+} &= \tilde P_0 \otimes \tilde Y_+ - \tilde Y_+ \otimes \tilde P_2 + \widetilde{JY_+} \otimes \tilde Q_0 - \tilde Q_2 \otimes \widetilde{JY_+}\\
&\quad + \sum_i \widetilde{Y_+ \circ X^i_-} \otimes \tilde X^i_- + \sum_i \widetilde{Y_+ \circ Z^i_-} \otimes \tilde Z^i_-\,,\\
T_{Z_+} &= \tilde P_0 \otimes \tilde Z_+ - \tilde Z_+ \otimes \tilde P_3 + \widetilde{JZ_+} \otimes \tilde Q_0 - \tilde Q_3 \otimes \widetilde{JZ_+}\\
&\quad + \sum_i \widetilde{Z_+ \circ X^i_-} \otimes \tilde X^i_- + \sum_i \widetilde{Z_+ \circ Y^i_-} \otimes \tilde Y^i_-\,,\\
T_{X_-} &= -\tilde P_2 \otimes \tilde X_- - \tilde X_- \otimes \tilde P_3 - \widetilde{JX_-} \otimes \tilde Q_2 - \tilde Q_3 \otimes \widetilde{JX_-}\\
&\quad + \sum_i \widetilde{X_- \circ Y^i_-} \otimes \tilde Y^i_- + \sum_i \widetilde{X_- \circ Z^i_+} \otimes \tilde Z^i_+\,,\\
T_{Y_-} &= -\tilde P_1 \otimes \tilde Y_- - \tilde Y_- \otimes \tilde P_3 - \widetilde{JY_-} \otimes \tilde Q_1 - \tilde Q_3 \otimes \widetilde{JY_-}\\
&\quad + \sum_i \widetilde{Y_- \circ X^i_-} \otimes \tilde X^i_- + \sum_i \widetilde{Y_- \circ Z^i_+} \otimes \tilde Z^i_+\,,\\
T_{Z_-} &= -\tilde P_1 \otimes \tilde Z_- - \tilde Z_- \otimes \tilde P_2 - \widetilde{JZ_-} \otimes \tilde Q_1 - \tilde Q_2 \otimes \widetilde{JZ_-}\\
&\quad + \sum_i \widetilde{Z_- \circ X^i_-} \otimes \tilde X^i_- + \sum_i \widetilde{Z_- \circ Y^i_+} \otimes \tilde Y^i_+\,.
\end{aligned}
$$

Remark 1.3. The bilinear mapping "$\circ$" introduced above is completely determined by the following relations which are obtained from the Koszul formula. The first three formulas are, as usual, to be read selecting either all the upper or all the lower signs and $\epsilon \in \{-, +\}$.

$$
\begin{aligned}
X_\epsilon \circ Y_{\pm\epsilon} &= \pm Y_{\pm\epsilon} \circ X_\epsilon \ \in \ z_{\mp}\,,\\
X_\epsilon \circ Z_{\pm\epsilon} &= \pm\epsilon Z_{\pm\epsilon} \circ X_\epsilon \ \in \ y_{\mp}\,,\\
Y_\epsilon \circ Z_{\pm\epsilon} &= Z_{\pm\epsilon} \circ Y_\epsilon \ \in \ x_{\mp}\,,\\
\langle X_+ \circ Y_+, Z_-\rangle &= -\tfrac{1}{\sqrt{2}}\langle JY_+, \psi(JX_+, Z_-)\rangle\,,\\
\langle X_- \circ Y_-, Z_-\rangle &= -\tfrac{1}{\sqrt{2}}\langle Y_-, \psi(X_-, Z_-)\rangle\,,\\
\langle X_+ \circ Y_-, Z_+\rangle &= \tfrac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,,\\
\langle X_- \circ Y_+, Z_+\rangle &= -\tfrac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle X_+ \circ Z_+, Y_-\rangle &= -\tfrac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,,\\
\langle X_- \circ Z_-, Y_-\rangle &= \tfrac{1}{\sqrt{2}}\langle Y_-, \psi(X_-, Z_-)\rangle\,,\\
\langle X_+ \circ Z_-, Y_+\rangle &= \tfrac{1}{\sqrt{2}}\langle JY_+, \psi(JX_+, Z_-)\rangle\,,\\
\langle X_- \circ Z_+, Y_+\rangle &= \tfrac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle Y_+ \circ Z_+, X_-\rangle &= -\tfrac{1}{\sqrt{2}}\langle JY_+, \psi(X_-, JZ_+)\rangle\,,\\
\langle Y_- \circ Z_-, X_-\rangle &= \tfrac{1}{\sqrt{2}}\langle Y_-, \psi(X_-, Z_-)\rangle\,,\\
\langle Y_+ \circ Z_-, X_+\rangle &= \tfrac{1}{\sqrt{2}}\langle JY_+, \psi(JX_+, Z_-)\rangle\,,\\
\langle Y_- \circ Z_+, X_+\rangle &= \tfrac{1}{\sqrt{2}}\langle Y_-, \psi(JX_+, JZ_+)\rangle\,.
\end{aligned}
$$

(The last formula in [3, (1) p. 334] contains a misprint and the factor $1/\sqrt{2}$ is missing at the end of the computation in [3, (3) p. 334].)

The classification of isometry classes of Alekseevskian $\mathcal{V}$-spaces can now be reduced to the classification of equivalence classes of special isometric mappings (see Prop. 2.23). The latter and its consequences will be presented in Sect. 2.2. This will correct/refine [3, Thm. 10.1].

2. Classification and basic facts

The theoretical physicists de Wit and Van Proeyen have discovered that Alekseevsky's classification [3] is not complete (a very rough sketch of their argument is given in the appendix of [10]). In [12] they classify certain homogeneous $N = 2$ supergravity models in $d = 5$ spacetime dimensions. Owing to a connection between $N = 2$ supergravity and Alekseevsky spaces, which was used earlier by Cecotti [8], they conclude from the result of their classification that there should exist Alekseevskian spaces missing in [3]. Furthermore they conjecture that the absence of these spaces is merely due to an error in the calculations of [3].

Motivated by [12] we have examined [3] with the intention of smoothing out possible errors and thus completing the original classification. Our results confirm de Wit's and Van Proeyen's discovery.

We found that the classification of Alekseevskian spaces of rank 3 is indeed flawed by a miscalculation. Correcting it and completing the classification guided by Alekseevsky's example one arrives at the result that the original classification includes only the first member of a family $\mathcal{T}(p)$, $p = 0, 1, 2 \ldots$, of rank 3 spaces (see Sect. 2.1).

The classification of rank 4 spaces is slightly obscured by a subtle conceptual error which leads to the identification of non-isomorphic spaces (see Sect. 2.2).

We give the new examples explicitly as Alekseevskian Lie algebras, in contrast to [12], where the corresponding $N = 2$ supergravity theories in $d = 5$ are discussed.

2.1. Rank 3

Alekseevsky's classification of the Kählerian Lie algebras of type 2 that admit a Q-representation (see [3, Sect. 8 Prop. on p. 329]) is not complete. Before completing it by Prop. 2.1, we remind the reader of the pertinent (correct) results obtained in [3, Sect. 8] and point out the error in computation which cut short Alekseevsky's arguments.

From [3, Sect. 8] we know that every Kählerian Lie algebra (u, J) of type 2 which posses a Q-representation is of the form $u = f_0 + u_0$, where u_0 is a semi-direct sum of two elementary Kählerian Lie algebras $f_1 + x_1$ and f_2, $[f_1, f_2] = 0$ and $\mathrm{ad}_{f_2}|_{x_1}$ is a non-degenerate symplectic representation. One of the two key algebras f_1, f_2 has root 1 and will be denoted by f', in accordance with [3, Sect. 8], the other has root $1/\sqrt{2}$ and will be denoted by f.

Under the assumption that u has a Q-representation T, it is shown in [3, Sect. 8] that the $T_U|_{\tilde{x}_1^-}$ for $U \in f_0 + f_1 + f_2$ are uniquely determined by the properties of a Q-representation. With respect to the orthonormal basis

$$
\begin{aligned}
\tilde{P}_+ &:= \tfrac{1}{2}(\tilde{H}_0 + \tilde{H}' + \sqrt{2}\tilde{H}), & \tilde{Q}_+ &:= \hat{J}\tilde{P}_+ = \tfrac{1}{2}(\tilde{G}_0 - \tilde{G}' + \sqrt{2}\tilde{G}), \\
\tilde{P}_0 &:= \tfrac{1}{\sqrt{2}}(\tilde{G}_0 + \tilde{G}'), & \tilde{Q}_0 &:= \hat{J}\tilde{P}_0 = \tfrac{1}{\sqrt{2}}(-\tilde{H}_0 + \tilde{H}'), \\
\tilde{P}_- &:= \tfrac{1}{2}(\tilde{H}_0 + \tilde{H}' - \sqrt{2}\tilde{H}), & \tilde{Q}_- &:= \hat{J}\tilde{P}_- = \tfrac{1}{2}(\tilde{G}_0 - \tilde{G}' - \sqrt{2}\tilde{G})
\end{aligned}
$$

of $\tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2$ they are given by the following formulas, valid on $\tilde{x}_1^-$.

$$
\begin{aligned}
T_{H'} &= \tfrac{1}{2}(\tilde{P}_+ \otimes \tilde{P}_+ + \tilde{P}_0 \otimes \tilde{P}_0 + \tilde{P}_- \otimes \tilde{P}_-) \\
&\quad - \tfrac{1}{2}(\tilde{Q}_+ \otimes \tilde{Q}_+ + \tilde{Q}_0 \otimes \tilde{Q}_0 + \tilde{Q}_- \otimes \tilde{Q}_-), \\
T_{G'} &= -\tilde{P}_0 \otimes \tilde{Q}_0 - \tilde{P}_+ \otimes \tilde{Q}_+ - \tilde{P}_- \otimes \tilde{Q}_-, \\
T_H &= \tfrac{1}{\sqrt{2}}(\tilde{P}_+ \otimes \tilde{P}_+ - \tilde{P}_- \otimes \tilde{P}_- + \tilde{Q}_+ \otimes \tilde{Q}_+ - \tilde{Q}_- \otimes \tilde{Q}_-), \\
T_G &= \tilde{Q}_0 \otimes \tilde{Q}_- - \tilde{Q}_+ \otimes \tilde{Q}_0 + \tilde{P}_0 \otimes \tilde{P}_- - \tilde{P}_+ \otimes \tilde{P}_0.
\end{aligned} \tag{4}
$$

In addition, it is shown that $T_{f'}\tilde{x}_1 = 0$. Finally, $T_{H_0} = \frac{1}{2}\mathrm{Id}$ and $T_{G_0} = 0$ holds for every Q-representation (Q1).

Under [3, point 8 on p. 328] it is (erroneously) claimed that the above formulas cannot be extended to define a Q-representation unless $x := x_1 = 0$. Accordingly Alekseevsky finds only one Kählerian Lie algebra of type 2 which has a Q-representation (defined by the above formulas).

The proof of this claim contains a miscalculation. The first line of the computation should read:

$$0 = (T_{G_i} T_{X_+} - T_{X_+} T_{G_i})\, \tilde{G}_0 = \tfrac{1}{2}(T_{G_i} \widetilde{J_1 X_+} + T_{X_+} \tilde{H}_i) - \omega(G_i)\, \tilde{X}_+ .$$

Remark 2.1. Apart from condition (Q2) of the definition of Q-representation (Def. 1.8) we only use that $\omega(X_+) = 0$ and $\omega(G_i) = -1/4\mu_i$, which follows from the structure equation $4\,d\omega = \rho = \langle J\cdot, \cdot\rangle$ (see [3, Lemma 5.3]).

Now, the last line should read:

$$T_{G_i} \widetilde{J_1 X_+} + T_{H_i} \tilde{X}_+ = \tfrac{1}{2}\mu \tilde{X}_+ + 2\omega(G_i)\, \tilde{X}_+ = \frac{1}{2}\left(\mu_i - \frac{1}{\mu_i}\right) \tilde{X}_+ .$$

For $f_i = f'$, i. e. $\mu_i = 1$, this does not imply anything new, since we know already that $T_{f'}\tilde{x} = 0$. In particular, we cannot conclude that $x = 0$. The proposition in [3, Sect. 8, p. 329] should be replaced with the following.

Proposition 2.1. *Every Kählerian Lie algebra (u, J) of type 2 which admits a Q-representation is (up to scaling) of the form $u = f_0 + u_0$, where*

$$u_0 = f_1 + x_1 + f_2$$

is a semi-direct orthogonal sum of the elementary Kählerian Lie algebras $f_1 + x_1$ and f_2 with root 1 and $1/\sqrt{2}$ respectively. The key algebras $f_i = \operatorname{span}\{G_i, H_i\}$ commute. $\mathrm{ad}_{f_2}|_{x_1}$ is a non-degenerate symplectic representation with weight decomposition $x_1 = x_+ + x_-$, $x_- = Jx_+$.

Furthermore, u has a unique Q-representation $T : u \to \mathrm{End}(\tilde{u})$. Let $\sim : u \to \tilde{u}$ denote, as usual, the corresponding isometry of Euclidean vector spaces. Then, with respect to the orthonormal basis

$$\begin{aligned}
\tilde{P}_+ &:= \tfrac{1}{2}(\tilde{H}_0 + \tilde{H}_1 + \sqrt{2}\tilde{H}_2), & \tilde{Q}_+ &:= \hat{J}\tilde{P}_- = \tfrac{1}{2}(\tilde{G}_0 - \tilde{G}_1 + \sqrt{2}\tilde{G}_2),\\
\tilde{P}_0 &:= \tfrac{1}{\sqrt{2}}(\tilde{G}_0 + \tilde{G}_1), & \tilde{Q}_0 &:= \hat{J}\tilde{P}_0 = \tfrac{1}{\sqrt{2}}(-\tilde{H}_0 + \tilde{H}_1),\\
\tilde{P}_- &:= \tfrac{1}{2}(\tilde{H}_0 + \tilde{H}_1 - \sqrt{2}\tilde{H}_2), & \tilde{Q}_- &:= \hat{J}\tilde{P}_+ = \tfrac{1}{2}(\tilde{G}_0 - \tilde{G}_1 - \sqrt{2}\tilde{G}_2)
\end{aligned}$$

of $\tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2$, T is given by

$$\begin{aligned}
T_{H_0} &= \tfrac{1}{2}\mathrm{Id},\\
T_{H_1} &= \tfrac{1}{2}(\tilde{P}_+ \otimes \tilde{P}_+ + \tilde{P}_0 \otimes \tilde{P}_0 + \tilde{P}_- \otimes \tilde{P}_-)\\
&\quad - \tfrac{1}{2}(\tilde{Q}_+ \otimes \tilde{Q}_+ + \tilde{Q}_0 \otimes \tilde{Q}_0 + \tilde{Q}_- \otimes \tilde{Q}_-),\\
T_{H_2}|_{\tilde{x}_1^\perp} &= \tfrac{1}{\sqrt{2}}(\tilde{P}_+ \otimes \tilde{P}_+ - \tilde{P}_- \otimes \tilde{P}_- + \tilde{Q}_+ \otimes \tilde{Q}_+ - \tilde{Q}_- \otimes \tilde{Q}_-),\\
T_{H_2}|_{\tilde{x}_+} &= \tfrac{1}{4}\sqrt{2}\mathrm{Id},\\
T_{H_2}|_{\tilde{x}_-} &= -\tfrac{1}{4}\sqrt{2}\mathrm{Id},
\end{aligned}$$

$$T_{G_0} = 0\,,$$
$$T_{G_1} = -\tilde{P}_0 \otimes \tilde{Q}_0 - \tilde{P}_+ \otimes \tilde{Q}_+ - \tilde{P}_- \otimes \tilde{Q}_-\,,$$
$$T_{G_2}|_{\tilde{x}_1^\perp} = \tilde{Q}_0 \otimes \tilde{Q}_- - \tilde{Q}_+ \otimes \tilde{Q}_0 + \tilde{P}_0 \otimes \tilde{P}_- - \tilde{P}_+ \otimes \tilde{P}_0\,,$$
$$T_{G_2}|_{\tilde{x}_+} = 0\,,$$
$$T_{G_2}|_{\tilde{x}_-} = -\tfrac{1}{2}\sqrt{2}J_1\,,$$
$$T_{X_+} = \tilde{P}_+ \otimes \tilde{X}_+ + \tfrac{1}{\sqrt{2}}\tilde{P}_0 \otimes \widetilde{JX_+} + \widetilde{JX_+} \otimes \tilde{Q}_- - \tfrac{1}{\sqrt{2}}\tilde{X}_+ \otimes \tilde{Q}_0\,,$$
$$T_{X_-} = \tilde{P}_- \otimes \tilde{X}_- + \tfrac{1}{\sqrt{2}}\tilde{P}_0 \otimes \widetilde{JX_-} + \widetilde{JX_-} \otimes \tilde{Q}_+ - \tfrac{1}{\sqrt{2}}\tilde{X}_- \otimes \tilde{Q}_0\,.$$

In particular, $T_{f_2}|_{\tilde{x}_1}$ *is a non-degenerate symplectic representation with weight decomposition* $\tilde{x}_1 = \tilde{x}_+ + \tilde{x}_-$.

