Many 3-colorings of triangle-free planar graphs

Carsten Thomassen

Department of Mathematics, Technical University of Denmark, DK-2800 Lyngby, Denmark
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Abstract

Grötzsch proved that every planar triangle-free graph is 3-colorable. We prove that it has at least \( 2^{3/12/n/20000} \) distinct 3-colorings where \( n \) is the number of vertices. If the graph has girth at least 5, then it has at least \( 2^{n/10000} \) distinct list-colorings provided every vertex has at least three available colors.

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1. Introduction

A classical theorem of Grötzsch [3] says that every planar triangle-free graph \( G \) is 3-colorable. Steinberg and Younger [7] gave a proof of the dual version and extended that to projective planar graphs. The dual version is interesting in that it may extend to nonplanar graphs. Indeed, a classical conjecture of Tutte (see e.g. [1, p. 252]) says that every 2-edge-connected graph with no 3-edge-cut has an orientation such that the outdegree of each vertex equals the indegree modulo 3. [9] gave a short proof of Grötzsch’s theorem and that was used to derive counterparts for projective planar graphs and toroidal graphs. That proof was recently used by Kowalik [5] to derive an almost linear time algorithm for 3-coloring a planar triangle-free graph. Voigt [16] gave examples of triangle-free planar graphs that are not 3-list-colorable. However, all planar graphs of girth at least 5 are 3-list-colorable as proved first in [10] and then by a much simpler argument in [12]. These results also provide proofs of Grötzsch’s theorem as that can easily be reduced to graphs of girth 5. The result in [12] was then used to derive a general 3-color theorem for each fixed surface in [13]. We shall here refine the proof in [12] to prove that a planar graph of girth at least 5 has not only exponentially many 3-colorings but also exponentially many list-colorings.
provided all lists have at least three colors. We then prove that every planar triangle-free graph has at least $2^{n^{1/12}/20000}$ distinct 3-colorings where $n$ is the number of vertices. First we relate the problem to the 3-color matrix.

The notation and terminology are the same as in [6,8,9,12].

2. The 3-color matrix and many 3-colorings

Let $G$ be a graph with vertices $v_1, v_2, \ldots, v_n$. Let $c$ be a 3-coloring such that each color is thought of as an element in the field with 3 elements. This coloring may be described by the vector $c(v_1), c(v_2), \ldots, c(v_n)$. The 3-color matrix of $G$ is the matrix whose rows are all these vectors. In [4] it was proved that the 3-color matrix of a planar graph has full column rank if and only if the graph is triangle-free. That result implies that the number of 3-colorings of a planar triangle-free graph with $n$ vertices is at least $n/6$. Then the question was raised if full column rank of the 3-color matrix implies exponentially many 3-colorings. We shall here answer this in the negative.

Let $m$ be a natural number. Consider the complete bipartite graph with $m > 3$ vertices in each class. Consider a perfect matching $M$ in this graph. We delete the edges of $M$ and call the resulting graph $Q$. For each edge $e$ in $M$, we add $m$ independent vertices to $Q$. Each of these $m$ vertices has degree 2 and is joined to the ends of $e$. The resulting graph $H$ has $n = m^2 + 2m$ vertices. There are $m$ distinct 3-colorings of $Q$ such that two ends of some edge of $M$ have the same color. Each such coloring can be extended to a 3-coloring of $H$ in $2^m$ different ways. There are $2^{m+1}$ distinct 3-colorings of $Q$ such that one partite class is monochromatic, and any such coloring can be extended to a unique 3-coloring of $H$. So, the total number of 3-colorings of $H$ is less than $\sqrt{n}2^{\sqrt{n}}$. It follows that, for any vertex $v$ in $H$, there exists a 3-coloring of $H$ such that all neighbors of $v$ have the same color. Thus the space generated by the rows of the 3-color matrix of $H$ contains the vector which is 1 at the entry corresponding to $v$ and zero everywhere else. Therefore, the 3-color matrix of $H$ has full column rank.

In view of this example, we replace the conjecture in [4] by the following.

**Conjecture 2.1.** There exists a positive number $\epsilon$ such that the following holds. Let $G$ be a graph on $n$ vertices whose 3-color matrix has full column rank.

(a) Then $G$ has at least $2^{n^\epsilon}$ distinct 3-colorings.
(b) If $G$ is planar, then $G$ has at least $(1 + \epsilon)^n$ distinct 3-colorings.

We shall here prove that if $G$ is planar and triangle-free, then it satisfies (a). If, in addition, it has girth at least 5, then it also satisfies (b).

3. Exponentially many 3-colorings in graphs of girth 5 on a fixed surface

In [14] it was pointed out that it is a consequence of Euler’s formula that the number of $k$-colorings of a $k$-colorable graphs with $n$ vertices on a surface $S$ of (fixed) Euler genus $g$ increases exponentially as a function of $n$ when $k$ is fixed and $k \geq 5$. A graph of girth at least 5 with $n$ vertices on a surface of Euler genus $g$ has at most $(5n - 10 + 5g)/3$ edges by Euler’s formula. This implies the following.
Theorem 3.1. Let \( G \) be a 3-colorable graph of girth at least 5 with \( n \) vertices embedded in a surface \( S \) of Euler genus \( g \). Then the number of 3-colorings of \( G \) is at least \( 2^{(n-5g)/9} \).

Proof. The proof is a repetition of the proof of Theorem 2.1 in [14]. The upper bound on the number of edges implies that some two color classes induce a graph with so many connected components that all color exchanges in these components give rise to an exponential number of colorings.

In [14] a general 5-color result was obtained: The minimum number of 5-colorings of a 5-colorable graph with \( n \) vertices on a fixed surface is a constant (depending only of the surface) multiplied by \( 2^n \). The proof depends on the general 5-color theorem in [11] saying that there are only finitely many obstructions for 5-coloring graphs on a fixed surface. For graphs of girth 5 there is a general 3-color theorem in [13] corresponding to the result in [11]. So it would be interesting to apply the result in [13] to determine the minimum number of 3-colorings of a 3-colorable graph of girth 5 with \( n \) vertices on a fixed surface. However, it is not clear if there is a nice answer to this problem.

Instead we pursue the 3-color problem in another direction. We derive a counterpart of Theorem 3.1, namely that there are exponentially many 3-list colorings of planar graphs of girth 5. (The analogous result that there are exponentially many 5-list colorings of planar graphs with no girth restriction has recently been established in [15].)

We also show that planar triangle-free graphs have many 3-colorings. Here we must confine ourselves to ordinary colorings, as there exist triangle-free planar graphs that are not 3-list-colorable, as shown by Voigt [16].

4. List-coloring planar graphs of girth 5

In this section we prove that a planar graph of girth at least 5 has not only exponentially many 3-colorings but also exponentially many list-colorings provided all lists have at least three colors.

The following result was proved in [12].

Theorem 4.1. Let \( G \) be a plane graph of girth at least 5. Let \( c \) be a coloring of a path or cycle \( P : v_1v_2\cdots v_q, \ 1 \leq q \leq 6 \), such that all vertices of \( P \) are on the outer face boundary. For each vertex \( v \) in \( G \), let \( L(v) \) be a list of colors. If \( v \) is in \( P \), then \( L(v) \) consists of \( c(v) \). Otherwise, \( L(v) \) has at least two colors. If \( v \) is not on the outer face boundary, then \( L(v) \) has three colors. Assume furthermore that there is no edge joining vertices whose lists have at most two colors except for the edges in \( P \). Then \( c \) can be extended to an \( L \)-coloring of \( G \), that is, a coloring such that neighbors have distinct colors, and every vertex \( v \) has a color in \( L(v) \).