Proof. We take up our previous exposition. Next, we show that the conditions (Q1–8) for Q-representations uniquely determine T_X and $T_F\tilde{X}$ for $X \in x_1$ and $F \in f$. First, we compute the covariant derivative ∇_X of u for $X \in x_1 = x_+ + x_-$:

$$\nabla_{X_\pm} = \tfrac{1}{2}\,(\mu_1 H_1 \pm \mu_2 H_2) \wedge X_\pm + \tfrac{1}{2}\,(\mu_1 G_1 \pm \mu_2 G_2) \wedge JX_\pm\,.$$

Where $\mu_1, \mu_2 \in \{1, 1/\sqrt{2}\}$ are the corresponding roots of the key algebras f_1, f_2. $V \wedge W$ $(V, W \in u)$ denotes the skew endomorphism $V \otimes W - W \otimes V$ of u (see Sect. 1, Remark 1.2). From this we obtain immediately the skew part of the operators T_X by (Q4) observing that $\omega(X) = 0$ (see structure equation in Remark 2.1).

$$T^{\mathrm{sk}}_{X_\pm} = \tfrac{1}{2}(\mu_1\tilde{H}_1 \pm \mu_2\tilde{H}_2) \wedge \tilde{X}_\pm + \tfrac{1}{2}(\mu_1\tilde{G}_1 \pm \mu_2\tilde{G}_2) \wedge \widetilde{JX}_\pm \tag{5}$$

Now we can already compute T_F for $F \in f$:

$$\begin{aligned}
T_H\tilde{X}_\pm &\overset{(a)}{=} T^{\mathrm{sym}}_H \tilde{X}_\pm \overset{(Q6)}{=} T^{\mathrm{sym}}_{X_\pm}\tilde{H} \overset{(a)}{=} c_\pm[T_H, T^{\mathrm{sk}}_{X_\pm}]\tilde{H}\\
&= c_\pm\left\{T_H\left(-\frac{1}{c_\pm}\tilde{X}_\pm\right) - T^{\mathrm{sk}}_{X_\pm}\tfrac{1}{2}(\tilde{H}_0 + \tilde{H}')\right\}\\
&= -T_H\tilde{X}_\pm - \tfrac{1}{2}c_\pm T^{\mathrm{sk}}_{X_\pm}\tilde{H}'\,,
\end{aligned}$$

where

$$c_\pm = \begin{cases} 4/\sqrt{2} & \text{if } f = f_1,\\ \pm 4/\sqrt{2} & \text{if } f = f_2\,. \end{cases}$$

(a) holds, because due to [3, Sect. 7(A1) and Prop. 7.1d]) we have $T_{H_1} = T^{\mathrm{sym}}_{H_1}$. We conclude

$$8T_H\tilde{X}_\pm = -2c_\pm T^{\mathrm{sk}}_{X_\pm}\tilde{H}' = \left\{\begin{array}{ll} \pm c_\pm\tilde{X}_\pm & \text{if } f = f_1\\ c_\pm\tilde{X}_\pm & \text{if } f = f_2 \end{array}\right\} = \pm 2\sqrt{2}\tilde{X}_\pm\,. \tag{6}$$

According to [3, Table A, Cor. 7.2],

$$T^{\mathrm{sk}}_G = -\tfrac{1}{4}\sqrt{2}J_1 \quad \text{on} \quad \tilde{x}_1\,.$$

(Q5) and equation (6) give

$$T_G^{\mathrm{sym}}\tilde{X}_\pm = J_1 T_H^{\mathrm{sym}}\tilde{X}_\pm \overset{(a)}{=} \pm\tfrac{1}{4}\sqrt{2}J_1\tilde{X}_\pm\,.$$

Adding these two expressions yields

$$T_G|_{\tilde{x}_+} = 0\,,\qquad T_G|_{\tilde{x}_-} = -\tfrac{1}{2}\sqrt{2}J_1\,.$$

The operators T_X^{sym} can now be computed using the following equation

$$T_X^{\mathrm{sym}} = \frac{2}{\mu_1}T^{\mathrm{sym}}_{[H_1,X]} \overset{(a)}{=} \frac{2}{\mu_1}\left[T_{H_1},T_X^{\mathrm{sk}}\right]\,. \tag{7}$$

We have to discuss two cases.

1) First, we consider the case when $f = f_1$ and $f' = f_2$. In this case we will see that u has no Q-representation, if $x_1 \neq 0$.

With the help of the equations (5), (7) and the knowledge of $T_{H_1} = T_H$ (equations (4) and (6)) we compute:

$$\begin{aligned} T_{X_+} &= -\tfrac{1}{2}\sqrt{2}\tilde{X}_+\otimes\tilde{Q}_0 + \widetilde{JX_+}\otimes\tilde{Q}_- + \tilde{P}_+\otimes\tilde{X}_+ + \tfrac{1}{2}\sqrt{2}\tilde{P}_0\otimes\widetilde{JX_+},\\ T_{JX_+} &= \widetilde{JX_+}\otimes\tilde{P}_- - \tfrac{1}{2}\sqrt{2}\tilde{X}_+\otimes\tilde{P}_0 - \tfrac{1}{2}\sqrt{2}\tilde{Q}_0\otimes\widetilde{JX_+} - \tilde{Q}_+\otimes\tilde{X}_+\,. \end{aligned}$$

From this it follows that

$$[T_{X_+},T_{JX_+}]\,\widetilde{JX_+} = \tilde{X}_+\,.$$

But T being a representation implies

$$[T_{X_+},T_{JX_+}]\,\widetilde{JX_+} = T_{[X_+,JX_+]}\widetilde{JX_+} = \tfrac{1}{2}\sqrt{2}T_{G_1}\widetilde{JX_+} = -\tfrac{1}{2}\tilde{X}_+\,.$$

This is only possible, if $x_1 = 0$.

2) Suppose now that $f = f_2$ and $f' = f_1$. From the equations (5), (7) and the knowledge of $T_{H_1} = T_{H'}$ we obtain this time

$$T_{X_\pm} = \tilde{P}_\pm\otimes\tilde{X}_\pm + \tfrac{1}{2}\sqrt{2}\tilde{P}_0\otimes\widetilde{JX_\pm} + \widetilde{JX_\pm}\otimes\tilde{Q}_\mp - \tfrac{1}{2}\sqrt{2}\tilde{X}_\pm\otimes\tilde{Q}_0\,.$$

Choose a Euclidean vector space $\tilde{u}$ isometric to u and an isometry $\sim: u \to \tilde{u}$. Then the formulas just derived for T define a linear mapping $T : u \to \mathrm{End}(\tilde{u})$. It is straightforward to check that T is a representation and that it fulfills the conditions (Q1) and (Q7). Therefore, we can make a metric Lie algebra out of $t := u + \tilde{u}$ using the procedure explained in Thm. 1.3.

According to Thm. 1.3, T is a Q-representation, if and only if (t, q) is an Alekseevskian Lie algebra. The latter statement is contained in the more precise information which we will derive in the following (see Cor. 2.4 and 2.10(ii)). □

Definition 2.1. Let $u = f_0 + f_1 + x_1 + f_2$ be an admissible Kählerian Lie algebra as in Prop. 2.1, $p = \dim_{\mathbb{C}} x_1 = \dim_{\mathbb{R}} x_+ \in \{0, 1, 2, \ldots\}$ and $T : u \to \mathrm{End}(\tilde{u})$ the representation given there. We will denote by $(t(p), \langle\cdot\,,\cdot\rangle, \mathrm{q})$ or short by (t, q) the metric Lie algebra with quaternionic structure defined at the end of the proof of Prop. 2.1 on $u + \tilde{u}$ with the help of T. The corresponding family of simply connected Lie groups with left-invariant metric will be denoted by $\mathcal{T}(p)$ $(p = 0, 1, 2, \ldots)$, and we will speak of $\mathcal{T}$-spaces.

Proposition 2.2. *$t = t(p)$ is 4-step solvable. The derived Lie algebra $n = [t, t]$ is 7-step nilpotent, if $p \neq 0$. For $t(0)$ it is 5-step nilpotent.*

Proof. Derived series $\mathcal{D}^{j+1}t = [\mathcal{D}^j t, \mathcal{D}^j t]$ ($\mathcal{D}^0 t = t$, $j = 0, 1, 2, \ldots$) of t:

$$\mathcal{D}^1 t = n = a^{\perp}, \quad \text{where } a = \mathrm{span}\{H_i \mid i = 0, 1, 2\}.$$

For $p \neq 0$ we get

$$\mathcal{D}^2 t = [n, n] = \mathrm{span}\{G_0, G_1\} + x_+ + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{P}_-, \tilde{Q}_0, \tilde{Q}_+\} + \tilde{x}_1,$$

$$\mathcal{D}^3 t = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{P}_-\} + \tilde{x}_+, \quad \mathcal{D}^4 t = 0.$$

For $p = 0$ we obtain

$$\mathcal{D}^2 t = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{P}_-, \tilde{Q}_0, \tilde{Q}_+\},$$

$$\mathcal{D}^3 t = \mathrm{span}\{G_0\}, \quad \mathcal{D}^4 t = 0.$$

Central series $\mathcal{Z}^{j+1}n = [n, \mathcal{Z}^j n]$ ($\mathcal{Z}^0 n = n$, $j = 0, 1, 2, \ldots$) of n:

$$\mathcal{Z}^1 n = [n, n] = \mathcal{D}^2 t.$$

For $p \neq 0$:

$$\mathcal{Z}^2 n = [n, \mathcal{Z}^1 n] = \mathrm{span}\{G_0, G_1\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{P}_-, \tilde{Q}_+\} + \tilde{x}_1,$$

$$\mathcal{Z}^3 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{P}_-\} + \tilde{x}_+,$$

$$\mathcal{Z}^4 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+\},$$

$$\mathcal{Z}^5 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_+\},$$

$$\mathcal{Z}^6 n = \mathrm{span}\{G_0\},$$

$$\mathcal{Z}^7 n = 0.$$

For $p = 0$:

$$\mathcal{Z}^2 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_+, \tilde{Q}_+\},$$

$$\mathcal{Z}^3 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_+\},$$

$$\mathcal{Z}^4 n = \mathrm{span}\{G_0\},$$

$$\mathcal{Z}^5 n = 0. \qquad \square$$

Definition 2.2. (see [24, Introduction]) A solvable Lie algebra s is of *Iwasawa type*, if

1. $s = a + n$, where a is an Abelian subalgebra and $n = [s, s]$.
2. All ad_A, $A \in a$ are semisimple.
3. There is an $A^0 \in a$ such that $\mathrm{ad}_{A^0}|_n$ has only eigenvalues with positive real part.

Remark 2.2. The classic example of a Lie algebra of Iwasawa type is the solvable part of the Iwasawa decomposition of the isometry algebra of a symmetric space of non-compact type (cf. [14, Ch. VI, Sect. 3, Ch. VII, Lemma 2.20 and Ch. IX, Sect. 1]).

Proposition 2.3. *$t(p)$ is of Iwasawa type and all operators ad_A, $A \in a$, are diagonalizable over $\mathbb{R}$.*

Proof. We set $a = \text{span}\{H_i \mid i = 0, 1, 2\}$. The proof of Prop. 2.2 shows that t is solvable and that $n = a^\perp$. Let $A \in a$, i.e. $A = \sum_{i=0}^{2} a_i H_i$, $a_i \in \mathbb{R}$, and let $\mu_i \in \{1, 1/\sqrt{2}\}$ be the root of the key algebra f_i. ad_A has the following eigenspace decomposition:

$$\begin{aligned}
\text{ad}_A|_a &= 0, \\
\text{ad}_A G_i &= a_i \mu_i G_i, \\
\text{ad}_A|_{x_\pm} &= \tfrac{1}{2}(a_1 \pm \tfrac{1}{2}\sqrt{2}a_2)\text{Id}, \\
\text{ad}_A \tilde{P}_0 &= \tfrac{1}{2}(a_0 + a_1)\tilde{P}_0, \\
\text{ad}_A \tilde{P}_\pm &= \tfrac{1}{2}(a_0 + a_1 \pm \sqrt{2}a_2)\tilde{P}_\pm, \\
\text{ad}_A \tilde{Q}_0 &= \tfrac{1}{2}(a_0 - a_1)\tilde{Q}_0, \\
\text{ad}_A \tilde{Q}_\pm &= \tfrac{1}{2}(a_0 - a_1 \pm \sqrt{2}a_2)\tilde{Q}_\pm, \\
\text{ad}_A|_{\tilde{x}_\pm} &= \tfrac{1}{2}(a_0 \pm \tfrac{1}{2}\sqrt{2}a_2)\text{Id}.
\end{aligned}$$

Finally, $A^0 := 3H_0 + H_2 + H_1$ satisfies the third condition of Def. 2.2. □

Corollary 2.4. *$t(p)$ is real solvable.*

Proof. We give a basis $\mathcal{B}$ with respect to which the ad_A, $A \in a$, are real diagonal matrices and the ad_N, $N \in n$, are lower triangular matrices with zeroes on the diagonal. For a subspace $y \subset t$ let $\mathcal{B}_y$ denote any basis of y.

$$\mathcal{B} := \left(\mathcal{B}_a, \mathcal{B}_{x_-}, G_2, \mathcal{B}_{x_+}, G_1, \tilde{Q}_-, \tilde{P}_-, \tilde{Q}_0, \tilde{Q}_+, \mathcal{B}_{\tilde{x}_-}, \mathcal{B}_{\tilde{x}_+}, \tilde{P}_0, \tilde{P}_+, G_0\right). \qquad \square$$

We describe the decomposition $t = a + n$ more accurately. For $b \subset t$ let $\text{zent}(b)$ (resp. $\text{norm}(b)$) be the centralizer (resp. normalizer) of b in t.

Proposition 2.5. *The following holds for t:*
(i) *a is an (Abelian) Cartan subalgebra.*
(ii) *n is the nilradical.*
(iii) $\text{zent}(t) = 0$. *In particular, $t(p)$ has rank* 3.

Proof. (i) follows e.g. from Def. 2.2.
(ii) : None of the endomorphisms $\text{ad}_A|_n$, $A \in a - \{0\}$, is nilpotent, therefore n is the maximal nilpotent ideal.
(iii) : On the one hand $\text{zent}(t) \subset \text{zent}(a) = a$ by (i), on the other hand $\text{zent}(t) \subset n$ by (ii), so $\text{zent}(t) = 0$. □

The roots of the Lie algebra $t(p)$ with respect to the Cartan subalgebra a (see Prop. 2.5(i)) can be read from the eigenspace decomposition given in the proof of Prop. 2.3.

Definition 2.3. Let s be a Lie algebra of Iwasawa type and Ω_+ its set of roots with respect to the Cartan subalgebra a. We will say that the roots of *s form a root system*, if $\Omega := \Omega_+ \cup -\Omega_+$ is a root system.