We shall modify the proof in [12], and we use the same notation. The main new idea in [12] is the actual formulation of the theorem, as the length of the proof is very sensitive to alterations in the formulation, as a comparison of the proofs in [10,12] shows.

The proof in [12] begins with some standard reductions and ends with a case that required new ideas. Ironically, the most difficult part of the proof in [12] is easily translated to the present proof. It is the simple reductions in the proof in [12] which are problematic in the present proof. For example, if we encounter a vertex of degree at most 2, we would just delete it in [12] and then use induction. However, there may not be any color choice when we put it back, so in the
present proof vertices of degree 2 need special attention. Also, cutvertices are easy to get rid of in the proof of [12] but not in the present proof.

We shall assume that every list \( L(v) \) is a set of natural numbers. When we list color we assume that neighbors receive distinct colors. However, we shall also work with another labelling. Each vertex will be called either positive or negative.

We now define a coloring scheme as follows: It consists of an ordering of the vertices \( x_1, x_2, \ldots, x_n \). The precolored vertices of the graph appear first. They are all negative. Each other vertex is either positive or negative. The coloring scheme is called admissible if the vertices \( x_1, x_2, \ldots, x_n \) can be colored successively such that, whenever we color a positive vertex, we have at least two choices for the color. In particular, the total number of colorings is at least \( 2^p \) where \( p \) is the number of positive vertices. So our strategy is to find an admissible coloring scheme with many positive vertices.

**Theorem 4.2.** Let \( G \) be a plane graph with \( n \) vertices and of girth at least 5. Let \( c \) be a coloring of a path or cycle \( P : v_1v_2\cdots v_q \) with \( q \) vertices such that \( 1 \leq q \leq 6 \), and all vertices of \( P \) are on the outer face boundary. For each vertex \( v \) in \( G \), let \( L(v) \) be a list of colors. If \( v \) is in \( P \), then \( L(v) \) consists of \( c(v) \). Otherwise, \( L(v) \) has at least two colors. If \( v \) is not on the outer face boundary, then \( L(v) \) has three colors. Assume furthermore that there is no edge joining vertices whose lists have at most two colors except for the edges in \( P \). Finally, we assume that each vertex of \( G \) has degree at least 2 and that each vertex not on the outer face boundary has degree at least 3.

Then \( G \) has an admissible coloring scheme with \( p \) positive vertices and \( n' \) negative vertices such that one of (i)–(iv) below holds.

(i) \( p = 0 \), \( P \) is a cycle, and \( G = P \).

(ii) \( p = 0 \), \( P \) is a path, and \( G \) is a cycle with one more vertex than \( P \).

(iii) \( p = 0 \), \( P \) is a path with \( q \geq 5 \) vertices which is not contained in a cycle with \( q + 1 \) vertices, and \( n \leq 6q - 19 \).

(iv) \( p > 0 \) and \( n' \leq 1000(p - 1) + 200(q - 1) \).

**Proof.** We prove Theorem 4.2 by induction on the number of vertices. We assume that \( G \) is a smallest counterexample and shall reach a contradiction.

We shall sometimes break the graph up into two part \( G_1, G_2 \) where \( G_1, G_2 \) have a path in common, and \( G_1 \) contains most of \( P \). Then we use induction first to \( G_1 \) and then to \( G_2 \). If each of \( G_1, G_2 \) has at least one positive vertex, then the proof is complete because \( 1000(p - 1) + 200(q - 1) \) is less than or equal to \( 1000p \) regardless of \( q \). (This turns out to be convenient in the calculation.) On the other hand, if one of \( G_1, G_2 \) has no positive vertex, then that graph is small, and that makes it possible to complete the proof. Before we use induction we may delete as many as 6 vertices. This is why we have chosen the multiplicative constant 6 in the expression \( 6q - 19 \) in (iii).

Note that if \( q \leq 4 \), then there will be positive vertices unless \( G \) is a 5-cycle. Also note that, if \( P \) is a path with \( q \) vertices contained in a cycle with \( q + 1 \) vertices, and \( G \) contains more vertices than that cycle, then also there are positive vertices.

\( G \) is connected. (1)

For otherwise we apply successively the induction hypothesis to every connected component of \( G \) beginning with the one that contains \( P \). When we proceed to a new component we begin with a vertex \( u \) on the outer face boundary. If possible, we select \( u \) so that it has only two
available colors. Then there are at least two ways of coloring \( u \) so we call \( u \) positive. However, when we apply induction to that component we consider the vertex \( u \) as being precolored, so that component will contain other positive vertices.

\( G \) has no cutvertex on the outer face boundary. (2)

For suppose (reductio ad absurdum) that \( u \) is a cutvertex contained in an endblock \( B \) of the subgraph \( G' \) induced by the outer face boundary. We select \( B \) such that it contains as few vertices as possible in \( P \). Then \( B \) has at most three vertices of \( P \). Also, if \( u \) has only one neighbor not in \( B \), then \( G \) has a unique path \( Q : uu_1u_2 \cdots u_r \) such that \( u_r \) has degree at least 3 and each of \( u_1, u_2, \ldots, u_{r-1} \) has degree 2 in \( G \). If \( u \) has more than one neighbor not in \( B \), then we put \( Q = u \).

Consider first the case where \( P \) intersects \( G - B - (Q - u_r) \) (if \( Q \) has at least two vertices) or \( G - (B - u) \) (if \( Q = u \)). Then we first apply the induction hypothesis to the subgraph \( G - B - (Q - u_r) \) (if \( Q \) has at least two vertices) or \( G - (B - u) \) (if \( Q = u \)). If \( Q \) has more than one vertex and also has vertices not in \( P \), then we color successively \( u_{r-1}, u_{r-2}, \ldots, u \). Note that no two negative vertices on \( Q \) are consecutive (unless they are in \( P \)). Finally we apply the induction hypothesis to \( B \). Possibly, \( u \) is positive. But, when we apply induction to \( B \), then we think of \( u \) as precolored and negative. The only problem we may encounter is that \( u \) has neighbors with only two available colors. We have no freedom when we color these neighbors. These neighbors divide \( B \) into parts with only three precolored vertices, and we apply the induction hypothesis successively to these parts. These neighbors will all be negative in the color scheme and there may be many of them. But, between any two of them, there will be positive vertices.

Consider next the case where \( P \) does not intersect \( G - B - (Q - u_r) \) (if \( Q \) has at least two vertices) or \( G - (B - u) \) (if \( Q = u \)). The minimality property of \( B \) then implies that \( P \) is a subpath of \( Q - u - u_r \). We then extend the coloring of \( P \) to \( Q \). Again, no two negative vertices on \( Q \) are consecutive (unless they are in \( P \)). Then we use induction to \( B \) and \( G - B - (Q - u_r) \).

(If \( u_r \) is a cutvertex in \( G - B - (Q - u_r) \) incident to more than two bridges \( u_rx_i \), such that \( x_i \) has two available colors, then we consider another endblock \( B' \) instead of \( B \) so that we are back to the first case.)