Proposition 2.6. *The following holds in the sense of Def.* 2.3:

(i) *The roots of* $t(0)$ *form a root system isomorphic to the root system* b_3.

(ii) *For* $p \neq 0$ *the roots of* $t(p)$ *do not form a root system.*

(The isomorphism type of the root system of $t(0)$ is indicated in [3, Table 1].)

Proof. If we identify a and a^* by means of the metric Lie algebra's inner product, the roots $\Omega_+(p)$ of $t(p)$ will be

$$\Omega_+(0) = \{\mu_i H_i \ (i = 0, 1, 2),\ \tfrac{1}{2}(H_0 \pm H_1),\ P_\pm,\ -JQ_\pm\} \quad \text{and}$$
$$\Omega_+(p) = \{\mu_i H_i,\ \tfrac{1}{2}(\mu_i H_i \pm \mu_j H_j)\ (0 \leqslant i < j \leqslant 2),\ P_\pm,\ -JQ_\pm\},$$

if $p \neq 0$. We recall:

$$P_\pm = \tfrac{1}{2}(H_0 + H_1 \pm \sqrt{2}H_2), \qquad -JQ_\pm = \tfrac{1}{2}(H_0 - H_1 \pm \sqrt{2}H_2).$$

If $\Omega = \Omega_+ \cup -\Omega_+$ is a root system, then for every root $\lambda \in \Omega$ there is a *reflection along* λ that leaves Ω invariant (see [14, Ch. X, Sect. 3.1, Def. on p. 456]). We suppose that $\Omega(p)$, $p \neq 0$, is a root system and will derive a contradiction. Let σ be the reflection along $\lambda = \left(H_0 + 1/\sqrt{2}\, H_2\right)/2$ which leaves $\Omega(p)$ invariant. Since neither $\lambda + H_1$ nor $\lambda - H_1$ lies in Ω, $\sigma H_1 = H_1$ must hold (see [14, Ch. X, Lemma 3.4]). As well, $\lambda \pm P_- \notin \Omega$ and $\lambda \pm JQ_- \notin \Omega$ imply $\sigma P_- = P_-$ and $\sigma JQ_- = JQ_-$. Therefore, σ is the Euclidean reflection with respect to $\lambda^\perp$ and one obtains e.g. $\sigma H_0 = -\frac{1}{3}(H_0 + 2\sqrt{2}H_2) \notin \Omega$. This shows (ii).

(i): It is easy to check that the *Euclidean* reflection σ_λ with respect to $\lambda^\perp$ leaves the set $\Omega(0)$ invariant for all $\lambda \in \Omega(0)$. In addition, the numbers $c(\mu, \lambda)$ defined by

$$\sigma_\lambda \mu = \mu - c(\mu, \lambda)\lambda\,, \qquad \lambda, \mu \in \Omega(0) \tag{8}$$

are *integers*, which can be checked easily using the formula $c(\mu, \lambda) = 2\langle\mu, \lambda\rangle/|\lambda|^2$ (see [14, Ch. X, Sect. 3.1, Lemma 3.3]). Thus, we see that $\Omega(0)$ is a (reduced) root system. In order to determine its isomorphism type we give a basis $\mathcal{B}$ of the root system and compute the corresponding Cartan matrix $(c(\lambda, \mu))_{\lambda,\mu\in\mathcal{B}}$. As basis we choose the simple roots of Ω_+:

$$\mathcal{B} = \left(H_1, -JQ_-, \tfrac{1}{2}\sqrt{2}H_2\right).$$

The Cartan matrix of $\Omega(0)$ with respect to $\mathcal{B}$ reads

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Therefore, $\Omega(0) \cong b_3$ (see [14, Ch. X, Sect. 3.3, p. 463]). $\square$

Next, we study the Levi-Civita connection of t. From the formulas for the covariant derivative one sees, on the one hand, that (t, q) is quaternionic Kählerian, on the other hand, they serve as a basis for curvature computations in Ch. 3.

Proposition 2.7. *The following subalgebras of t are totally geodesic*:

$$a, \quad \sum_{i\in I} f_i, \quad u_0, \quad u \quad (I \subset \{0,1,2\}).$$

Proof. Trivial check using the Koszul formula. □

Proposition 2.8. *$t(p')$ is a quaternionic subalgebra of $t(p)$, if $p' \leqslant p$. In particular, the inclusion $t(p') \subset t(p)$ is totally geodesic.*

Proof. Choose $x'_- \subset x_-$ with $\dim x'_- = p'$. Set $x' := x'_- + Jx'_-$ and $u' := \sum_i f_i + x'$. Then $t(p') \cong u' + \tilde{u}'$ is a quaternionic subalgebra. Quaternionic subalgebras are totally geodesic, according to Thm. 1.2(i). □

Proposition 2.9. *The covariant derivative of t is given by* ($A \in a$, $X_\pm \in x_\pm$):

$$\begin{aligned}
\nabla_A &= 0\,,\\
\nabla_{G_0} &= -G_0 \wedge H_0 - \tfrac{1}{2}\hat{J} \circ \mathrm{Pr}_{\tilde{u}}\,,\\
\nabla_{G_1} &= -G_1 \wedge H_1 - \tfrac{1}{2}J \circ \mathrm{Pr}_x + \tfrac{1}{2}(\tilde{G}_0 \wedge \tilde{H}_0 - \tilde{G}_1 \wedge \tilde{H}_1 + \tilde{G}_2 \wedge \tilde{H}_2),\\
\nabla_{G_2} &= -\tfrac{1}{2}\sqrt{2}(G_2 \wedge H_2 + \tfrac{1}{2}J \circ \mathrm{Pr}_x - \tilde{G}_0 \wedge \tilde{H}_0 - \tilde{G}_1 \wedge \tilde{H}_1 + \tfrac{1}{2}J_1 \circ \mathrm{Pr}_{\tilde{x}}),\\
\nabla_{X_\pm} &= \tfrac{1}{2}((H_1 \pm \tfrac{1}{2}\sqrt{2}H_2) \wedge X_\pm + (G_1 \pm \tfrac{1}{2}\sqrt{2}G_2) \wedge JX_\pm)\\
&\quad + \tfrac{1}{2}((\tilde{G}_1 \pm \tfrac{1}{2}\sqrt{2}\tilde{G}_2) \wedge \widetilde{JX_\pm} + (\tilde{H}_1 \pm \tfrac{1}{2}\sqrt{2}\tilde{H}_2) \wedge \tilde{X}_\pm),\\
\nabla_{\tilde{H}_0} &= -\tfrac{1}{2}J_2\,,\\
\nabla_{\tilde{G}_0} &= \tfrac{1}{2}J_3\,,\\
\nabla_{\tilde{H}_1} &= -\tfrac{1}{2}(\tilde{H}_1 \wedge H_0 + \tilde{H}_0 \wedge H_1 + \tilde{H}_2 \wedge H_2 - \tilde{G}_1 \wedge G_0 - \tilde{G}_0 \wedge G_1 - \tilde{G}_2 \wedge G_2),\\
\nabla_{\tilde{H}_2} &= -\tfrac{1}{2}(\tilde{H}_2 \wedge (H_0 + H_1) + (\tilde{H}_0 + \tilde{H}_1) \wedge H_2 - \tilde{G}_2 \wedge (G_0 + G_1))\\
&\quad + \tfrac{1}{2}((\tilde{G}_0 + \tilde{G}_1) \wedge G_2 - \tfrac{1}{2}\sqrt{2}J_2 \circ (\mathrm{Pr}_{x_+ + \tilde{x}_+} - \mathrm{Pr}_{x_- + \tilde{x}_-})),\\
\nabla_{\tilde{G}_1} &= -\tfrac{1}{2}(\tilde{G}_1 \wedge H_0 + \tilde{G}_0 \wedge H_1 - \tilde{G}_2 \wedge H_2) - \tfrac{1}{2}(\tilde{H}_1 \wedge G_0 + \tilde{H}_0 \wedge G_1 - \tilde{H}_2 \wedge G_2),\\
\nabla_{\tilde{G}_2} &= -\tfrac{1}{2}(\tilde{G}_2 \wedge (H_0 - H_1) + (\tilde{G}_0 - \tilde{G}_1) \wedge H_2 + \tilde{H}_2 \wedge (G_0 - G_1))\\
&\quad - \tfrac{1}{2}((\tilde{H}_0 - \tilde{H}_1) \wedge G_2 + \tfrac{1}{2}\sqrt{2}J_3 \circ (\mathrm{Pr}_{x_+ + \tilde{x}_-} - \mathrm{Pr}_{x_- + \tilde{x}_+})),\\
\nabla_{\tilde{X}_\pm} &= -\tfrac{1}{2}(\tilde{X}_\pm \wedge (H_0 \pm \tfrac{1}{2}\sqrt{2}H_2) - \widetilde{JX_\pm} \wedge (G_0 \pm \tfrac{1}{2}\sqrt{2}G_2)\\
&\quad + (\tilde{H}_0 \pm \tfrac{1}{2}\sqrt{2}\tilde{H}_2) \wedge X_\pm - (\tilde{G}_0 \pm \tfrac{1}{2}\sqrt{2}\tilde{G}_2) \wedge JX_\pm).
\end{aligned}$$

Here Pr_v denotes the orthogonal projection on the subspace $v \subset t$, and $V \wedge W = V \otimes W - W \otimes V$ denotes the skew endomorphism $S \mapsto V\langle W, S\rangle - W\langle V, S\rangle$ of t.

Proof. We have computed the covariant derivatives ∇_U, $U \in u$, by means of the Koszul formula, observing Prop. 2.7 and 2.8. The operators $\nabla_{\tilde{U}}|_u$ result from the vanishing of the torsion ($U_1, U_2 \in u$):

$$\nabla_{\tilde{U}_1} U_2 = -T_{U_2}\tilde{U}_1 + \nabla_{U_2}\tilde{U}_1\,.$$

$[\tilde{u}, \tilde{u}] \subset u$ yields $\nabla_{\tilde{u}}\tilde{u} \subset u$, by the Koszul formula. Therefore, we can compute the remaining operators $\nabla_{\tilde{U}}|_{\tilde{u}}$ by using the compatibility of the covariant derivative and the metric ($U, U_1, U_2 \in u$):

$$\langle \nabla_{\tilde{U}_1}\tilde{U}_2, U\rangle = -\langle \tilde{U}_2, \nabla_{\tilde{U}_1}U\rangle. \qquad \square$$

Corollary 2.10. *The following holds in the sense of Def.* 1.5:

(i) (u, J) *is Kählerian.*

(ii) (t, q) *is quaternionic Kählerian.*

Proof. (i): From the formulas for the covariant derivative of the totally geodesic subalgebra u we see that $\mathrm{kos}(u) \subset \mathrm{zent}(J)$.

(ii) : The formulas in Prop. 2.9 allow us to infer immediately that

1. $\nabla_A = 0$ for $A \in a$,
2. $\nabla_{\tilde{H}_0}, \nabla_{\tilde{G}_0} \in \mathrm{q}$,
3. $\nabla_W \in \mathrm{zent}(\mathrm{q})$ for $W \in x_1 + \tilde{u}_0$,
4. $\nabla_{G_i} \in \mathrm{zent}(J_1)$, $i = 0, 1, 2$. In particular, (1–3), $\nabla_W \in \mathrm{q} + \mathrm{zent}(\mathrm{q})$ for $W \in (Ja)^\perp$.

Claim: $\nabla_{G_i} \in \mathrm{span}\,\{J_1\} + \mathrm{zent}(\mathrm{q}) \subset \mathrm{q} + \mathrm{zent}(\mathrm{q})$. We define $\bar{\nabla}_{G_i}$ by:

$$\nabla_{G_i} = -\tfrac{1}{4}J_1 + \bar{\nabla}_{G_i}, \quad i = 0, 1,$$
$$\nabla_{G_2} = -\tfrac{1}{4}\sqrt{2}J_1 + \bar{\nabla}_{G_2}.$$

We have to show that $\bar{\nabla}_{G_i} \in \mathrm{zent}(\mathrm{q})$. Because of 4. we only need to show $\bar{\nabla}_{G_i} \in \mathrm{zent}(J_2)$. Obviously it is sufficient to check $[\bar{\nabla}_{G_i}, J_2]|u = 0$.

$$[\nabla_{G_0}, J_2]|_u = -\tfrac{1}{2}\hat{J}J_2 - (-\tilde{G}_0 \wedge H_0) = -\tfrac{1}{2}J_3 = [-\tfrac{1}{4}J_1, J_2],$$
$$[\nabla_{G_1}, J_2]|_u = \tfrac{1}{2}(\tilde{G}_0 \wedge H_0 - \tilde{G}_1 \wedge H_1 + \tilde{G}_2 \wedge H_2) - (-\tilde{G}_1 \wedge H_1 + \tfrac{1}{2}J_3 \circ \mathrm{Pr}_x)$$
$$= -\tfrac{1}{2}J_3,$$
$$[\nabla_{G_2}, J_2]|_u = \tfrac{\sqrt{2}}{2}(\tilde{G}_0 \wedge H_0 + \tilde{G}_1 \wedge H_1 - \tfrac{1}{2}J_3 \circ \mathrm{Pr}_x) + \tfrac{\sqrt{2}}{2}(\tilde{G}_2 \wedge H_2 - \tfrac{1}{2}J_3 \circ \mathrm{Pr}_x)$$
$$= -\tfrac{\sqrt{2}}{2}J_3 = [-\tfrac{\sqrt{2}}{4}J_1, J_2]. \qquad \square$$

Cor. 2.4, Prop. 2.5 and Cor. 2.10 imply that (t, q) is an Alekseevskian Lie algebra of rank 3. That finishes the proof of Prop. 2.1, and the classification of Alekseevsky spaces of rank 3 goes as follows.

Theorem 2.11. *There exists a family* $\mathcal{T}(p)$, $p = 0, 1, 2, \ldots,$ *of Alekseevskian spaces of rank* 3 *and of quaternionic dimension* $p + 3$. *Any Alekseevsky space of rank* 3 *is (up to scaling) isometric to one of these spaces.*

$\mathcal{T}(0)$ *is isometric to the symmetric space* $SO_0(3, 4)/(SO(3) \times SO(4))$ *of scalar curvature* -30. *All the other* $\mathcal{T}(p)$ *are not symmetric.*

Proof. The only question that is left to be answered is which of the spaces $\mathcal{T}(p)$ are symmetric. The 12-dimensional non-compact Grassmann manifold $SO_0(3, 4)/(SO(3) \times SO(4))$ is the only symmetric space of non-compact type which is quaternionic Kählerian and has rank 3 (see [6, 14.52 Table]). According to Prop. 2.1, $\mathcal{T}(0)$ is the only Alekseevsky space of rank 3 and of real dimension $3 \cdot 4 = 12$. Thus, up to scaling,

$$\mathcal{T}(0) \cong \frac{SO_0(3,4)}{SO(3) \times SO(4)}$$

and all the other $\mathcal{T}(p)$ are not symmetric.

The reduced scalar curvature sk_r of $\mathcal{T}(0)$ coincides, due to Thm. 1.2 (i) and (iii), with the reduced scalar curvature $\mathrm{sk}_r = -1/2$ of the canonical onedimensional quaternionic subalgebra $f_0 + \tilde{f}_0$. So, in accordance with equation 3, the scalar curvature of $\mathcal{T}(0)$ is $\mathrm{sk} = 4 \cdot 3(3+2) \cdot (-\frac{1}{2}) = -30$. □

2.2. Rank 4

The Alekseevskian spaces of rank 4 are the $\mathcal{W}$- and $\mathcal{V}$-spaces reviewed in the Sections 1.2 and 1.3. The main aim of this section is to clarify the classification of $\mathcal{V}$-spaces. Nevertheless, with the purpose of rounding off our exposition, we begin with a discussion of $\mathcal{W}$-spaces similar to that of $\mathcal{T}$-spaces in the previous section.

The fact that the metric Lie algebras $w(p,q)$ and $v(\psi)$ are Alekseevskian is due to the circumstance that they are constructed from Q-representations (see Thm. 1.3 and Thm. 1.4). Alternatively one could check directly that they are real solvable and quaternionic Kählerian. We have pursued this approach for the $\mathcal{T}$-spaces in Sect. 2.1 (cf. Cor. 2.4, Prop. 2.9 and Prop. 2.10) and for the $\mathcal{W}$-spaces in [9, II.3–4]. We shall not insist on this stand-point in order to avoid too much redundancy.

Proposition 2.12. *$w = w(p,q)$ is 4-step solvable. The derived Lie algebra $n = [w,w]$ is 7-step nilpotent, if $(p,q) \neq (0,0)$. For $w(0,0)$ it is 5-step nilpotent.*

Proof. Derived series:

$$\mathcal{D}w = [w,w] = \mathrm{span}\{G_i \mid i = 0,1,2,3\} + x_1 + \tilde{u}.$$

$$\mathcal{D}^2 w = \mathrm{span}\{G_0, G_1\} + y_+ + z_+ + \mathrm{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0,1,2,3; \alpha = 1,2,3\} + \tilde{x}_1,$$

if $(p,q) \neq (0,0)$.

In the case $p = q = 0$ one obtains

$$\mathcal{D}^2 w = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0,1,2,3; \alpha = 1,2,3\}$$

and thus

$$\mathcal{D}^3 w = \mathrm{span}\{G_0\},$$

$$\mathcal{D}^4 w = 0.$$

For $(p,q) \neq (0,0)$ one has

$$\mathcal{D}^3 w = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{Q}_2, \tilde{Q}_3\} + \tilde{y}_+ + \tilde{z}_+,$$

$$\mathcal{D}^4 w = 0.$$

Central series of n:

$$\mathcal{Z}n = [n,n] = \mathcal{D}^2 w,$$

$$\mathcal{Z}^2 n = [n, \mathcal{Z}n].$$

For $p = q = 0$:

$$\mathcal{Z}^2 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{Q}_\alpha \mid \alpha = 1, 2, 3\},$$
$$\mathcal{Z}^3 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0\},$$
$$\mathcal{Z}^4 n = \mathrm{span}\{G_0\},$$
$$\mathcal{Z}^5 n = 0.$$

For $(p, q) \neq (0, 0)$:

$$\mathcal{Z}^2 n = \mathrm{span}\{G_0, G_1\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{Q}_\alpha \mid \alpha = 1, 2, 3\} + \tilde{x}_1,$$
$$\mathcal{Z}^3 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{P}_1, \tilde{Q}_2, \tilde{Q}_3\} + \tilde{y}_+ + \tilde{z}_+,$$
$$\mathcal{Z}^4 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{Q}_2, \tilde{Q}_3\},$$
$$\mathcal{Z}^5 n = \mathrm{span}\{G_0\} + \mathrm{span}\{\tilde{P}_0\},$$
$$\mathcal{Z}^6 n = \mathrm{span}\{G_0\},$$
$$\mathcal{Z}^7 n = 0. \qquad \square$$

Proposition 2.13. *$w(p, q)$ is of Iwasawa type and the ad_A, $A \in a$, have the following eigenspace decomposition*

$$\begin{aligned}
\mathrm{ad}_A a &= 0, \\
\mathrm{ad}_A G_i &= a_i G_i, \\
\mathrm{ad}_A|_{y_\pm} &= \tfrac{1}{2}(a_1 \pm a_3)\,\mathrm{Id}, \\
\mathrm{ad}_A|_{z_\pm} &= \tfrac{1}{2}(a_1 \pm a_2)\,\mathrm{Id}, \\
\mathrm{ad}_A \tilde{P}_0 &= \tfrac{1}{2}(a_0 + a_1 + a_2 + a_3)\,\tilde{P}_0, \\
\mathrm{ad}_A \tilde{P}_\alpha &= \tfrac{1}{2}(a_0 + a_\alpha - a_\beta - a_\gamma)\,\tilde{P}_\alpha, \\
\mathrm{ad}_A \tilde{Q}_0 &= \tfrac{1}{2}(a_0 - a_1 - a_2 - a_3)\,\tilde{Q}_0, \\
\mathrm{ad}_A \tilde{Q}_\alpha &= \tfrac{1}{2}(a_0 - a_\alpha + a_\beta + a_\gamma)\,\tilde{Q}_\alpha, \\
\mathrm{ad}_A|_{\tilde{y}_\pm} &= \tfrac{1}{2}(a_0 \pm a_2)\,\mathrm{Id}, \\
\mathrm{ad}_A|_{\tilde{z}_\pm} &= \tfrac{1}{2}(a_0 \pm a_3)\,\mathrm{Id},
\end{aligned}$$

where $A = \sum_{i=0}^{3} a_i H_i$, $a_i \in \mathbb{R}$, and $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$.

Proof. $A^0 := 7H_0 + 3H_1 + 2H_2 + H_3$ satisfies the third condition of Definition 2.2. $\square$

Proposition 2.14. *The following holds for w:*

1. *a is an (Abelian) Cartan subalgebra.*
2. *n is the nilradical.*
3. $\mathrm{zent}(w) = 0$. *In particular, w has rank* 4.

Proof. Same proof as for Prop. 2.5 $\square$.

The following information together with Prop. 2.22 serves to recognize the symmetric spaces in the list of Alekseevskian spaces of rank 4. The corresponding discussion is postponed until Thm. 2.28.

Proposition 2.15. (see [3, Table 1]) *The roots of $w(p, q)$ with respect to the Cartan subalgebra a (cf. Def. 2.3):*
(i) *form a root system isomorphic to d_4, if $p = q = 0$;*
(ii) *form a root system isomorphic to b_4, if either $p = 0$ or $q = 0$;*
(iii) *do not form a root system, if $p \neq 0$ and $q \neq 0$.*

Proof. The roots $\Omega_+(p, q)$ can be read from the eigenspace decomposition of Prop. 2.13. If $p \neq 0$ and $q \neq 0$ they are ($i = 0, 1, 2, 3$; $\alpha = 1, 2, 3$):

$$\{H_i, \tfrac{1}{2}(H_1 \pm H_3), \tfrac{1}{2}(H_1 \pm H_2), P_0, -P_\alpha, -JQ_0, JQ_\alpha, \tfrac{1}{2}(H_0 \pm H_2), \tfrac{1}{2}(H_0 \pm H_3)\}.$$

For $q = 0$ $\frac{1}{2}(H_1 \pm H_2)$ and $\frac{1}{2}(H_0 \pm H_3)$ have to be dropped. For $p = 0$ $\frac{1}{2}(H_1 \pm H_3)$ and $\frac{1}{2}(H_0 \pm H_2)$ have to be dropped.

First, we assume that $\Omega(p, q)$ ($p \neq 0$ and $q \neq 0$) forms a root system and deduce a contradiction. Let σ be the reflection along $\lambda := \frac{1}{2}(H_1 + H_3)$ which leaves $\Omega(p, q)$ invariant. Since neither $\lambda + \frac{1}{2}(H_1 + H_2)$ nor $\lambda - \frac{1}{2}(H_1 + H_2)$ lies in $\Omega(p, q)$, we must have $\sigma(H_1 + H_2) = H_1 + H_2$. By the same token $\sigma(H_1 - H_2) = H_1 - H_2$ and $\sigma(H_0 \pm H_3) = H_0 \pm H_3$. Thus, $\sigma = \mathrm{Id}$, which is not a reflection. This shows (iii).

For $\lambda \in a - \{0\}$ let σ_λ denote the Euclidean reflection with respect to $\lambda^\perp$. We observe ($i = 0, 1, 2, 3$):

1. σ_{H_i}, σ_{P_i} and σ_{JQ_i} leave $\Omega(p, q)$ invariant.
2. $\sigma_{H_1 \pm H_3}$ and $\sigma_{H_0 \pm H_2}$ leave $\Omega(p, 0)$, $p \neq 0$, invariant.
3. $\sigma_{H_1 \pm H_2}$ and $\sigma_{H_0 \pm H_3}$ leave $\Omega(0, q)$, $q \neq 0$, invariant. The condition $c(\mu, \lambda) \in \mathbb{Z}$ may be checked for $\lambda, \mu \in \Omega(p, 0)$ as indicated in the proof of Prop. 2.6. This proves that $\Omega(p, q)$ is a root system, if $p = 0$ or $q = 0$.

Next, we compute the isomorphism type of the root system. A basis of $\Omega(p, 0)$ ($p \neq 0$) is

$$\mathcal{B}_{1,0} := (H_1, -JQ_3, H_0, -\tfrac{1}{2}(H_0 - H_2))\,.$$

A basis of $\Omega(0, 0)$ is

$$\mathcal{B}_{0,0} := (-JQ_0, P_0, -JQ_1, -JQ_2)\,.$$

The Cartan matrix of $\mathcal{B}_{1,0}$ reads

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

and that of $B_{0,0}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

This determines the isomorphism type to be b_4 and d_4 respectively (see [14, Ch. X, Sect. 3.3, pp. 462–464]). □

Next we compute the Levi-Civita connection of w part of which we will use for curvature computations in Ch. 3.

Proposition 2.16. *The following subalgebras of w are totally geodesic*:

$$a, \quad \sum_{i \in I} f_i, \quad f_1 + f_3 + y, \quad f_1 + f_2 + z, \quad u_0, \quad u \quad (I \subset \{0, 1, 2, 3\}).$$

Proof. Easy computation using the Koszul formula. □

Proposition 2.17. *$w(p', q')$ is a quaternionic subalgebra of $w(p, q)$, if $p' \leqslant p$ and $q' \leqslant q$. In particular, the corresponding inclusion is totally geodesic.*

Proof. Cf. proof of Prop. 2.8. □

Proposition 2.18. *The covariant derivative of $w(p, q)$ is given by the following formulas ($A \in a$, $X_\pm \in x_\pm$, $Y_\pm \in y_\pm$ and $Z_\pm \in z_\pm$).*

$$\nabla_A = 0,$$

$$\begin{aligned}
&\nabla_{G_i}|_{f_i} = -J\,, && \nabla_{G_0}|_{u_0} = 0\,,\\
&\nabla_{G_1}|_{x_1} = -\tfrac{1}{2}J\,, && \nabla_{G_1}|_{f_0+f_2+f_3} = 0\,,\\
&\nabla_{G_2}|_{z} = -\tfrac{1}{2}J\,, && \nabla_{G_2}|_{f_0+f_1+f_3+y} = 0\,,\\
&\nabla_{G_3}|_{y} = -\tfrac{1}{2}J\,, && \nabla_{G_3}|_{f_0+f_1+f_2+z} = 0\,,\\
&\nabla_{G_i}|_{\tilde{f}_i} = \tfrac{1}{2}J_1\,, && \nabla_{G_0}|_{\tilde{u}_0} = -\tfrac{1}{2}J_1\,,\\
&\nabla_{G_1}|_{\tilde{x}_1} = 0\,, && \nabla_{G_1}|_{\tilde{f}_0+\tilde{f}_2+\tilde{f}_3} = -\tfrac{1}{2}J_1\,,\\
&\nabla_{G_2}|_{\tilde{z}} = 0\,, && \nabla_{G_2}|_{\tilde{f}_0+\tilde{f}_1+\tilde{f}_3+\tilde{y}} = -\tfrac{1}{2}J_1\,,\\
&\nabla_{G_3}|_{\tilde{y}} = 0\,, && \nabla_{G_3}|_{\tilde{f}_0+\tilde{f}_1+\tilde{f}_2+\tilde{z}} = -\tfrac{1}{2}J_1\,.
\end{aligned}$$

$\nabla_{X_1} \in \mathrm{zent}(\mathrm{q})$, *if $X_1 \in x_1 = y + z$. The operators ∇_{X_1}, $X_1 \in x_1$, are thus already determined by the following restrictions*:

$$\begin{aligned}
&\nabla_{Y_-}|_{a+y_-+z_-} = \tfrac{1}{2}(H_1 - H_3) \wedge Y_-\,, && \nabla_{Y_+}|_{a+y_++z_+} = \tfrac{1}{2}(H_1 + H_3) \wedge Y_+\,,\\
&\nabla_{Z_-}|_{a+y_-+z_-} = \tfrac{1}{2}(H_1 - H_2) \wedge Z_-\,, && \nabla_{Z_+}|_{a+y_++z_+} = \tfrac{1}{2}(H_1 + H_2) \wedge Z_+\,.\\
&\nabla_{\tilde{H}_0} = -\tfrac{1}{2}J_2\,, && \nabla_{\tilde{G}_0} = \tfrac{1}{2}J_3\,.
\end{aligned}$$

$\nabla_{\tilde{V}} \in \mathrm{zent}(\mathfrak{q})$, *if* $V \in u_0$. *Therefore, it suffices to give restrictions* ($\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$)

$$\begin{aligned}
&\nabla_{\tilde{H}_\alpha}|_a = -\tfrac{1}{2}(\tilde{H}_\alpha \otimes H_0 + \tilde{H}_0 \otimes H_\alpha + \tilde{H}_\gamma \otimes H_\beta + \tilde{H}_\beta \otimes H_\gamma),\\
&\nabla_{\tilde{H}_1}|_{y_-+z_-} = 0, \qquad \nabla_{\tilde{H}_2}|_{y_-} = \tfrac{1}{2}J_2, \qquad \nabla_{\tilde{H}_2}|_{z_-} = 0,\\
&\qquad\qquad\qquad\qquad \nabla_{\tilde{H}_3}|_{y_-} = 0, \qquad \nabla_{\tilde{H}_3}|_{z_-} = \tfrac{1}{2}J_2,\\
&\nabla_{\tilde{G}_\alpha}|_a = -\tfrac{1}{2}(\tilde{G}_\alpha \otimes H_0 + \tilde{G}_0 \otimes H_\alpha - \tilde{G}_\gamma \otimes H_\beta - \tilde{G}_\beta \otimes H_\gamma),\\
&\nabla_{\tilde{G}_1}|_{y_-+z_-} = 0, \qquad \nabla_{\tilde{G}_2}|_{y_-} = \tfrac{1}{2}J_3, \qquad \nabla_{\tilde{G}_2}|_{z_-} = 0,\\
&\qquad\qquad\qquad\qquad \nabla_{\tilde{G}_3}|_{y_-} = 0, \qquad \nabla_{\tilde{G}_3}|_{z_-} = \tfrac{1}{2}J_3,\\
&\nabla_{\tilde{Y}_-}|_{a+\tilde{y}_-+\tilde{z}_-} = \tfrac{1}{2}(H_0 - H_2) \wedge \tilde{Y}_-, \qquad \nabla_{\tilde{Y}_+}|_{a+\tilde{y}_++\tilde{z}_+} = \tfrac{1}{2}(H_0 + H_2) \wedge \tilde{Y}_+,\\
&\nabla_{\tilde{Z}_-}|_{a+\tilde{y}_-+\tilde{z}_-} = \tfrac{1}{2}(H_0 - H_3) \wedge \tilde{Z}_-, \qquad \nabla_{\tilde{Z}_+}|_{a+\tilde{y}_++\tilde{z}_+} = \tfrac{1}{2}(H_0 + H_3) \wedge \tilde{Z}_+.
\end{aligned}$$

Proof. Similar to that of Prop. 2.9. □

For more details concerning $\mathcal{W}$-spaces the reader is referred to [9], where e.g. the automorphism group of the metric Lie algebras $w(p, q)$ is determined.

We have explained in Sect. 1.3 how the Alekseevskian $\mathcal{V}$-spaces $\mathcal{V}(\psi)$ are constructed out of special isometric mappings ψ. It is clear that equivalent (see Def. 1.10) special isometric mappings yield isometric $\mathcal{V}$-spaces by this construction.

The classification of $\mathcal{V}$-spaces is based on that of special isometric mappings and this is in turn based on the classification of $\mathbb{Z}_2$-graded Clifford modules. The correction which we shall note concerns [3, Thm. 10.1] (not to be mixed up with Prop. 10.1), i.e., the classification of special isometric mappings up to equivalence which we carry out in Prop. 2.26 (cf. Prop. 2.25). Before going into that, we will mention some basic facts which are independent of the choice of special isometric mapping ψ.