This proves (2).

So, \( G \) has an outer cycle which we denote by \( C \). If some edge of \( P \) is a chord of \( C \), then that chord divides \( G \) into subgraphs \( G_1, G_2 \) having only the chord and its ends in common. We complete the proof by applying the induction hypothesis to each of \( G_1, G_2 \). If one of \( G_1, G_2 \) has positive vertices, then the proof is easy to complete because the other either has positive vertices as well or else it has less than 200 vertices and contains at least 4 vertices of \( P \). On the other hand, if none of \( G_1, G_2 \) has positive vertices, then they both contain at least 4 vertices of \( P \) which implies that they both contain precisely 4 vertices of \( P \), and hence each of \( G_1, G_2 \) is a 5-cycle by the induction hypothesis. This contradiction shows that \( P \) has no edge which is a chord of \( C \).

So the notation can be chosen such that \( C : v_1v_2 \cdots v_q \cdots v_kv_1 \).

A vertex \( v \) in \( G - P \) is joined to at most one vertex of \( P \) unless \( k = q + 1 \) and \( v = v_{q+1} \). (3)

For suppose (reductio ad absurdum) that \( v \) is inside \( C \) and is joined to more than one vertex of \( P \). (The proof is similar if \( v \) is on \( C \).) As \( G \) has girth at least 5, \( v \) is joined precisely to two vertices \( v_i, v_j \), where \( 1 \leq i < j \leq q \). Let \( G_1 \) denote the cycle \( v_i v_{i+1} \cdots v_j v_i \) and its interior. Let \( G_2 \) denote the cycle \( v_i v_{j-1} \cdots v_j v_i \) and its interior. We apply the induction hypothesis first to \( G_1 \). If \( v \) is joined to no vertices with precisely two available colors, then we apply the induction hypothesis to \( G_2 \). If \( v \) is joined to vertices with precisely two available colors, then we
color these after \( v \) has been colored. Then the colored vertices divide \( G_2 \) into parts and we apply the induction hypothesis to each of those parts.

If one (or more) of the graphs to which we apply induction has positive vertices, then the proof is easy to complete. (We leave the straightforward calculations for the reader.) So assume that when we apply induction, there are no positive vertices. Then \( G_1 \) satisfies (ii), and hence \( G_1 \) is a cycle with \( j - i + 2 \) vertices. As there are no positive vertices, \( v \) is joined to at most one vertex with precisely two available colors (or two vertices with precisely two available colors if \( v \) is on \( C \)). If \( v \) is joined to such a vertex, then we must have \( q = 6, i = 2 \) and \( j = 5 \) because the two parts that \( G_2 \) is divided into must have at least 4 precolored vertices (again because there are no positive vertices). By the induction hypothesis each of those two parts are 5-cycles. But then \( G \) has only 10 or 11 vertices, a contradiction. So \( v \) is not joined to a vertex with precisely two available colors. But then \( G_2 \) has at least 4 precolored vertices (in particular, \( (i, j) \neq (1, q) \)), and \( G_2 \) has at most \( 6(q - (j - i) + 2) - 19 \) vertices, and then \( G \) has at most \( 6(q - (j - i) + 2) - 19 + (j - i - 1) \) vertices and satisfies therefore (iii), a contradiction.

\[ P \text{ is a path, and } C - P \text{ has at least one vertex with precisely two available colors.} \]

\[ \text{In particular } q + 3 \leq k. \] (4)

For suppose (reductio ad absurdum) that there is no vertex with precisely two available colors. If \( G = C \), then all vertices of \( G - P \) except one are positive, in which case the proof is easy. So assume that at least one vertex of \( C \) has degree at least 3.

If all vertices of \( P \) have degree 2, then we delete the maximal path which contains \( P \) and whose vertices all have degree 2. Then we call some vertex on the outer face boundary of the remaining graph positive, we color that vertex, and then we use induction. If \( v_{q+1}, v_k \) both have degree at least 3 and are neighbors, we color both of them. If \( v_{q+1} = v_k \), and it has degree 1 in the resulting graph, then we color it and delete it and color its neighbor.

So assume some vertex of \( P \) has degree at least 3. If \( q \geq 2 \), we may assume that some vertex of \( P \) distinct from \( v_q \) has degree at least 3. Then we delete \( v_q \) and delete its color from the lists of all its neighbors.

Consider first the case where \( k \geq q + 2 \) or \( k = q \). Then by (3), we can apply the induction hypothesis to \( G - v_q \) unless there is a vertex of degree 1. If we create a vertex of degree 1, we delete it. If we create a new vertex of degree 1, we delete it, and we continue like this until all vertices have degree at least 2. Then we use induction. Note that there may be many vertices of degree 1 which were deleted. They are colored after we use induction. All of those (except \( v_{q+1} \)) are positive. \( v_{q+1} \) is negative but that is no problem as the value of \( q \) becomes smaller when we use induction.

Consider next the case where \( q = 1 \). Then we color the vertex \( v_2 \) and call it positive, and we repeat the argument in the previous paragraph. (Then \( q \) increases by 1 but we have already one positive vertex.)

Consider finally the case where \( k = q + 1 \). Then \( q \geq 4 \). Then we color \( v_k \) (and call it negative), we delete \( v_k, v_{k-1} \), and we delete their colors from the lists of all their neighbors, and then we use induction. There may be a vertex joined to both \( v_k \) and a vertex of \( P \). In fact, if \( k = 7 \), then there may be a vertex \( u \) joined to \( v_7 \) and \( v_3 \), and there may also be a vertex \( u' \) joined to \( v_7 \) and \( v_4 \). If one or both of \( u, u' \) exist, then we color \( u, u' \) before we use induction to the uncolored parts. The uncolored parts all have smaller values of \( q \). So, if there is at least one positive vertex we have reached a contradiction which proves (4). Assume therefore that there is no positive vertex. In particular, then not both of \( u, u' \) exist. Assume that \( u' \) does not exist. Also, we may assume
that no vertex inside $C$ is joined to $v_q, u$, since that also would imply the existence of positive vertices.

Now we color $v_k$ and delete $v_1, v_k, v_{k-1}$, and we use induction, if possible. As the number of precolored vertices is now at most 4, there will be positive vertices. The problem that may now occur is a path $v_1 u_1 u_2 v_q$. This path divides $G$ into graphs $G_1, G_2$ where $G_1$ contains $P$. Then we apply induction first to $G_1$ and then to $G_2$. Before we apply induction to $G_1$ we color $u_1$ (which we call it positive) and $u_2$ (which we call it negative), we delete $u_1, u_2$, and then we also color those (possible three) vertices whose color is forced before we apply induction to what remains of $G_1$.

This contradiction proves (4).

If $v_j v_i$ ($i < j$) is a chord of $C$, then $v_i, v_j$ both have three available colors. Moreover, $j = i + 4$, and each of $v_{i+1}, v_{j-1}$ has precisely two available colors. (5)

To prove (5) we assume (reductio ad absurdum) that it is false. The edge $v_j v_i$ divides $G$ into graphs $G_1, G_2$.