Proposition 2.19. *The Lie algebra* $v = v(\psi)$ *is* 5*-step solvable. Its derived Lie algebra* $n = [v, v]$ *is* 11*-step nilpotent.*

Proof. Derived series:

$$\mathcal{D}^1 v = [v, v] = a^\perp, \quad \text{where } a = \mathrm{span}\,\{H_i \mid i = 0, 1, 2, 3\}.$$

In the following we use that $\psi(X_-, \cdot) : z_- \to y_-$ ($X_- \in x_- - \{0\}$) is an isomorphism and that the bilinear mappings $(V, W) \mapsto [V, W]$ and $(V, W) \mapsto V \circ W = 2\nabla_V W$, from the product of two of the spaces x, y, z into the third, are completely determined by ψ, according to the Remark 1.3. We use e.g. $[x_-, z_-] = x_- \circ z_- = y_-$.

$$\begin{aligned}
\mathcal{D}^2 v &= [n, n] = \mathrm{span}\,\{G_i \mid i = 0, 1, 2\} + y + z_+ + x_+\\
&\qquad + \mathrm{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0, 1, 2, 3;\ \alpha = 1, 2, 3\} + \tilde{y} + \tilde{z} + \tilde{x},\\
\mathcal{D}^3 v &= \mathrm{span}\{G_0, G_1\} + z_+ + \mathrm{span}\{\tilde{P}_\alpha, \tilde{Q}_\beta \mid \alpha = 0, 1;\ \beta = 1, 2, 3\} + \tilde{y} + \tilde{z} + \tilde{x}_+,\\
\mathcal{D}^4 v &= \mathrm{span}\,\{G_0\} + \mathrm{span}\{\tilde{P}_0, \tilde{Q}_3\} + \tilde{x}_+,\\
\mathcal{D}^5 v &= 0.
\end{aligned}$$

Central series of n:

$$
\begin{aligned}
\mathcal{Z}^1 n &= [n, n] = \mathcal{D}^2 v\,,\\
\mathcal{Z}^2 n &= [n, \mathcal{D}^2 v] = \operatorname{span}\{G_i \mid i = 0, 1, 2\} + y_+ + z_+\\
&\qquad + \operatorname{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0, 1, 2;\ \alpha = 1, 2, 3\} + \tilde{y} + \tilde{z} + \tilde{x}\,,\\
\mathcal{Z}^3 n &= \operatorname{span}\{G_0, G_1\} + z_+ + \operatorname{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0, 1, 2;\ \alpha = 1, 2, 3\} + \tilde{y} + \tilde{z} + \tilde{x}_+\,,\\
\mathcal{Z}^4 n &= \operatorname{span}\{G_0, G_1\} + \operatorname{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0, 1;\ \alpha = 1, 2, 3\} + \tilde{y}_+ + \tilde{z} + \tilde{x}_+\,,\\
\mathcal{Z}^5 n &= \operatorname{span}\{G_0\} + \operatorname{span}\{\tilde{P}_i, \tilde{Q}_\alpha \mid i = 0, 1;\ \alpha = 2, 3\} + \tilde{y}_+ + \tilde{z}_+ + \tilde{x}_+\,,\\
\mathcal{Z}^6 n &= \operatorname{span}\{G_0\} + \operatorname{span}\{\tilde{P}_0, \tilde{Q}_\alpha \mid \alpha = 2, 3\} + \tilde{y}_+ + \tilde{x}_+\,,\\
\mathcal{Z}^7 n &= \operatorname{span}\{G_0\} + \operatorname{span}\{\tilde{P}_0, \tilde{Q}_3\} + \tilde{x}_+\,,\\
\mathcal{Z}^8 n &= \operatorname{span}\{G_0\} + \operatorname{span}\{\tilde{P}_0, \tilde{Q}_3\}\,,\\
\mathcal{Z}^9 n &= \operatorname{span}\{G_0\} + \operatorname{span}\{\tilde{P}_0\}\,,\\
\mathcal{Z}^{10} n &= \operatorname{span}\{G_0\}\,,\\
\mathcal{Z}^{11} n &= 0\,. \qquad \square
\end{aligned}
$$

Proposition 2.20. *$v = a + [v, v]$ is of Iwasawa type and the ad_A, $A \in a$, have the following eigenspace decomposition*

$$
\begin{aligned}
&\mathrm{ad}_A a = 0\,, && \mathrm{ad}_A G_i = a_i G_i\,,\\
&\mathrm{ad}_A|_{y_\pm} = \tfrac{1}{2}(a_1 \pm a_3)\,\mathrm{Id}\,, && \mathrm{ad}_A|_{\tilde{y}_\pm} = \tfrac{1}{2}(a_0 \pm a_2)\,\mathrm{Id}\,,\\
&\mathrm{ad}_A|_{z_\pm} = \tfrac{1}{2}(a_1 \pm a_2)\,\mathrm{Id}\,, && \mathrm{ad}_A|_{\tilde{z}_\pm} = \tfrac{1}{2}(a_0 \pm a_3)\,\mathrm{Id}\,,\\
&\mathrm{ad}_A|_{x_\pm} = \tfrac{1}{2}(a_2 \pm a_3)\,\mathrm{Id}\,, && \mathrm{ad}_A|_{\tilde{x}_\pm} = \tfrac{1}{2}(a_0 \pm a_1)\,\mathrm{Id}\,,\\
&\mathrm{ad}_A \tilde{P}_0 = \tfrac{1}{2}(a_0 + a_1 + a_2 + a_3)\,\tilde{P}_0\,, &&\\
&\mathrm{ad}_A \tilde{P}_\alpha = \tfrac{1}{2}(a_0 + a_\alpha - a_\beta - a_\gamma)\,\tilde{P}_\alpha\,, &&\\
&\mathrm{ad}_A \tilde{Q}_0 = \tfrac{1}{2}(a_0 - a_1 - a_2 - a_3)\,\tilde{Q}_0\,, &&\\
&\mathrm{ad}_A \tilde{Q}_\alpha = \tfrac{1}{2}(a_0 - a_\alpha + a_\beta + a_\gamma)\,\tilde{Q}_\alpha\,. &&
\end{aligned}
$$

Here $A = \sum_{i=0}^{3} a_i H_i$, $a_i \in \mathbb{R}$, and $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$.

Proof. $A^0 := 7H_0 + 3H_1 + 2H_2 + H_3$ satisfies the third condition of Definition 2.2. $\square$

Proposition 2.21. *The following holds for v:*

1. *a is an (Abelian) Cartan subalgebra.*
2. *n is the nilradical.*
3. $\mathrm{zent}(v) = 0$. *In particular, v has rank* 4.

Proof. Same proof as for Prop. 2.5. $\square$

The remark preceding Prop. 2.15 refers to the following proposition as well.

Proposition 2.22. (see [3, Table 1]) *The roots of $v(\psi)$ with respect to the Cartan subalgebra a form a root system isomorphic to f_4.*

Proof. The roots Ω_+ of $v(\psi)$ can be read from the eigenspace decomposition of Prop. 2.20:

$$\Omega_+ = \{H_i,\ \tfrac{1}{2}(H_i \pm H_j)\ (0 \leqslant i < j \leqslant 3),\ P_0,\ -JQ_0, -P_\alpha,\ JQ_\alpha\ (\alpha = 1, 2, 3)\}.$$

As in the proof of Prop. 2.6, one can check that for every $\lambda \in \Omega_+$ the Euclidean reflection with respect to $\lambda^\perp$ leaves $\Omega = \Omega_+ \cup -\Omega_+$ invariant and that $c(\mu, \lambda) \in \mathbb{Z}$ for all $\mu, \lambda \in \Omega$. Thus, Ω is a root system.

If we choose the simple roots

$$\mathcal{B} = \left(-JQ_0,\ H_3,\ \tfrac{1}{2}(H_2 - H_3),\ \tfrac{1}{2}(H_1 - H_2)\right)$$

of Ω_+ as basis of the root system, then the Cartan matrix reads

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Therefore, $\Omega \cong f_4$ (see [14, Ch. X, Sect. 3.3, p. 474]). □

Next, we settle the question in which cases two special isometric mappings define isometric $\mathcal{V}$-spaces. The adequate concept is that of equivalence (see Def. 1.10).

Proposition 2.23. *Two $\mathcal{V}$-spaces $\mathcal{V}(\psi)$ and $\mathcal{V}(\psi')$ are isometric, if and only if the special isometric mappings ψ and ψ' are equivalent.*

Proof. The notation for vectors $H_i, G_i, \ldots$ and subspaces $f_i, u, \ldots$ of $v(\psi)$ will be as up to now. Vectors and subspaces of $v(\psi')$ will be marked out by a prime: $H_i', G_i', f_i', u', \ldots$

As already mentioned (see Sect. 1.1 p. 130) $\mathcal{V}(\psi)$ and $\mathcal{V}(\psi')$ are isometric, if and only if the corresponding metric Lie algebras $v = v(\psi)$ and $v' = v(\psi')$ are isomorphic. It is clear that equivalent special isometric mappings yield isomorphic metric Lie algebras.

Conversely, let $\phi : v \to v'$ be an isomorphism of *metric* Lie algebras. We show that ψ and ψ' are equivalent via ϕ. $a = [v, v]^\perp$ (together with $a' = [v', v']^\perp$) implies $\phi(a) = a'$, and ϕ maps the root spaces of v with respect to a onto the root spaces of v'.

Claim 1. $\phi(G_0) \in \{-G_0', G_0'\}$, $\phi(H_0) = H_0'$ and $\phi(u_0) = u_0'$.

Proof: $\mathcal{Z}^{10}n = \operatorname{span}\{G_0\}$ (see proof of Prop. 2.19) implies $\phi(G_0) = \pm G_0'$ and, as well, $\phi(H_0) = H_0'$, due to $\operatorname{span}\{H_0\} = \left(\ker \operatorname{ad}_{G_0}\right)^\perp$ and $[H_0, G_0] = G_0$. $\phi(u_0) = u_0'$ follows from $u_0 = \left(\ker \operatorname{ad}_{H_0}\right) \cap H_0^\perp$ (we recall: $u_0 = f_0^\perp \cap u$).

Claim 2. $\phi(G_1) \in \{-G_1', G_1'\}$ and $\phi(H_1) = H_1'$.

Proof: From $\operatorname{span}\{G_1\} = \left(\mathcal{Z}^4 n\right) \cap u_0$ we obtain $\phi(G_1) = \pm G_1'$ and by virtue of $\operatorname{span}\{H_1\} = \left(\ker \operatorname{ad}_{G_1}\right)^\perp \cap u$ (and $[H_1, G_1] = G_1$) we get $\phi(H_1) = H_1'$ as well.

Claim 3. $\phi(G_2) \in \{-G_2', G_2'\}$, $\phi(H_2) = H_2'$, $\phi(y) = y'$, $\phi(z_\pm) = z_\pm'$ and $\phi(x) = x'$.

Proof: $y + z$ is the eigenspace of $\mathrm{ad}_{H_1}|_u$ corresponding to the eigenvalue $\frac{1}{2}$, thus $\phi(y + z) = y' + z'$. $(\mathbb{Z}^2 n) \cap u_0 \cap (G_1)^\perp \cap (y + z)^\perp = \mathrm{span}\{G_2\}$ implies $\phi(G_2) = \pm G'_2$, and $(\ker \mathrm{ad}_{G_2})^\perp \cap u \cap (y + z)^\perp = \mathrm{span}\{H_2\}$ implies $\phi(H_2) = H'_2$. Further, $z_\pm$ is the eigenspace of $\mathrm{ad}_{H_2}|_y + z$ corresponding to the eigenvalue $\pm\frac{1}{2}$ and hence $\phi(z_\pm) = z'_\pm$ and $\phi(y) = y'$. Finally, x is the eigenspace of $\mathrm{ad}_{H_2}|_{z^\perp} \cap u$ corresponding to the eigenvalue $1/2$.

Claim 4. $\phi(G_3) \in \{-G'_3, G'_3\}, \phi(H_3) = H'_3, \phi(y_\pm) = y'_\pm, \phi(z_\pm) = z'_\pm$ and $\phi(x_\pm) = x'_\pm$.

Proof: The two first relations result from $\mathrm{span}\{G_3\} = u_0 \cap (a + x + y + z)^\perp \cap G_1^\perp \cap G_2^\perp$ and $\mathrm{span}\{H_3\} = \mathrm{span}\{H_i \mid i = 0, 1, 2\}^\perp \cap a$. The rest follows from the three last equations of 3. and from the fact that $y = y_+ + y_-$ and $x_+ + x_-$ are weight decompositions for the representations $\mathrm{ad}_{H_3}|_y$ and $\mathrm{ad}_{H_3}|_x$.

The special isometric mapping $\psi : x_- \times z_- \to y_-$ (analogously ψ') is proportional to the restriction of the Lie product:

$$\psi(X_-, Z_-) = \sqrt{2}[X_-, Z_-], \quad X_- \in x_-, \ Z_- \in z_- .$$

From the fact that ϕ is a Lie algebra isomorphism which transfers y_-, z_- and x_- to y'_-, z'_- and x'_- respectively we can now conclude

$$\phi^{-1}(\psi'(\phi(X_-), \phi(Z_-))) = \psi(X_-, Z_-) . \qquad \square$$

Prop. 2.23 should be ascribed to D.V. Alekseevsky, although it is not proved in [3], since the one-to-one correspondence between isometry classes of Alekseevskian $\mathcal{V}$-spaces on the one hand and isomorphism classes of special isometric mappings on the other hand is needed to conclude the classification of $\mathcal{V}$ spaces up to *isometry*.

One may prove similarly that two $\mathcal{W}$-spaces $\mathcal{W}(p, q)$ and $\mathcal{W}(p', q')$ are isometric, if and only if $\{p, q\} = \{p', q'\}$.

We shall now derive the classification of special isometric mappings up to equivalence from the well known classification of $\mathbb{Z}_2$-graded Clifford modules. For that we will state the connection between special isometric mappings and $\mathbb{Z}_2$-graded Clifford modules discovered by Alekseevsky.

Besides the notion of equivalence we introduce the finer notion of isomorphism for didactic reasons.

Definition 2.4. Let $\phi : x \to x'$ be an isomorphism of Euclidean vector spaces. Two isometric mappings $\psi : x \times z \to y$ and $\psi' : x' \times z' \to y'$ are said to be *ϕ-isomorphic*, if there are isomorphisms $\tau : z \to z'$ and $\upsilon : y \to y'$ of Euclidean vector spaces such that the following diagram commutes:

$$\begin{array}{ccc} x \times z & \xrightarrow{\psi} & y \\ {\scriptstyle \phi \times \tau} \downarrow & & \downarrow {\scriptstyle \upsilon} \\ x' \times z' & \xrightarrow{\psi'} & y' \end{array}$$

If $x = x'$ and $\phi = \mathrm{Id}$, we simply speak of *isomorphic* isometric mappings $\psi : x \times z \to y$ and $\psi' : x \times z' \to y'$.

Consequently two isometric mappings $\psi : x \times z \to y$ and $\psi' : x' \times z' \to y'$ are equivalent, if and only if there is an isomorphism $\phi : x \to x'$ such that ψ and ψ' are ϕ-isomorphic. If we consider a fixed isomorphism ϕ, then the relation of ϕ-isomorphism implies that of equivalence, but, in general, we cannot expect equivalent isometric mappings to be ϕ-isomorphic for the given choice of ϕ.

As Alekseevsky has observed, a special isometric mapping $\psi : x \times z \to y$ defines in a canonical way a $\mathbb{Z}_2$-graded module $M_\psi = M_0 \oplus M_1$ over the Clifford algebra $C\ell(x)$. In fact, set $M_0 := z$, $M_1 := y$ and define $\Psi : x \to \mathrm{End}(M_\psi)$ by

$$\Psi(X)Z := \psi(X, Z), \quad \Psi(X)Y := -\psi^t(X, Y) \quad (X \in x,\ Y \in y,\ Z \in z)$$

(see Def. 2.6), then Ψ satisfies the relation $\Psi(X)^2 = -\langle X, X\rangle \mathrm{Id}$.

Definition 2.5. (cf. [3, Sect. 10]) Let x and x' be Euclidean vector spaces of dimension $k \geqslant 1$. Two $\mathbb{Z}_2$-graded modules $M = M_0 \oplus M_1$ and $M' = M'_0 \oplus M'_1$ over $C\ell(x)$ and $C\ell(x')$ respectively are *equivalent*, if there is an isomorphism $\sigma : C\ell(x) \to C\ell(x')$ of Clifford algebras and an isomorphism $\tau : M \to M'$ of $\mathbb{Z}_2$-graded vector spaces such that the following diagram commutes:

$$\begin{array}{ccc} C\ell(x) \times M & \longrightarrow & M \\ {\scriptstyle \sigma\times\tau} \downarrow & & \downarrow {\scriptstyle \tau} \\ C\ell(x') \times M' & \longrightarrow & M' \end{array}$$

k is the *order* of the Clifford module M. Let $\phi : x \to x'$ be an isomorphism of Euclidean vector spaces. M and M' are *ϕ-isomorphic*, if there is an isomorphism $\tau : M \to M'$ of $\mathbb{Z}_2$-graded vector spaces such that the following diagram commutes:

$$\begin{array}{ccc} x \times M_0 & \longrightarrow & M_1 \\ {\scriptstyle \phi\times\tau|_{M_0}} \downarrow & & \downarrow {\scriptstyle \tau|_{M_1}} \\ x' \times M'_0 & \longrightarrow & M'_1 \end{array}$$

The horizontal arrows denote the corresponding restriction of Clifford multiplication. If $x = x'$ and $\phi = \mathrm{Id}$, then the $\mathbb{Z}_2$-graded $C\ell(x)$-modules M and M' are said to be *isomorphic*.

We recall the classification (up to isomorphism) of $\mathbb{Z}_2$-graded Clifford modules. Let $k \in \{1, 2, \ldots\}$. We set $C\ell_k := C\ell(\mathbb{R}^k)$, where $\mathbb{R}^k$ is equipped with the standard scalar product. It is well known (see e.g. [19, Ch. I, Thm. 5.8, Prop. 5.20 and Prop. 5.4]) that for every $k \not\equiv 0 \bmod 4$ there is (up to isomorphism) a unique irreducible $\mathbb{Z}_2$-graded module M_k over $C\ell_k$ and that for every $k \equiv 0 \bmod 4$ there are exactly two irreducible $\mathbb{Z}_2$-graded modules M_k^+, M_k^- over $C\ell_k$. Every $\mathbb{Z}_2$-graded Clifford module over $C\ell_k$ is a sum of irreducible $\mathbb{Z}_2$-graded Clifford modules. As a result, an arbitrary (non-trivial) $\mathbb{Z}_2$-graded module over $C\ell_k$ is of the form ($p, q \in \{0, 1, \ldots\}$; $l, p+q \in \{1, 2, \ldots\}$):

1. $M_{l,k} := l \cdot M_k = M_k \oplus \cdots \oplus M_k$, if $k \not\equiv 0 \bmod 4$;
2. $M_{p,q;k} := p \cdot M_k^+ \oplus q \cdot M_k^-$, if $k \equiv 0 \bmod 4$.

The relation between the notions of equivalence and ϕ-isomorphism is the same as for special isometric mappings.

Proposition 2.24. *Let x and x' be Euclidean vector spaces of dimension $\geqslant 1$. Two $\mathbb{Z}_2$-graded modules M and M' over $C\ell(x)$ and $C\ell(x')$ respectively are equivalent, if and only if there is an isomorphism $\phi : x \to x'$ such that M and M' are ϕ-isomorphic.*

Proof. It is clear that ϕ-isomorphic modules are equivalent. Conversely, suppose M and M' are equivalent and $\sigma : C\ell(x) \to C\ell(x')$ is an algebra isomorphism establishing the equivalence (see Def. 2.5).

If $k = \dim x \not\equiv 0 \bmod 4$, then, according to our previous remarks, the isomorphism type of the module M is already determined by its dimension. Consequently, M and M' are ϕ-isomorphic for every isomorphism $\phi : x \to x'$, simply for dimensional reasons.

If $k \equiv 0 \bmod 4$, the automorphism $-\mathrm{Id} : x \to x$ induces an automorphism α of the Clifford algebra $C\ell(x)$ which interchanges the two isomorphism classes of irreducible $\mathbb{Z}_2$-graded $C\ell(x)$-modules (cf. [19, Prop. 5.9]). We decompose M' into irreducible summands M'^j. Let $\sigma^* M'^j$ denote the (irreducible) $\mathbb{Z}_2$-graded $C\ell(x)$-module induced by σ. Let $\phi : x \to x'$ be an isomorphism of Euclidean vector spaces and $\phi_\# : C\ell(x) \to C\ell(x')$ the induced algebra isomorphism. According to what we have just said, it is evident that either the $\mathbb{Z}_2$-graded $C\ell(x)$-module $\phi_\#{}^* M'^j$ or $(-\phi)_\#{}^* M'^j$ is isomorphic to $\sigma^* M'^j$. Without restriction of generality, we assume that $\phi_\#{}^* M'^j$ is isomorphic to $\sigma^* M'^j$ for the summand M'^j in question. Let M'^i be an other summand and $\theta : C\ell(x) \to C\ell(x')$ an algebra isomorphism.

If M'^i and M'^j are isomorphic, then so are $\theta^* M'^i$ and $\theta^* M'^j$. In particular, $\phi_\#^* M'^i$ is isomorphic to $\phi_\#^* M'^j$ and $\sigma^* M'^i$ is isomorphic to $\sigma^* M'^j$, so $\phi_\#^* M'^i$ and $\sigma^* M'^i$ are isomorphic.

If M'^i and M'^j are not isomorphic, then neither are $\theta^* M'^i$ and $\theta^* M'^j$. In particular, $\phi_\#^* M'^i$ is not isomorphic to $\phi_\#^* M'^j$ and $\sigma^* M'^i$ is not isomorphic to $\sigma^* M'^j$. Since there are only two isomorphism types of $\mathbb{Z}_2$-graded $C\ell(x)$-modules, $\phi_\#^* M'^i$ and $\sigma^* M'^i$ have to be isomorphic in this case as well. In summary, $\phi_\#{}^* M'$, $\sigma^* M'$ and M are isomorphic, hence M' and M are ϕ-isomorphic. $\square$

Now we give a slightly refined version of [3, Prop. 10.1] which reduces the classification of special isometric mappings up to equivalence to the classification of $\mathbb{Z}_2$-graded Clifford modules up to equivalence. Clifford modules will be tacitly assumed to be non-trivial and of order $k > 0$.

Proposition 2.25. (cf. [3, Prop. 10.1]) (i) *Fix a Euclidean vector space x. There is a one-to-one correspondence between special isometric mappings $\psi : x \times z \to y$ considered up to isomorphism and $\mathbb{Z}_2$-graded $C\ell(x)$-modules up to isomorphism.*

(ii) *There is a one-to-one correspondence between special isometric mappings of order k considered up to equivalence and $\mathbb{Z}_2$-graded Clifford modules of order k up to equivalence. In both cases the correspondence is induced by the map $\psi \mapsto M_\psi$.*

Proof. Let $\Psi : C\ell(x) \to \mathrm{End}(M)$, $M = M_0 \oplus M_1$, be a $\mathbb{Z}_2$-graded Clifford module. We choose a (positive definite) scalar product $\langle \cdot, \cdot \rangle$ on M invariant under the compact group $\Psi(\mathrm{Pin}(x))$. (We recall: $\mathrm{Pin}(x)$ is the subgroup of the multiplicative group of invertible elements of $C\ell(x)$ generated by the elements $X \in x$ with $|X| = 1$.) Set $z := (M_0, \langle \cdot, \cdot \rangle | M_0 \times M_0)$ and $y := (M_1, \langle \cdot, \cdot \rangle | M_1 \times M_1)$, then

$$\psi_M(X, Z) := \Psi(X)Z \quad (X \in x,\ Z \in z)$$

defines a special isometric mapping $\psi_M : x \times z \to y$ (cf. pf. of [3, Prop. 10.1]). The isomorphism type of ψ_M does not depend on the choice of invariant scalar product which can be seen by decomposing $\Psi|_{\mathrm{Pin}(x)}$ into irreducible summands. What we have just described is a map which assigns an isomorphism class (x is fixed) $[\psi_M]$ of special isometric mappings to every $\mathbb{Z}_2$-graded

$\mathcal{C}\ell(x)$-module M. $\psi \mapsto M_\psi$ and $M \mapsto [\psi_M]$ factorize modulo isomorphism relation to mutual inverses, from which (i) follows.

(ii) results from the following remark. Let $\phi : x \to x'$ be an isomorphism of Euclidean vector spaces. Two special isometric mappings $\psi : x \times z \to y$ and $\psi' : x' \times z' \to y'$ are ϕ-isomorphic, if and only if M_ψ and $M_{\psi'}$ are. $\square$

Corresponding to the operations defined for $\mathbb{Z}_2$-graded Clifford modules, sums and $\mathbb{Z}_2$-graded tensor products of special isometric mappings can be defined.

Definition 2.6. (cf. [3, Sect. 10]) Let $\psi : x \times z \to y$ be a special isometric mapping. The *transpose* of ψ is the special isometric mapping $\psi^t : x \times y \to z$ defined by

$$\langle \psi^t(X,Y), Z\rangle = \langle Y, \psi(X,Z)\rangle .$$

Let $\bar{\psi} : \bar{x} \times \bar{z} \to \bar{y}$ be a second special isometric mapping and $\bar{\bar{x}} := x + \bar{x}$, $\bar{\bar{z}} := z \otimes \bar{z} + y \otimes \bar{y}$ and $\bar{\bar{y}} := z \otimes \bar{y} + y \otimes \bar{z}$. The $\mathbb{Z}_2$-graded *tensor product* $\psi \hat{\otimes} \bar{\psi}$ is the special isometric mapping $\psi \hat{\otimes} \bar{\psi} : \bar{\bar{x}} \times \bar{\bar{z}} \to \bar{\bar{y}}$ defined by

$$\begin{aligned}\psi \hat{\otimes} \bar{\psi}(X + \bar{X}, Z \otimes \bar{Z} + Y \otimes \bar{Y}) \\ = \psi(X,Z) \otimes \bar{Z} - \psi^t(X,Y) \otimes \bar{Y} + Z \otimes \bar{\psi}(\bar{X}, \bar{Z}) + Y \otimes \bar{\psi}^t(\bar{X}, \bar{Y}).\end{aligned}$$

Suppose now that $\dim x = \dim \bar{x}$ and fix an isomorphism $\phi : x \xrightarrow{\sim} \bar{x}$. The *sum* $\psi \oplus_\phi \bar{\psi}$ is the special isometric mapping $\psi \oplus_\phi \bar{\psi} : x \times (z + \bar{z}) \to y + \bar{y}$ defined by

$$\psi \oplus_\phi \bar{\psi}(X, Z + \bar{Z}) := \psi(X,Z) + \bar{\psi}(\phi(X), \bar{Z}) .$$

The equivalence class of $\psi \oplus_\phi \bar{\psi}$ depends, in general, on the isomorphism ϕ. In particular, the sum is not a well defined operation between equivalence classes of special isometric mappings of a fixed order. This detail has been overlooked in [3, Sect. 10], for which reason the statement in Thm. 10.1b) is not well defined and misleading. Herewith, we have essentially anticipated the refinement of [3, Thm. 10.1] which will be presented subsequently.

Now we give the correct classification of $\mathbb{Z}_2$-graded Clifford modules up to equivalence. Due to Prop. 2.25, this canonically provides the classification of special isometric mappings up to equivalence, which in turn implies the classification of $\mathcal{V}$-spaces up to isometry, thanks to Prop. 2.23. We replace [3, Thm. 10.1] by the following proposition.

Proposition 2.26. *Let* $k \in \{1, 2, \ldots\}$.

(i) *Two* $\mathbb{Z}_2$*-graded modules over* $\mathcal{C}\ell_k$, $k \not\equiv 0 \bmod 4$, *are equivalent, if and only if they are isomorphic.*

(ii) *Two* $\mathbb{Z}_2$*-graded modules* $M_{p,q;k}$ *and* $M_{p',q';k}$ *over* $\mathcal{C}\ell_k$, $k \equiv 0 \bmod 4$, *are equivalent, if and only if* $\{p,q\} = \{p',q'\}$. *In particular, for every* $k \in \{1, 2, \ldots\}$ *there is exactly one equivalence class of irreducible* $\mathbb{Z}_2$*-graded modules over* $\mathcal{C}\ell_k$.

Proof. (i) is true, since for $k \not\equiv 0 \bmod 4$ non-isomorphic $\mathbb{Z}_2$-graded modules over $\mathcal{C}\ell_k$ differ in dimension.

(ii) As the categories of $\mathbb{Z}_2$-graded modules over $\mathcal{C}\ell_k$ and of ungraded modules over $\mathcal{C}\ell_{k-1}$ are equivalent (see [19, Ch. I, Prop. 5.20]), we can equally well discuss the induced notion of equivalence for ungraded Clifford modules.

Assume $k \equiv 0 \bmod 4$, so according to [19, Ch. I, Thm. 5.8]

$$\mathcal{C}\ell_{k-1} \cong K(n) \oplus K(n) \quad \text{(as algebras)},$$

where $K(n)$ denotes the algebra of $n \times n$-matrices over the field K, and both n and $K \in \{\mathbb{R}, \mathbb{H}\}$ depend on k as indicated there. We denote by $M^{\pm}_{k-1}$ and $M_{p,q;k-1}$ the ungraded Clifford modules corresponding to $M^{\pm}_k$ and $M_{p,q;k}$ respectively. M^{+}_{k-1} resp. M^{-}_{k-1} is given by the standard representation of the first resp. the second $K(n)$-summand and the trivial representation of the second resp. the first $K(n)$-summand on K^n.

The automorphism $A \oplus B \mapsto B \oplus A$ of $\mathcal{C}\ell_{k-1}$ interchanges the $K(n)$-summands, consequently $M_{p,q;k-1}$ and $M_{q,p;k-1}$ are equivalent. On the other hand, every automorphism of $\mathcal{C}\ell_{k-1}$ permutes the summands $K(n)$, because $K(n)$ is simple. Therefore, it is sufficient to discuss the effect of an automorphism $\phi : K(n) \to K(n)$ on the standard representation ρ of $K(n)$ on K^n. $\rho \circ \phi$ is irreducible, hence isomorphic to ρ, since simple algebras have only one irreducible representation (cf. [18, Ch. XVII, Sect. 4, Cor. 2]). So, for every automorphism $\alpha : \mathcal{C}\ell_{k-1} \to \mathcal{C}\ell_{k-1}$ the Clifford module $\alpha^* M_{p,q;k-1}$ induced by α is isomorphic to $M_{p,q;k-1}$ or $M_{q,p;k-1}$, depending on whether α preserves or interchanges the $K(n)$-summands. $\square$

It should be taken into account that, in line with Prop. 2.26, sums of $\mathbb{Z}_2$-graded $\mathcal{C}\ell_{4m}$-modules with equivalent summands may fail to be equivalent, e.g. it is true that

$$M^{+}_{4m} \oplus M^{+}_{4m} \sim M^{-}_{4m} \oplus M^{-}_{4m}, \text{ but } \not\sim M^{+}_{4m} \oplus M^{-}_{4m}.$$

This has been apparently overlooked in [3], which results in not distinguishing between modules $M_{p,q;4m}$ and $M_{p',q';4m}$ with $p + q = p' + q' = l$. Consequently, the corresponding Alekseevsky spaces are denoted by $\mathcal{V}(l, 4m)$ in [3, Sect. 10]. This notation is ambiguous for $l = p+q \geqslant 2$ and should be replaced by $\mathcal{V}(p, q; 4m)$. Therefore, in the following, we will denote by $\mathcal{V}(p, q; 4m)$ the $\mathcal{V}$-space corresponding to $M_{p,q;4m}$. We set $\mathcal{V}(1, 4m) := \mathcal{V}(0, 1; 4m)$ in harmony with [3]. We conclude this section with a precise formulation of this observation.