Suppose first that $v_i, v_j$ both have at least two available colors. Then we apply induction first to the part (say $G_1$) which contains $P$ and then to $G_2$. Before we apply induction to $G_2$ we color those vertices on $C$ whose color is forced by the coloring of $v_i, v_j$. If $G_2$ has positive vertices, we are done. On the other hand, if $G_2$ has no positive vertices then it is a 5-cycle by the induction hypothesis, and it has four precolored vertices. This is the conclusion of (5). So we may assume that $1 \leq i \leq 3$ and that $G_1$ contains $v_q$. If $i = 3$, we apply induction first to $G_2$ (which has three precolored vertices) and then to $G_1$ which has at most $q$ precolored vertices. If $i < 3$, we apply induction first to $G_1$ (which has $q - i + 1$ precolored vertices) and then to $G_2$ which has at most 4 precolored vertices.

This contradiction proves (5).

In what follows we may assume that $q \geq 2$. For if $q = 1$, then we call $v_2$ positive and we color $v_2$. If $v_3$ has only two available colors, we color that, too, and we call it negative. In the resulting graph $q$ equals 2 or 3 and it suffices to prove the theorem for that graph. ($q$ has increased by 1 or 2, but we have already obtained a positive vertex.)

$G$ has no path of the form $v_j u v_i$ where $u$ is in $int(C)$ and $v_i, v_j$ both have precisely two available colors. (6)

To prove (6) we define $G_1, G_2$ as in the proof of (3). We choose $v_i, v_j$ such that $G_2$ is minimal. Then we apply induction first to $G_1$ and then to $G_2$ which has only three precolored vertices.

$G$ has no path of the form $v_j u v_i$ where $u$ is in $int(C)$, $v_j$ is in $P$, and $v_i$ has precisely two available colors. (7)

To prove (7) we define $G_1, G_2$ as in the proof of (3). Assume that $j \leq (q + 1)/2$ and that $G_1$ contains $v_q$. We then apply induction first to the graph $G_1$ and then to $G_2$. If one (or both) of $G_1, G_2$ has at least one positive vertex, then the proof is easy to complete. So assume that both of $G_1, G_2$ has no positive vertex. Then $j > 1$. Moreover, $G_1$ has at most $6(q - j + 1) - 19$ vertices, and $G_2$ has at most $6(j + 2) - 19 - 3 < 6q - 19$ vertices. Hence $G$ has at most $6(q - j + 1) - 19 + 6(j + 2) - 19 - 3 < 6q - 19$ vertices.

$L(v_{q+2})$ has precisely two colors. This is the only neighbor of $v_{q+1}$ which has precisely two available colors. (8)

For, if $L(v_{q+2})$ has three colors, then we complete the proof by deleting $v_q$ and deleting the color of $v_q$ from the list of each neighbor of $v_q$, and we apply the induction hypothesis to $G - v_q$.
and obtain thereby a contradiction. (If deleting \( v_q \) results in a vertex of degree 1, we repeat the argument disposing of that situation in the proof of (4).) So we assume that \( L(v_{q+2}) \) has at most two colors. By (4), \( v_{q+2} \) is not part of \( P \), so \( L(v_{q+2}) \) has precisely two colors. This proves the first statement in (8). The second statement follows from (5).

For later purposes we observe that the proof of (8) is carried out with \( q - 1 \) instead of \( q \). This gives a better upper bound on the number of negative vertices. This will be used later when we repeat the proof of (8).

\[ G \text{ has no path of the form } v_{q+1}uv_i \text{ where } u \neq v_q \text{ and } v_i \text{ has at most two available colors unless } q = 6 \text{ and } i = 4. \]

(9)

Suppose (reductio ad absurdum) that (9) is false. We define \( G_1, G_2 \) as in the proof of (3) such that \( G_1 \) contains \( v_q \). We choose \( v_i \) such that \( G_2 \) is minimal.

Consider first the case where \( v_i \) is not in \( P \). Then we apply induction first to \( G_1 - v_q \) (in order to get a smaller \( q \)) and then to \( G_2 \). If \( u \) is on \( C \), then \( u = v_{q+5} \), by (5), and the proof is easily completed. So assume that \( u \) is in \( \text{int}(C) \).

Before we apply induction to \( G_2 \) we color \( v_{q+2} \) and call it negative. An easy count now completes the proof.

Consider next the case where \( v_i \) is in \( P \), and \( 1 \leq i \leq 3 \). If \( 1 = i \), we repeat the previous proof so assume \( 2 \leq i \leq 3 \). We color \( v_{q+1} \) and call it positive. Then we apply induction first to \( G_1 \) and then to \( G_2 \). As we have at least one positive vertex, we easily get a contradiction which proves (9).

\[ L(v_{q+4}) \text{ has less than three colors.} \]

(10)

By (8), \( L(v_{q+3}) \) has three colors. If also \( L(v_{q+4}) \) has three colors, then we first color \( v_{q+1} \) (which we call positive) and \( v_{q+2} \) (which we call negative), then we delete them and also delete the color of \( v_i \) from the list of each neighbor of \( v_i \) for \( i = q + 2, q + 1 \). If possible, we obtain a contradiction by applying the induction hypothesis to the resulting graph. If this is not possible, then \( \text{int}(C) \) has a vertex \( u \) such that \( u \) is joined to one of \( v_{q+1}, v_{q+2} \) and to another vertex \( v_j \) with less than three available colors. By (6), (7), \( u \) is not joined to \( v_{q+2} \) (which has precisely two available colors). So \( u \) is joined to \( v_{q+1} \). By (9), \( q = 6 \), and \( u \) is joined to \( v_4 \) and to \( v_{q+1} = v_7 \). Then we also color \( u \) (and call it negative) and we apply induction to the graph inside \( v_1v_2v_3v_4uv_7v_8 \cdots v_1 \) after having deleted \( v_{q+1}, v_{q+2} \). This proves (10).

There is no path \( v_{q+3}uv_i \) where \( v_i \) has precisely two available colors. \( \) (11)

Again the proof is carried out by defining \( G_1, G_2 \) (where \( G_1 \) contains \( v_q \)) and using induction to these graphs in that order. Possibly, \( G_2 \) has precisely 5 vertices and no positive vertex. In that case we repeat the proof of (10) to \( G_1 \).

There is no path \( v_{q+3}uu'v_j \) where \( u, u' \) are inside \( C \) and \( v_j \) has precisely two available colors and \( i \in \{1, 2, 3\} \). \( \) (12)

The proof is the similar to that of (11). More precisely, if \( i = 1 \), then we color \( v_{q+1} \) (and call it positive), and we delete it before we apply induction to \( G_1 \).

If \( i = 2 \) or \( i = 3 \), then we repeat the proof of (10). That is, we color \( v_{q+1}, v_{q+2} \) (and call \( v_{q+1} \) positive) before we use induction to \( G_1 \).

This completes the proof of (12).

Finally, we may assume that

\[ q > 4, \text{ and some vertex of } P \text{ distinct from } v_q \text{ has degree at least 3.} \]

(13)
For, if \( q \leq 4 \), then we color \( v_{q+1}, v_{q+2} \) and call \( v_{q+1} \) positive, and we prove the theorem for this extended coloring. So we may assume that \( q > 4 \). If all vertices of \( P \) have degree 2, except possibly \( v_q \), then we delete \( v_3 \) and add an edge between \( v_2 \) and \( v_4 \) if their colors are distinct. Otherwise we identify them, and then we use induction.