Corollary 2.27. *For $m, l \in \{1, 2, \ldots\}$ let $\mathcal{V}_{l,4m}$ denote the family (of isometry classes) of Alekseevskian $\mathcal{V}$-spaces corresponding to $\mathbb{Z}_2$-graded $\mathcal{C}\ell_{4m}$-modules of (real) dimension $2lN(4m)$, where $2N(k)$ is the dimension of an irreducible $\mathbb{Z}_2$-graded module over $\mathcal{C}\ell_k$. Then*

$$\mathcal{V}_{l,4m} = \{\mathcal{V}(p, q; 4m) \mid 0 \leqslant p \leqslant q,\ p + q = l\}, \quad \sharp\mathcal{V}_{l,4m} = 1 + \left[\tfrac{1}{2}l\right].$$

In particular, $\sharp\mathcal{V}_{l,4m} \geqslant 2$ for $l \geqslant 2$.

Proof. According to Prop. 2.26, the modules $M_{p,q;4m}$ with $p+q = l$ and $0 \leqslant p \leqslant q$ are pairwise inequivalent, and altogether they represent all the equivalence classes of $\mathbb{Z}_2$-graded $\mathcal{C}\ell_{4m}$-modules of dimension $2lN(4m)$. $\square$

2.3. Summary

We summarize the analysis undertaken in the Sections 2.1 and 2.2 in the form of a corrected classification theorem.

Alekseevsky space	symm. space?	rk.	$\dim_{\mathbb{H}}$	root system?
$H_{\mathbb{H}}^m$	$\frac{Sp(m,1)}{Sp(m)Sp(1)}$	1	$m \geqslant 1$	a_1
hermitian symm.	$\frac{SU(1,2)}{U(2)}$	1	1	a_1
Alekseevsky spaces	$\frac{SU(m,2)}{S(U(m)\times U(2))}$	2	$m \geqslant 2$	b_2
"type 3" (see [3, Sect. 8])	$\frac{G_2^{(2)}}{SO(4)}$	2	2	g_2
$\mathcal{T}(0)$	$\frac{SO_0(3,4)}{SO(3)\times SO(4)}$	3	3	b_3
$\mathcal{T}(p),\ p \geqslant 1$	no	3	$p+3$	no
$\mathcal{W}(0,0)$	$\frac{SO_0(4,4)}{SO(4)\times SO(4)}$	4	4	d_4
$\mathcal{W}(0,q),\ q \geqslant 1$	$\frac{SO_0(q+4,4)}{SO(q+4)\times SO(4)}$	4	$q+4$	b_4
$\mathcal{W}(p,q),\ 1 \leqslant p \leqslant q$	no	4	$4+p+q$	no
$\mathcal{V}(1,1)$	$\frac{F_4^{(4)}}{Sp(3)Sp(1)}$	4	7	f_4
$\mathcal{V}(1,2)$	$\frac{E_6^{(2)}}{SU(6)Sp(1)}$	4	10	f_4
$\mathcal{V}(1,4)$	$\frac{E_7^{(-5)}}{\mathrm{Spin}(12)Sp(1)}$	4	16	f_4
$\mathcal{V}(1,8)$	$\frac{E_8^{(-24)}}{E_7 Sp(1)}$	4	28	f_4
$\mathcal{V}(l,k),\ k \not\equiv 0 \pmod 4,$ $(l,k) \notin \{(1,1);(1,2)\}$	no	4	$4+k+2lN(k)$	f_4
$\mathcal{V}(p,q;k),\ k \equiv 0 \pmod 4,$ $(p+q,k) \notin \{(1,4);(1,8)\}$	no	4	$4+k+2lN(k)$ $(l=p+q)$	f_4

Table 1.

Theorem 2.28 (Classification of Alekseevsky spaces).

(i) *Alekseevskian spaces have rank* $\leqslant 4$. *If their rank is* $\leqslant 2$, *they are symmetric.*

(ii) *The spaces of rank* 3 *are* (*up to scaling*) *exactly the spaces*

$$\mathcal{T}(p), \quad p \geqslant 0.$$

(iii) *The spaces of rank* 4 *are exactly the spaces*

$$\begin{aligned} &\mathcal{W}(p,q), && 0 \leqslant p \leqslant q; \\ &\mathcal{V}(l,k), && l,k \geqslant 1, \quad k \not\equiv 0 \bmod 4; \\ &\mathcal{V}(p,q;4m), && q,m \geqslant 1, \quad 0 \leqslant p \leqslant q. \end{aligned}$$

(iv) *The symmetric Alekseevsky spaces of rank* $\geqslant 3$ *are* $\mathcal{T}(0)$, $\mathcal{W}(0,q)$ $(q \geqslant 0)$, $\mathcal{V}(1,1)$, $\mathcal{V}(1,2)$, $\mathcal{V}(1,4) = \mathcal{V}(0,1;4)$ *and* $\mathcal{V}(1,8) = \mathcal{V}(0,1;8)$. *This and further information is compiled in Table* 1. *The rank and the quaternionic dimension* ($\dim_{\mathbb{R}}/4$) *of the Alekseevskian spaces is listed in the third and in the fourth columns respectively.* $2N(k)$ *denotes the dimension of an irreducible* $\mathbb{Z}_2$*-graded* $\mathcal{C}\ell_k$*-module and is computable from the recursion formula*

$$\begin{aligned} &N(8s+t) = (16)^s N(t) \quad (1 \leqslant t \leqslant 8,\ s \geqslant 0), \\ &N(1) = 1, \\ &N(2) = 2, \\ &N(3) = N(4) = 4, \\ &N(5) = N(6) = N(7) = N(8) = 8. \end{aligned}$$

From the last column one can decide whether the roots of the Alekseevskian Lie algebra in question form a root system in the sense of Def. 2.3. *If this is the case, the isomorphism type of the root system is indicated in the usual notation* (*see e.g.* [14, *Ch. X, Sect.* 3.3]).

Proof. First of all Alekseevsky's classification [3], (cf. Sect. I of the present exposition) together with the refinements noted in Thm. 2.11 and Cor. 2.27) provides the Alekseevskian spaces as list of Alekseevskian Lie algebras. Some of these metric Lie algebras are the solvable part of the Iwasawa decomposition of the isometry algebra of a symmetric space of non-compact type; they correspond to the symmetric Alekseevsky spaces. For deciding which Alekseevsky spaces are symmetric the following argument due to Alekseevsky is enough (cf. [9, II.3.4] for details). Quaternionic Kählerian symmetric spaces of non-compact type (see [6, 14.52 Table] for a list) are Alekseevskian (owed to the Iwasawa decomposition), so for every such symmetric space there must be *at least* one Alekseevskian Lie algebra s of same dimension, same rank and same root system. The last condition means that the roots of s have to form a root system (in the sense of Def. 2.3) which is isomorphic to the *restricted root system* of the symmetric space. In particular, an Alekseevsky space whose roots do not form a root system cannot be symmetric. A list of the restricted root systems of the symmetric spaces of non-compact type is printed in [14, Ch. X, Table VI on pp. 532–534].

From the list of Alekseevskian Lie algebras one gathers (having determined the root systems) that for every quaternionic Kählerian symmetric space of non-compact type there is always *exactly* one Alekseevsky space with the same dimension, rank and root system. (In some cases, an

Alekseevsky space is already uniquely determined by its rank and its dimension alone, e.g. in the case of the spaces of rank 3 (cf. Thm. 2.11).)

The remaining Alekseevskian Lie algebras necessarily correspond to the non-symmetric Alekseevsky spaces (which also follows from our curvature computations in Sect. 3). In [3, Table 1], Alekseevsky gives (without computation) the isomorphism type of the root systems for the Lie algebras he constructed, in case a root system is on hand. For reasons of completeness we discussed the roots for the $\mathcal{T}$-spaces in Prop. 2.6 and even for the $\mathcal{W}$- and $\mathcal{V}$-spaces in Prop. 2.15 and 2.22. □

Proposition 2.29. *Alekseevskian Lie algebras are of Iwasawa type. In particular, they admit—independently of the scalar product $\langle\cdot\,,\cdot\rangle$ considered up to now—scalar products $\langle\cdot\,,\cdot\rangle_{np}$ of non-positive sectional curvature. The $\mathcal{T}$- and $\mathcal{W}$-spaces are 4-step, the $\mathcal{V}$-spaces 5-step solvable Lie groups.*

Proof. Since Lie algebras of Iwasawa type admit non-positively curved scalar products (see [23, pp. 17–20]), the first statements flow from Prop. 2.3, 2.13 and 2.20. The derived series are computed in Prop. 2.2, 2.12 and 2.19. □

Nevertheless we will go on studying $(t(p), \langle\cdot\,,\cdot\rangle)$, because at present we are not interested in non-positively curved metrics but in quaternionic Kählerian ones. We will see in Ch. 3 that the non-symmetric Alekseevsky spaces have both negatively curved and positively curved tangent planes.

3. Characterization of symmetric spaces

The main result of this chapter is that the *symmetric* Alekseevsky spaces are completely characterized by the property of being non-positively curved (Theorem 3.10).

First, we make use of the formulas provided for the covariant derivative of the $\mathcal{T}$- and $\mathcal{W}$-spaces to compute curvatures. Let s be a metric Lie algebra. We use the abbreviation R_S for the operator $R_S : s \to s$ defined as $R_S T := R(T, S)S$ $(S, T \in s)$ by means of the curvature tensor.

Proposition 3.1. *The curvature of $\mathcal{T}(p)$ satisfies:*

$$R_A = -\mathrm{ad}_A^2 \quad \textit{for } A \in a\,.$$

In particular, the eigenspace decomposition of the operators R_A ensues from the root space decomposition in the proof of Prop. 2.3.

Let $p \geqslant 1$. The operators R_{X_+} $(X_+ \in x_+, |X_+| = 1)$ have the following eigenspace decomposition:

$$R_{X_+}|_{(H_1+\frac{\sqrt{2}}{2}H_2)^\perp\cap a} = 0\,, \qquad R_{X_+}(H_1 + \tfrac{\sqrt{2}}{2}H_2) = -\tfrac{3}{8}(H_1 + \tfrac{\sqrt{2}}{2}H_2),$$

$$R_{X_+}|_{(G_1-\frac{\sqrt{2}}{2}G_2)^\perp\cap Ja} = 0\,, \qquad R_{X_+}(G_1 - \tfrac{\sqrt{2}}{2}G_2) = -\tfrac{3}{8}(G_1 - \tfrac{\sqrt{2}}{2}G_2),$$

$$R_{X_+}|_{(\tilde H_0+\frac{\sqrt{2}}{2}\tilde H_2)^\perp\cap \tilde a} = 0, \qquad R_{X_+}(\tilde H_0 + \tfrac{\sqrt{2}}{2}\tilde H_2) = -\tfrac{3}{8}(\tilde H_0 + \tfrac{\sqrt{2}}{2}\tilde H_2),$$

$$R_{X_+}|_{(\tilde G_0-\frac{\sqrt{2}}{2}\tilde G_2)^\perp\cap \widetilde{Ja}} = 0, \qquad R_{X_+}(\tilde G_0 - \tfrac{\sqrt{2}}{2}\tilde G_2) = -\tfrac{3}{8}(\tilde G_0 - \tfrac{\sqrt{2}}{2}\tilde G_2),$$

$$R_{X_+}X_+ = R_{X_+}\widetilde{JX_+} = 0\,,$$
$$R_{X_+}JX_+ = -\tfrac{3}{4}JX_+\,, \qquad R_{X_+}\tilde{X}_+ = -\tfrac{3}{4}\tilde{X}_+\,,$$
$$R_{X_+}X'_+ = -\tfrac{3}{8}X'_+\,, \qquad R_{X_+}\widetilde{JX'_+} = \tfrac{1}{8}\widetilde{JX'_+}\,,$$
$$R_{X_+}JX'_+ = -\tfrac{1}{8}JX'_+\,, \qquad R_{X_+}\tilde{X}'_+ = -\tfrac{1}{8}\tilde{X}'_+\,,$$

where $X'_+ \in x_+$ and $X'_+ \perp X_+$. In particular, if $X'_+ \neq 0$ (which requires $p \geqslant 2$),

$$K\left(X_+ \wedge \widetilde{JX'_+}\right) = +\tfrac{1}{8}\,. \tag{9}$$

Analogous formulas for the operators $R_{J_\alpha A}$ and $R_{J_\alpha X_+}$ ($\alpha = 1, 2, 3$) arise from the $Sp(1)$-invariance of the curvature.

Proof. The first statement follows immediately from $\nabla_A = 0$. The operators $R_X|_a$ ($X \in x_+ \cup x_- \cup \tilde{x}_+ \cup \tilde{x}_-$) are easy to compute, due to $\nabla_A = 0$ ($A \in a$). From this one obtains expressions for $R_X|_{J_\alpha a}$ ($\alpha = 1, 2, 3$) using the $Sp(1)$-invariance of the curvature (Eq. (2)). The remaining formulas result from Prop. 2.9 after a short computation. □

Corollary 3.2. *The sectional curvature of $\mathcal{T}(1)$ satisfies the sharp estimate*

$$-1 \leqslant K(S \wedge T) \leqslant 0$$

for all planes with $S \in t(1)$ and $T \in t(0)$.

Proof. Since $\mathcal{T}(0)$ is totally geodesic, it is enough to show the estimate for $S \in t(0) \cup t(0)^{\perp}$. The $Sp(1)$-invariance of the curvature allows us further to assume that $S \in a \cup x_+$. Now the estimate follows from Prop. 3.1. □

Corollary 3.3. *The sectional curvature of the symmetric space $\mathcal{T}(0)$ satisfies the sharp estimate*

$$-1 \leqslant K \leqslant 0\,.$$

Proof. This results e.g. from Cor. 3.2, as $\mathcal{T}(0)$ is totally geodesic in $\mathcal{T}(1)$. □

Now we repeate the same kind of computation for the $\mathcal{W}$-spaces (for more details concerning their curvature see [10, II.6]).

Proposition 3.4. *The curvature of $\mathcal{W}(p, q)$ satisfies:*

$$R_A = -\mathrm{ad}_A^2 \quad \textit{for} \quad A \in a\,.$$

In particular, the eigenspace decomposition of the operators R_A ensues from the root space decomposition in Prop. 2.13.

Let $p \geqslant 1$. The operators R_{Y_-} ($Y_- \in y_-$, $|Y_-| = 1$) have the following eigenspace decomposition:

$$\begin{aligned}
&R_{Y_-}(H_1 - H_3) = -\tfrac{1}{2}(H_1 - H_3)\,, && R_{Y_-}(H_1 - H_3)^\perp \cap a = 0\,,\\
&R_{Y_-}(G_1 + G_3) = -\tfrac{1}{2}(G_1 + G_3)\,, && R_{Y_-}(G_1 + G_3)^\perp \cap Ja = 0\,,\\
&R_{Y_-}(\tilde H_0 - \tilde H_2) = -\tfrac{1}{2}(\tilde H_0 - \tilde H_2)\,, && R_{Y_-}(\tilde H_0 - \tilde H_2)^\perp \cap \tilde a = 0\,,\\
&R_{Y_-}(\tilde G_0 + \tilde G_2) = -\tfrac{1}{2}(\tilde G_0 + \tilde G_2)\,, && R_{Y_-}(\tilde G_0 + \tilde G_2)^\perp \cap \widetilde{Ja} = 0\,,
\end{aligned}$$

$$\begin{aligned}
&R_{Y_-} Y_- = 0\,, && R_{Y_-} J Y_- = -\tfrac{1}{2} J Y_-\,,\\
&R_{Y_-}|_{y_- \cap Y_-^\perp} = -\tfrac{1}{2}\,\mathrm{Id}, && R_{Y_-}|_{y_+ \cap J Y_-^\perp} = 0\,, && R_{Y_-}|_z = -\tfrac{1}{4}\,\mathrm{Id},\\
&R_{Y_-}\tilde Y_- = -\tfrac{1}{2}\tilde Y_-\,, && R_{Y_-}\widetilde{JY_-} = -\tfrac{1}{2}\widetilde{JY_-}\,,\\
&R_{Y_-}|_{\tilde Y_-^\perp \cap \widetilde{JY_-}^\perp \cap \tilde y} = 0\,, && R_{Y_-}|_{\tilde z_\pm} = \pm\tfrac{1}{4}\,\mathrm{Id}\,.
\end{aligned}$$

In particular,

$$K(Y_- \wedge \tilde Z_+) = +\tfrac{1}{4}\,, \tag{10}$$

if $Y_- \in y_- - \{0\}$ and $Z_- \in z_- - \{0\}$, which requires $p \geqslant 1$ and $q \geqslant 1$.