We are now ready for the final contradiction following the proof in [12].

We give \( v_{q+3} \) a color not in \( L(v_{q+4}) \), then we color \( v_{q+2}, v_{q+1} \), and finally we delete \( v_i \) and also delete the color of \( v_i \) from the list of each neighbor of \( v_i \) for \( i = q + 3, q + 2, q + 1, q \). (Note that we also delete \( v_q \).) The vertices \( v_i, i = q + 3, q + 2, q + 1, q \), are negative and appear immediately after the precolored vertices in the coloring scheme.

If possible, we apply the induction hypothesis to the resulting graph and obtain a contradiction. (As we also delete \( v_q \) it follows that \( q \) is replaced by \( q - 1 \) in the resulting graph.) If we create vertices of degree 1, we delete them successively and color them at the end of the proof. No two consecutive vertices among those deleted vertices are negative unless they are precolored. (So, if we create a vertex of degree 1, then in the worst case there is only one such vertex, namely \( v_{q+4} \), and we delete altogether 5 vertices before we use induction.)

We shall now discuss the cases that may complicate the induction.

If \( G \) has a vertex \( u \) in \( \text{int}(C) \) joined to \( v_q \) and \( v_{q+3} \), then the path \( v_q u v_{q+3} \) divides \( G \) into graphs \( G_1, G_2 \) where \( G_1 \) contains \( P \). Then we apply induction first to \( G_1 - u \) and then to \( G_2 \).

So assume that \( G \) has no vertex \( u \) in \( \text{int}(C) \) joined to \( v_q \) and \( v_{q+3} \).

There may be a path \( v_i u v_j \) where \( v_j \) is one of \( v_q, v_{q+1}, v_{q+2}, v_{q+3} \) and \( v_i \) has precisely two available colors. By (6), \( (7), v_j \) is one of \( v_{q+1}, v_{q+3} \). But, \( v_j \neq v_{q+1} \), by (9), and \( v_j \neq v_{q+3} \), by (11). So we may assume that there is no path \( v_i u v_j \) where \( v_j \) is one of \( v_q, v_{q+1}, v_{q+2}, v_{q+3} \) and \( v_i \) has precisely two available colors.

By (7), \( G \) has no vertex \( u \) in \( \text{int}(C) \) joined to \( v_{q+2} \) and a precolored vertex.

If \( G \) has a vertex \( u \) in \( \text{int}(C) \) joined to \( v_{q+3} \) and a precolored vertex \( v_j \), then the path \( v_{q+3} u v_j \) divides \( G \) into two graphs. We apply the induction hypothesis to one of these, say \( G_1 \) and then to the other, say \( G_2 \). The order depends on \( j \). We make sure that there are never more than 6 precolored vertices. That is, \( G_1 \) contains the largest part of \( P \). If \( j \leq 3 \), then we repeat the argument of (10) to \( G_1 \) in order to insure that \( G_1 \) has positive vertices.

And if \( j > 3 \), then we color \( u \) (and call it positive) before we apply induction to \( G_1 \), again in order to insure that \( G_1 \) has positive vertices. So we may assume that \( G \) has no vertex \( u \) in \( \text{int}(C) \) joined to \( v_{q+3} \) and a precolored vertex.

If \( G \) has a vertex \( u_0 \) in \( \text{int}(C) \) joined to \( v_{q+1} \) and a precolored vertex, then that vertex must be \( v_4 \), by (9). Then we give \( u_0 \) a color before we use induction to \( G - v_q - v_{q+1} - v_{q+2} - v_{q+3} \).

So we are left only with the problem that there may be a path \( v_{q+1} w z v_{q+3} \) or \( v_q w z v_{q+3} \) or \( v_q w z v_{q+2} \). We select one such that the cycle which consists of this path and a path of \( G \) containing the edge \( v_{q+1} v_{q+2} \) has as many vertices as possible in its interior. We may assume that this path is of the form \( v_{q+1} w z v_{q+3} \). (For otherwise, it divides \( G \) into graphs \( G_1, G_2 \) where \( G_1 \) contains \( P \). Then we apply induction first to \( G_1 - v_q \) and then to \( G_2 \).)

Consider first the case where \( w \) is joined to \( v_q \), that is, \( u_0 \) exists and is equal to \( w \). Then we also color \( w, z \), and we delete \( z \) (in order to get a smaller \( q \)) and complete the proof by induction. By (12), there is no path \( w z v_j \) where \( v_j \) has precisely two available colors. For any path \( z w v_j \) where \( v_j \) is precolored, we color \( u \) before we use induction. So, we may assume that, if \( u_0 \) exists, then it is distinct from \( w \) (and of course also distinct from \( z \)).

Then we also color \( w, z \) and delete them before we apply the induction hypothesis. By the reasoning above, none of \( w, z \) is joined to \( P \). So when we color, we have two possible choices, and therefore one of \( w, z \) is positive.
The coloring and deletion of \( w, z \) may create additional complications which we now discuss. These complications occur outside the cycle \( v_{q+1}v_{q+2}v_{q+3}zwv_{q+1} \), because the interior of this cycle can be colored by induction, and if it is nonempty, then it will contain positive vertices.

First, there may be a path \( zxyv_{q} \) or \( zxyv_{q+1} \) or \( wxyv_{q+3} \) or \( wxyv_{q} \). (In that case \( x, y \) will have only two available colors after \( z, w \) have been colored and deleted.) In that case the above path divides \( G - v_{q} - v_{q+1} - v_{q+2} - v_{q+3} \) into graphs \( G_{1} \) (containing \( v_{1} \)) and \( G_{2} \). We apply induction first to \( G_{1} \) and then to \( G_{2} \). Before we apply induction to \( G_{2} \), we color \( z, w \). This works unless both of \( z, w \) are in \( G_{1} \), that is, the path \( wxyv_{q} \) is present. In that case we color \( v_{q+3}, v_{q+2}, v_{q+1}, z, w \) (and call \( z \) positive), we delete these vertices, and then we use induction and obtain a contradiction. Assume therefore that there is no path of length 3 as described above.

There may also be a path \( uu_{i}v_{i} \) or \( zuv_{i} \) where \( v_{i} \) has precisely two available colors. Then that path together with a vertex on \( C \) divides \( G \) into graphs \( G_{1} \) (containing \( u_{1} \)) and \( G_{2} \). We apply induction first to \( G_{1} \) and then to \( G_{2} \). The argument of (10) shows that \( G_{1} \) has at least one positive vertex, and therefore the proof is easy to complete. Assume therefore that there is no path of length 2 as described above.

There may be vertices in the interior of \( C \) which are joined to both one of the deleted vertices \( w, z \) and also to a vertex on \( P \). There may be as many as 6 such vertices. For example, there may be paths \( zu_{i}v_{i} \) where \( i \in \{1, 2, 3, 4, 5, 6\} \), and there are other possibilities. If one of \( u_{1}, u_{2} \), say \( u_{2} \), exists, then the path \( v_{2}u_{2}zv_{q+3} \) divides \( G \) into graphs \( G_{1}, G_{2} \) where \( G_{1} \) contains \( v_{q} \). We apply induction first to \( G_{1} \) and then to \( G_{2} \). The argument of (10) shows that \( G_{1} \) has at least one positive vertex, and therefore the proof is easy to complete.

Another extreme case occurs when \( q = 6 \) and the above \( u_{6} \) exists. Then the path \( v_{6}u_{6}zv_{q+3} \) divides \( G \) into graphs \( G_{1}, G_{2} \) where \( G_{1} \) contains \( v_{1} \). We apply induction first to \( G_{1} - v_{q} \) (which has a smaller value of \( q \)) and then to \( G_{2} \). If some (or all) of \( u_{3}, u_{4}, u_{5} \) are in \( G_{1} \), we can just color all these vertices (and also color \( u_{0} \) if it exists) after having deleted \( v_{q+3}, v_{q+2}, v_{q+1}, v_{q}, z, w \). Then we can apply induction separately to each part of \( G \) on which it is divided by \( P \) and the paths containing \( u_{3}, u_{4}, u_{5}, u_{0} \) (as each such part contains at most 5 precolored vertices).

This contradiction completes the proof. \( \square \)

**Theorem 4.3.** Let \( G \) be a plane graph with \( n \) vertices and of girth at least 5. If \( G \) has an outer cycle of length 5, then we denote it by \( P \). Otherwise, we let \( P \) denote a path with one or two vertices such that each vertex of \( P \) is on the outer face boundary. Let \( c \) be a coloring of \( P \). For each vertex \( v \) in \( G \), let \( L(v) \) be a list of colors. If \( v \) is in \( P \), then \( L(v) \) consists of \( c(v) \). Otherwise, \( L(v) \) has three colors. Assume furthermore that \( G \) has at least one vertex not in \( P \).

Then \( G \) has at least \( 2^{n/10000} \) distinct \( L \)-colorings extending \( c \).

**Proof.** The proof is by induction on \( n \). It is easy to dispose of the cases where \( G \) is disconnected or contains a cutvertex. So assume that \( G \) is 2-connected with outer cycle \( C \). We may also assume that \( P \) is not a chord of \( C \). If all vertices not in \( C \) have degree at least 3, then the result follows from Theorem 4.2. So assume that \( \text{int}(C) \) has a vertex \( u_{1} \) of degree 2. We delete that vertex. If \( G - u_{1} \) has a vertex \( u_{2} \) in \( \text{int}(C) \) such that \( u_{2} \) either has degree 1 or has degree 2 and is not a cutvertex, then we delete that. We proceed in this way defining a sequence \( u_{1}, u_{2}, \ldots, u_{r} \). Let \( p \) denote the number of vertices of degree 1 in this sequence. Then \( G' = G - u_{1} - u_{2} - \cdots - u_{r} \) is a connected graph which contains \( C \). Moreover, each vertex of \( G' \) which is inside \( C \) and which has degree 2 is a cutvertex. Let \( t \) be the number of vertices in \( G' \) which are inside \( C \) and have degree 2 (in \( G' \)) and both neighbors have degree at least 3 (in \( G' \)). Let \( n' \) be the number of vertices of \( G' \). It is easy to prove that \( t < n'/2 \). By Theorem 4.2, \( G' \) has at least \( 2^{(n' - 1)/1001} \).
5. 3-Coloring planar graphs of girth 4

We now turn to ordinary 3-colorings, that is, we use only the colors 1, 2, 3.

Let $G$ be a plane triangle-free graph with outer cycle $C$ of length 4 or 5 or 6. Let $c$ be a 3-coloring of $C$. If $C$ has length 6, we assume that the coloring of $C$ is not of the form 1, 2, 3, 1, 2, 3. Then there is at least one 3-colorings of $G$ extending $c$, by Theorem 5.3 in [2]. (If $C$ has length 4 or 5, then we can insert vertices of degree 2 on $C$ so that it becomes a cycle of length 6 as required in Theorem 5.3 in [2].)

There need not be more than one such 3-coloring. Indeed, if $C$ is not a 4-cycle with only two distinct colors, then it is easy to successively add vertices joined to two vertices with distinct colors such that the resulting graph is planar and triangle-free, and of course it has only one 3-coloring extending $c$. However, for each other plane triangle-free graph with precolored outer cycle of length at most 6, there are at least two extensions of the precoloring as the next result shows.

**Theorem 5.1.** Let $G$ be a connected plane triangle-free graph with outer cycle $C : v_1v_2 \cdots v_kv_1$ of length 4 or 5 or 6 such that there is at least one vertex inside $C$. Let $c$ be a 3-coloring of $C$. If $C$ has length 6, then we assume that the coloring of $C$ is not of the form 1, 2, 3, 1, 2, 3. Then there are at least two 3-colorings of $G$ extending $c$ unless there is a vertex inside $C$ joined to two vertices of $C$ with distinct colors.

**Proof.** The proof is by induction on the number of vertices of $G$.

If $G$ is not 2-connected, then we consider an endblock $B$ not containing $C$ and with a cutvertex $u$. We extend $c$ to $G$ and obtain a new coloring by permuting the two colors in $B$ distinct from the color of $u$. So assume that $G$ is 2-connected.

If $G$ has a vertex $v$ inside $C$ joined to two vertices of $C$, then we may assume that these two neighbors on $C$ have the same color. Now there are two ways of coloring $v$, and each of these colorings can be extended to $G$. So we may assume that $G$ has no vertex inside $C$ joined to two vertices of $C$. 

$L$-colorings extending $c$ unless $G' = P$ and $P$ is a 5-cycle. (Before we apply Theorem 4.2, we suppress the vertices in $int(C)$ of degree 2. When we put them back we have color choices except for those $t$ of them which have two neighbors of degree at least 3. There may be paths with at least two vertices such that all vertices have degree 2. For each such path we have two choices for each vertex except one.) We also put back the vertices $u_r, u_{r-1}, \ldots, u_1$. For $p$ of these we have a color choice so the total number of extensions of $c$ is at least $2^{(n'-1)/1001+p}$ which is greater than $2^{n'/2002+p}$. (For notational convenience we assume here that $G'$ has at least one uncolored vertex. If $G' = P$ and $P$ is a 5-cycle, the proof is similar and slightly simpler.) So there only remains to be proved that $n'/2002 + p > n/10000$. Clearly, $n = n' + r$. Let $e, f$ be the number of edges and faces of $G$, respectively. By Euler's formula, $n - e + f = 2$. Let $e_1, e_2, \ldots, e_f$ be the numbers of edges in the facial walks of $G$. Define $\alpha(G)$ as the sum of $e_i - 5$ taken over all the facial walks. As $G$ has girth at least 5, $0 \leq \alpha(G) \leq 2e - 5f < 2e < 10n/3$. Similarly, $0 \leq \alpha(G') < 10n'/3$.