For $q \geqslant 1$ the eigenspace decomposition of R_{Z_-} ($Z_- \in z_-$, $|Z_-| = 1$) can be obtained by simultaneously interchanging the roles of y and z and the indices 2 and 3 in the expressions for R_{Y_-}. Further eigenspace decompositions flow immediately from the $Sp(1)$-invariance of the curvature.

Proof: The proof is similar to that of Prop. 3.1. The penultimate statement arises from the standard isomorphism $w(p,q) \xrightarrow{\sim} w(q,p)$. □

Corollary 3.5. *The sectional curvature of $w(p,q)$ satisfies the following sharp estimates for $A \in a$, $Y_- \in y_-$ and $W \in w$ arbitrary*

$$\begin{aligned}
&-1 \leqslant K(A \wedge W) \leqslant 0\,,\\
&-\tfrac{1}{2} \leqslant K(Y_- \wedge W) \leqslant \tfrac{1}{4}\,.
\end{aligned}$$

As before Y_- may be replaced by $Z_- \in z_-$ and further estimates follow from $Sp(1)$-invariance.

Proof. This follows from the eigenspace decomposition of Prop. 3.4 (cf. [9, p. 45] for trivial details). □

Corollary 3.6. *The sectional curvature of the symmetric spaces $\mathcal{W}(p,0) \cong \mathcal{W}(0,p)$ satisfies the sharp estimate*

$$-1 \leqslant K \leqslant 0\,.$$

Proof. The linear isotropy representation of a symmetric space is transitive on the flat totally geodesic (infinitesimal) subspaces of maximal dimension (see [14, Ch.V, Sect. 6, Theorem 6.2]). Therefore, it is sufficient to consider sectional curvatures $K(A \wedge W)$ for $A \in a$. Now the claim follows from Cor. 3.5. □

In dealing with the $\mathcal{V}$-spaces we will make use of the following two lemmas.

Lemma 3.7. *With the notations introduced in Sect. 1.3, the* $\mathcal{V}$*-spaces satisfy*

$$K(Y_- \wedge \tilde{Z}_+) = \tfrac{1}{4}(1 - 3\,|Y_- \circ Z_+|^2)$$

for $Y_- \in y_-$ *and* $Z_+ \in z_+$ *with* $|Y_-| = |Z_+| = 1$. *Furthermore,*

$$|Y_- \circ Z_+|^2 = \tfrac{1}{2}\sum_i \langle Y_-, \psi(X_-^i, JZ_+)\rangle^2$$

for every orthonormal basis $(X_-^i)_{i=1,\dots,k}$ *of* x_-.

Proof. The following holds because the covariant derivative is compatible with the metric:

$$K(Y_- \wedge \tilde{Z}_+) = -\langle \nabla_{\tilde{Z}_+}\tilde{Z}_+, \nabla_{Y_-}Y_-\rangle + \langle \nabla_{Y_-}\tilde{Z}_+, \nabla_{\tilde{Z}_+}Y_-\rangle - \langle \nabla_{T_{Y_-}\tilde{Z}_+}\tilde{Z}_+, Y_-\rangle\,.$$

Using the Koszul formula we compute

$$\nabla_{Y_-}Y_- = \tfrac{1}{2}(H_1 - H_3)$$

and then

$$-\langle \nabla_{\tilde{Z}_+}\tilde{Z}_+, \nabla_{Y_-}Y_-\rangle = \tfrac{1}{4}\,.$$

In dealing with the second and the third term we use the vanishing of the covariant derivative's torsion, the definition of the representation T (see Thm. 1.4) and, at the places labeled by $(*)$, the identity $\nabla_{Y_-}\tilde{Z}_+ = \widetilde{\nabla_{Y_-}Z_+}$, which results from the Koszul formula.

$$\begin{aligned}\langle \nabla_{Y_-}\tilde{Z}_+, \nabla_{\tilde{Z}_+}Y_-\rangle &= |\nabla_{Y_-}\tilde{Z}_+|^2 - \langle \nabla_{Y_-}\tilde{Z}_+, T_{Y_-}\tilde{Z}_+\rangle \\ &\overset{(*)}{=} |\nabla_{Y_-}Z_+|^2 - \langle \nabla_{Y_-}\tilde{Z}_+, \widetilde{Y_- \circ Z_+}\rangle \overset{(*)}{=} -|\nabla_{Y_-}Z_+|^2 = -\tfrac{1}{4}\,|Y_- \circ Z_+|^2\,.\end{aligned}$$

In addition, for the last term we use again the compatibility of the covariant derivative with the metric and, at the place labeled by (a), the fact that $Y_- \circ Z_+ \in x_+$ and $T_{Y_-}x_+ = 0$.

$$\begin{aligned}-\langle \nabla_{T_{Y_-}\tilde{Z}_-}\tilde{Z}_+, Y_-\rangle &= -\langle \nabla_{\widetilde{Y_- \circ Z_+}}\tilde{Z}_+, Y_-\rangle = \langle \tilde{Z}_+, \nabla_{\widetilde{Y_- \circ Z_+}}Y_-\rangle \\ &= -\langle \tilde{Z}_+, T_{Y_-}\widetilde{Y_- \circ Z_+}\rangle + \langle \tilde{Z}_+, \nabla_{Y_-}\widetilde{Y_- \circ Z_+}\rangle \\ &\overset{(a)}{=} -\langle \nabla_{Y_-}\tilde{Z}_+, \widetilde{Y_- \circ Z_+}\rangle \overset{(*)}{=} -\tfrac{1}{2}\,|Y_- \circ Z_+|^2\,.\end{aligned}$$

The second statement is a trivial consequence of the Remark 1.3:

$$|Y_- \circ Z_+|^2 = \sum_i \langle Y_- \circ Z_+, JX_-^i\rangle^2 = \tfrac{1}{2}\sum_i \langle Y_-, \psi(X_-^i, JZ_+)\rangle^2\,. \qquad \square$$

Lemma 3.8. $\mathcal{V}$*-spaces which correspond to sums or* $\mathbb{Z}_2$*-graded tensor products of (at least two non-trivial)* $\mathbb{Z}_2$*-graded Clifford modules (of order* $k \geqslant 1$*) show positively curved tangent planes.*

Proof. Sums: In this case we have $z_- = v^0 + w^0$ and $y_- = v^1 + w^1$ (orthogonal sums), where $v^0 + v^1$ and $w^0 + w^1$ are $\mathbb{Z}_2$-graded modules over $\mathcal{C}\ell(x_-) \cong \mathcal{C}\ell_k$. $\psi : x_- \times z_- \to y_-$ is the restriction of Clifford multiplication. Choosing $JZ_+ \in v^0 - \{0\}$ and $Y_- \in w^1 - \{0\}$ entails $\psi(X_-, JZ_+) \in v^1 \perp w^1$ and hence $K(Y_- \wedge \tilde{Z}_+) = \frac{1}{4} > 0$, due to Lemma 3.7.

Tensor products: In this case the special isometric mapping ψ defining the $\mathcal{V}$-space is of the form

$$\psi = \bar{\psi}\hat{\otimes}\bar{\bar{\psi}} : x_- \times z_- \to y_- ,$$

where $\bar{\psi} : \bar{x} \times \bar{z} \to \bar{y}$ and $\bar{\bar{\psi}} : \bar{\bar{x}} \times \bar{\bar{z}} \to \bar{\bar{y}}$ are special isometric mappings and

$$x_- = \bar{x} + \bar{\bar{x}} , \quad z_- = \bar{z} \otimes \bar{\bar{z}} + \bar{y} \otimes \bar{\bar{y}} , \quad y_- = \bar{z} \otimes \bar{\bar{y}} + \bar{y} \otimes \bar{\bar{z}} .$$

If we choose

$$Y_- = \bar{Y} \otimes \bar{\bar{Z}} \in \bar{y} \otimes \bar{\bar{z}} \subset y_- \quad \text{and} \quad JZ_+ = \bar{Z} \otimes \bar{\bar{Z}}' \in \bar{z} \otimes \bar{\bar{z}} \subset z_-$$

such that $Y_- \neq 0$, $Z_+ \neq 0$ and $\bar{\bar{Z}} \perp \bar{\bar{Z}}'$, then the following holds for every $X_- = \bar{X} + \bar{\bar{X}} \in \bar{x} + \bar{\bar{x}} = x_-$, according to the definition of $\bar{\psi}\hat{\otimes}\bar{\bar{\psi}}$ (see Def. 2.6)

$$\langle Y_-, \psi\,(X_-, JZ_+)\rangle = \langle \bar{Y} \otimes \bar{\bar{Z}}, \bar{\psi}\,(\bar{X}, \bar{Z}) \otimes \bar{\bar{Z}}'\rangle = 0 .$$

Hence, $K(Y_- \wedge \tilde{Z}_+) = \frac{1}{4} > 0$, again by the previous lemma. $\square$

Corollary 3.9. *The following spaces show positively curved tangent planes*:
(i) $\mathcal{V}(l, k)$, *if* $l > 1$ *or* $k > 8$.
(ii) $\mathcal{V}(p, q; 4m)$, *if* $p + q > 1$ *or* $4m > 8$.

Proof. The $\mathbb{Z}_2$-graded Clifford module defining $\mathcal{V}(l, k)$ resp. $\mathcal{V}(p, q; 4m)$ is, by definition, a *sum* of l resp. $p + q$ irreducible $\mathbb{Z}_2$-graded Clifford modules. Therefore if $l > 1$ resp. $p + q > 1$, then the corresponding $\mathcal{V}$-space shows positively curved tangent planes, owing to Lemma 3.8.

By virtue of the periodicity modulo 8 of $\mathbb{Z}_2$-graded Clifford modules (see [5, Cor. 6.6]), every irreducible $\mathbb{Z}_2$-graded module over $C\ell_k$ for $k > 8$ is a non-trivial $\mathbb{Z}_2$-graded *tensor product* of $\mathbb{Z}_2$-graded Clifford modules. Therefore, again by Lemma 3.8, $\mathcal{V}$-spaces corresponding to Clifford modules of order > 8 show positively curved tangent planes. $\square$

According to a theorem of J.E. D'Atri and I. Dotti Miatello [11], a bounded homogeneous domain is symmetric, if and only if its sectional curvature (with respect to the Bergmann metric) is non-positive. We proof an analogue of this statement for Alekseevsky spaces.

Theorem 3.10. *Let* $\mathcal{S}$ *be an Alekseevsky space. Then the following conditions are equivalent*:
(i) $\mathcal{S}$ *is symmetric.*
(ii) *The sectional curvature of* $\mathcal{S}$ *is non-positive.*

Proof. For every non-symmetric Alekseevsky space we will exhibit tangent planes of positive curvature. There are three families to treat; $\mathcal{T}$-, $\mathcal{W}$- and $\mathcal{V}$-spaces.

$\mathcal{T}$-spaces: Let $\mathcal{T}(p)$ be non-symmetric, i.e. $p \geqslant 1$. We can choose $X_+ \in x_+$ with $|X_+| = 1$, since $p = \dim x_+$. We claim that the tangent plane

$$\tau = (H_0 + X_+) \wedge (\tilde{G}_0 - \widetilde{JX_+})$$

is positively curved. With the help of the propositions 3.1 and 2.9 and considering the fact that

$t(0)$ is totally geodesic in $t(1)$ we obtain

$$\begin{aligned}4K(\tau) &= |\tau|^2 K(\tau)\\ &= K(H_0 \wedge \tilde{G}_0) + K(H_0 \wedge \widetilde{JX_+}) + K(\tilde{G}_0 \wedge X_+) + K(X_+ \wedge \widetilde{JX_+})\\ &\quad - 2(\langle R(H_0, \tilde{G}_0)\widetilde{JX_+}, X_+\rangle + \langle R(H_0, \widetilde{JX_+})\tilde{G}_0, X_+\rangle)\\ &= -\tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} + 0 - 2(-\tfrac{1}{4} - \tfrac{1}{4}) = \tfrac{1}{4} > 0\,.\end{aligned}$$

$\mathcal{W}$-spaces: Let $\mathcal{W}(p, q)$ be non-symmetric, i.e. $p \geqslant 1$ and $q \geqslant 1$. We can choose $Y_- \in y_- - \{0\}$ and $Z_+ \in z_+ - \{0\}$, since $p = \dim y_-$ and $q = \dim z_+$. According to eq. 10 in Prop. 3.4, the plane $Y_- \wedge \tilde{Z}_+$ is positively curved.

$\mathcal{V}$-spaces: Thanks to Cor. 3.9, we only need to exhibit positively curved tangent planes in the case of $v(1, k)$ for $k = 3, 5, 6$ and 7. Let ψ_k be the special isometric mapping determined up to equivalence by $v(1, k)$, i.e. one which defines an irreducible $\mathbb{Z}_2$-graded module over $C\ell_k$.

It is well known ([3, Sect. 10]) and obvious that $\psi_3 : x_- \times z_- \to y_-$ can be realized restricting quaternion multiplication:

$$z_- = y_- = \mathbb{H} \quad \text{and} \quad x_- = \operatorname{span}\{1, \mathrm{i}, \mathrm{j}\} \subset \mathbb{H}\,.$$

Where $\mathbb{H}$ is supplied with the standard scalar product. Set $Y_- = \mathrm{k}$, $JZ_+ = 1$ and $(X_-^1, X_-^2, X_-^3) = (1, \mathrm{i}, \mathrm{j})$. Then Lemma 3.7 gives

$$|Y_- \circ Z_+|^2 = \tfrac{1}{2}\left(\langle \mathrm{k}, 1\rangle + \langle \mathrm{k}, \mathrm{i}\rangle + \langle \mathrm{k}, \mathrm{j}\rangle\right) = 0\,,$$

hence $K(Y_- \wedge \tilde{Z}_+) = \frac{1}{4} > 0$.

The cases $k = 5, 6, 7$ will be treated analogously. Let **Ca** denote the Cayley octaves provided with the standard scalar product,

$$z_- = y_- = \mathbf{Ca} \quad \text{and} \quad x_- \subset \mathbf{Ca} \ \text{ a } k\text{-dimensional subspace}\,.$$

$\psi_k : x_- \times z_- \to y_-$ is obtained by restricting the Cayley product. Choosing $Y_- \in x_-^\perp - \{0\}$, $JZ_+ = 1$ and any orthonormal basis $(X_-^i)_{i=1,\dots,k}$ of x_- and applying Lemma 3.7 gives

$$|Y_- \circ Z_+|^2 = \tfrac{1}{2}\sum_{i=1}^{k}\langle Y_-, X_-^i\rangle = 0\,,$$

hence again $K(Y_- \wedge \tilde{Z}_+) = \frac{1}{4} > 0$. □

Remark 3.1. In eq. (9) of Prop. 3.1 we had already exhibited positively curved tangent planes of $\mathcal{T}(p)$ for $p \geqslant 2$. For $\mathcal{T}(1)$ such planes, as given in the proof of Thm. 3.10, are less easy to find.

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