If we add a vertex of degree 1 to a graph then $\alpha$ increases by 2. This happens $p$ times when we extend $G'$ to $G$. On the other hand, if we add a vertex of degree 2 to a graph such that the resulting graph has girth at least 5, then $\alpha$ decreases by 1. This happens $r - p$ times when we extend $G'$ to $G$. Hence $r - p < 2p + 10n'/3$, and so $n = n' + r < 3p + 13n'/3$ which implies that $n'/2002 + p > n/10000$ as required. \( \square \)
If $G$ contains no 4-cycle distinct from $C$, then we consider an edge $e$ of $C$ such that the largest block $B$ of $G - e$ is as large as possible. Then $G$ consists of $B$ and a path $P : v_1v_2 \cdots v_m$ containing $e$. By Theorem 4.1, the coloring of $C$ can be extended to a 3-coloring of $B$. But that coloring cannot be unique. For if the outer face boundary of $B$ contains a vertex $u$ which is not joined to a vertex of $C$, then we let $L(u)$ consist of the colors 1, 2, 3 minus the color of $u$. Now we use Theorem 4.1 again to extend $c$ such that $u$ receive another color. So suppose that such a vertex $u$ does not exist. Let $v_1w_1 \cdots w_rv_m \cdots v_1$ be the outer cycle of $B$.

If that cycle has at most 5 vertices in $C$, then we change the color of $w_1$. By Theorem 4.1 applied to $B$, the colored vertices of $C$, and the new coloring of $w_1$, we obtain a second coloring of $B$. On the other hand, if the outer cycle of $B$ has more than 5 vertices in $C$, then $C$ has length $k = 6$, and $m = 2$. The assumption that each of $w_1, w_2, \ldots, w_r$ is joined to a vertex of $C$ implies that $r = 3$, and $w_2$ is joined to one of $v_4, v_5$, say $v_4$. Now we change the color of $w_2$. By Theorem 4.1 applied to two subgraphs of $B$, the coloring of $C$ and the new coloring of $w_2$ can be extended to $B$. This gives a second coloring of $G$. So we may assume that $G$ contains a 4-cycle $C_1$ distinct from $C$.

If $C_1$ is separating, we delete its interior and use induction. By the same argument we may assume that $G$ has no separating 5-cycle. So assume that $C_1$ is facial. Then $C_1 \cap C$ is either empty or consists of one vertex or two vertices joined by an edge (since otherwise there would be a separating cycle of length 4 or 5). There are two ways of identifying two opposite vertices of $C_1$. We may assume that either way gives rise to a path of length at most 2 joining two vertices of $C$. (Otherwise the proof is completed by induction.) This implies in particular that $C_1$ cannot have precisely one vertex $u$ in common with $C$ because each of the two neighbors in $C_1$ of $u$ has only one neighbor (namely $u$) on $C$.

If $C_1 \cap C$ is empty, then we let $H$ denote the subgraph of $G$ induced by $C_1 \cup C$. Each facial cycle of $H$ has length at most 6 and there is at most one (distinct from $C$) of length precisely 6. It is easy to extend $c$ to a coloring of $H$ in two ways. If one of these induces the coloring 1, 2, 3, 1, 2, 3 on a facial cycle of length 6, then it is easy to find two extensions of $c$ where this does not happen. So there are at least two extensions of $c$ to $G$ in this case. So we may assume that every 4-cycle of $G$ is facial and has an edge in common with $C$.

Let $v_1v_2xuv_1$ be a 4-cycle. Consider first the case where there exists another 4-cycle having an edge in common with this 4-cycle, say $v_2xuv_3v_2$. We may identify $x$ with $v_1$. Instead of that we may also identify $x$ with $v_3$. In either case the coloring can be extended to $G$. So we may assume that $v_1, v_3$ have the same color, say 1. We delete $v_2$ and identify $v_1, x$. Now we may choose the color of $u$ in two ways, and either way can be extended by Theorem 5.3 in [2]. So we may assume that there exists no other 4-cycle having an edge in common with the 4-cycle $v_1v_2xuv_1$.

Again, we may identify $x$ with $v_1$. Instead of that we may also identify $y$ with $v_2$. We can complete the proof by induction unless there exists a vertex $u$ joined to $x$ and a vertex $v_i$ where $i > 3$, and there also exists a vertex $v$ joined to $y$ and a vertex $v_j$, where $j < k$. Now we can extend the coloring of $C$ to $x, y, u, v$ in at least three distinct ways. Each of these, except possibly one, can be extended to $G$ by the induction hypothesis. This completes the proof.

If the outer cycle has length 4 or 5 we can be more precise.

The cycle of length 5 has precisely one 3-coloring (up to rotation and permutation of colors). One vertex has a color distinct from the color of the other vertices. We call that the special vertex. The edge of distance 2 to the special vertex is called the special edge.
Theorem 5.2. Let $G$ be a connected plane triangle-free graph with outer cycle $C : v_1 v_2 \cdots v_q v_1$ of length 4 or 5.

Let $c$ be a 3-coloring of $C$.

If $q = 4$, then there are at least two 3-colorings of $G$ extending $c$ unless $G$ consists of a collection of paths of length 2 joining two opposite vertices of $C$ with distinct colors.

If $q = 5$, then there are at least two 3-colorings of $G$ extending $c$ unless either $G = C$ or else all facial cycles of $G$, except $C$ and precisely one more, say $C'$, have length 4. The cycle $C'$ is a face boundary of length 5 containing the special edge of $C$.

Proof. The proof is by induction on the number of vertices. If there is a vertex in the interior of $C$, then we may assume that there is one, say $u$ which is joined to two vertices of $C$ with distinct colors, by Theorem 5.1. We color $u$. Then the colored vertices divide $G$ into two parts. We apply the induction hypothesis to each part. This completes the proof when $q = 4$. If $q = 5$, then the part containing the special edge is bounded by a 5-cycle, and the special edge of that 5-cycle is also the special edge of $G$. This completes the proof.

We now proceed to many 3-colorings. When we prove Grötzsch’s theorem, the 5-cycles give us problems. Indeed, it is easy to reduce the proof of Grötzsch’s theorem to the case where all facial cycles have length 5. Ironically, the 5-cycles are very helpful when we seek many 3-colorings as we show below. First we prove a result on shortest paths and use that to prove a Ramsey type result for the 5-cycles.

Lemma 5.3. Let $q$ be a natural number $> 1$ and let $s, t$ be vertices at distance $m$ in a graph $G$.

(a) If there are at least $q^m$ edges incident with $s$ which are contained in shortest paths from $s$ to $t$, then $G$ contains two vertices $u, v$ which are connected by $q$ internally disjoint paths of the same length ($\leq m$).

(b) If there are at least $q^{m(m+1)/2}$ distinct shortest paths from $s$ to $t$, then $G$ contains two vertices $u, v$ which are connected by $q$ internally disjoint paths of the same length ($\leq m$).

Proof. The proof is by induction on $m$. We first prove (a). We may assume that $G$ consists of the shortest paths from $s$ to $t$. Let $D_1, D_2, \ldots, D_m$ be the distance classes from $s$. For every vertex in $D_1$ we select an edge to $D_2$. If $q$ of these edges go to the same vertex, there are $q$ internally disjoint paths of length 2 from $s$ to that vertex. So assume that there is no such vertex. Then there are at least $q^{m-1}$ paths of length 2 from $s$ to distinct vertices in $D_2$ such that any two of these paths have only $s$ in common. We think of these paths as edges and use induction. This proves (a).

We now prove (b). We may assume that $G$ does not satisfy the hypothesis of (a). But then some edge $ss_1$ from $s$ to $D_1$ is contained in at least $q^{m(m-1)/2}$ shortest paths from $s$ to $t$. Now we apply the induction hypothesis to the pair $s_1, t$.

Lemma 5.4. Let $G$ be a plane triangle-free graph having $10q^{12}$ distinct cycles of length 5.

Then either

(a) $G$ contains a collection of at least $q$ paths of length 2 joining the same two vertices $u, v$ or

(b) $G$ contains a collection of at least $q$ internally disjoint paths of length 3 joining the same two vertices $u, v$. 

Lemma 5.3. \(G\) contains a collection of at least \(q\) internally disjoint paths of length 4 joining the same two adjacent vertices \(u, v\) or
\(G\) contains a collection of at least \(q\) pairwise edge-disjoint 5-cycles such that if \(C_1, C_2\) are two of these 5-cycles, then all edges of one of them are in the exterior of the other.

**Proof.** We define a graph \(F\) as follows: The vertices of \(F\) are the 5-cycles in \(G\). Two vertices are nonadjacent if and only if they are edge-disjoint and one of them is contained in the exterior of the other. If \(F\) has \(q\) pairwise nonadjacent vertices then (d) holds. So we may assume that this is not the case. Then Brooks’ theorem implies that \(F\) has a vertex, say \(C\), of degree at least \(10q^{11}\). Then \(C\) contains two vertices \(s, t\) such that at least \(q^{11}\) other 5-cycles contain \(s, t\).

If \(s, t\) are nonadjacent, then either \(G\) contains \(q\) paths of length 2 from \(s\) to \(t\) or else there are less than \(q\) such paths but then there are at least \(q^{10}\) paths of length 3 from \(s\) to \(t\) which are shortest paths in the subgraph which is the union of these paths. Now the proof is completed by Lemma 5.3.

If \(s, t\) are adjacent, then \(G\) contains \(q^{10}\) paths of length 4 from \(s\) to \(t\) which are shortest paths in the subgraph which is the union of these paths. Again, the proof is completed by Lemma 5.3. \(\square\)

**Lemma 5.5.** Let \(G\) be a plane triangle-free graph. If \(G\) satisfies one of (a), (b) in Lemma 5.4, then \(G\) has at least \(2^q\) distinct 3-colorings.

If \(G\) satisfies (c) in Lemma 5.4, then \(G\) has at least \(2^{q/2}\) distinct 3-colorings.

**Proof.** If \(G\) satisfies (a) in Lemma 5.4, then we give \(u, v\) the same color. There are \(2^q\) ways of coloring their common neighbors, and each such coloring can be extended to \(G\) by Grötzsch’s theorem.

If \(G\) satisfies (b) in Lemma 5.4, then we give \(u, v\) distinct colors. There are more than \(2^q\) ways of coloring the paths of length 3 between them, and each such coloring can be extended to \(G\) by Theorem 5.3 in [2].

Suppose now that \(G\) satisfies (c) in Lemma 5.4. Let \(C_1, C_2, \ldots\) be a collection of \(q\) 5-cycles having the edge \(uv\) and nothing else in common such that \(\text{Int}(C_1) \supseteq \text{Int}(C_2) \supseteq \cdots\). By Grötzsch’s theorem, any 3-coloring of \(C_{i-1}\) can be extended to \(C_i\) and the graph between \(C_{i-1}, C_i\). If at least \(q/2\) of the cycles in \(C_1, C_2, \ldots\) have the property that any 3-coloring of \(C_{i-1}\) can be extended to \(C_i\) and the graph between \(C_{i-1}, C_i\) in at least two different ways, then clearly \(G\) has at least \(2^{q/2}\) distinct 3-colorings.

So assume that there are more than \(q/2\) cycles in the sequence \(C_1, C_2, \ldots\) which have the property that some 3-coloring of \(C_{i-1}\) cannot be extended to \(C_i\) and the graph between \(C_{i-1}, C_i\) in more than one way. Theorem 5.2 tells us when a 3-coloring of a 5-cycle cannot be extended to its interior in more than one way. The special edge must be \(uv\). Hence the vertex of \(C_{i-1}\) opposite to \(uv\) (that is, the vertex of \(C_{i-1}\) which is not a neighbor of any of \(u, v\)) is the special vertex, that is the only vertex of \(C_{i-1}\) whose color is distinct from the colors of \(u, v\).

Now let us change the coloring of \(C_1, C_2, \ldots\) such that, in each \(C_j\), the vertex opposite to \(uv\) has the same color as either \(u\) or \(v\).

By Theorem 5.2, the 3-coloring of \(C_{i-1}\) can then be extended to \(C_i\) and the graph between \(C_{i-1}, C_i\) in more than one way when that graph has the structure in Theorem 5.2. (Knowing the structure of the graph between \(C_{i-1}\) and \(C_i\) it is easy to see that the two distinct colorings can be chosen such that the vertex on \(C_{i-1}\) opposite to the edge \(uv\) has the same color as one of \(u, v\). This is the case for more than \(q/2\) values of \(i\).)
So it only remains to be shown that the color extension is possible when the graph between \( C_{i-1}, C_i \) does not have the structure described by Theorem 5.2. In this case we are satisfied with just one color extension. We want a vertex \( w \) in \( C_i \) to get the same color as one of \( u, v \). We can achieve that by Theorem 5.3 in [2] by first adding a vertex \( z \) of degree 2 on the edge \( uv \). Then we give \( z \) the color distinct from those of \( u, v \), and then we add the edge \( zw \).

This completes the proof. \( \square \)

**Theorem 5.6.** Let \( G \) be a connected plane triangle-free graph with outer cycle \( C \) of length 4 or 5 and with \( q \) face boundaries of length at least 5 distinct from \( C \). Let \( c \) be a 3-coloring of \( C \).

If \( C \) has length 4, then there are at least \( 2^{q/10000} \) distinct 3-colorings of \( G \) extending \( c \).

If \( C \) has length 5, then there are at least \( 2^{q(1-1)/10000} \) distinct 3-colorings of \( G \) extending \( c \).

If \( C \) has length 5, and at least one of the \( q \) face boundaries is edge-disjoint from \( C \), then there are at least \( 2^{q/10000} \) distinct 3-colorings of \( G \) extending \( c \).

**Proof.** The proof is by induction on \( n, q \) ordered lexicographically.

If \( q \leq 10000 \), then Theorem 5.6 follows from Theorem 5.2. So assume that \( q > 10000 \).

If \( G \) has a vertex of degree 1 we delete it and use induction. So assume that all vertices have degree at least 2.

By adding edges if necessary, we may assume that no facial walk has length 8 or more. In particular, \( G \) is 2-connected.

If \( G \) has a separating cycle \( C_1 \) of length 4 or 5, then we apply the induction hypothesis first to \( Ext(C_1) \) and then \( Int(C_1) \). So assume that \( G \) has no separating cycle \( C_1 \) of length 4 or 5.

If \( G \) has no cycle of length 4 distinct from \( C \), then we use Theorem 4.3 which implies that \( G \) has at least \( 2^{q/10000} \) 3-colorings extending \( c \). (If \( C \) has length 4, we subdivide one of its edges before we use Theorem 4.3.) By Euler’s formula, \( q \leq 2n/3 \), so the proof is complete if \( G \) has no cycle of length 4 distinct from \( C \). So we assume that \( G \) has a 4-cycle \( C_1 \) distinct from \( C \), and that \( C_1 \) is facial. We identify two opposite vertices of \( C_1 \) (not both on \( C \)) and complete the proof by induction, if possible. Identifying two opposite vertices of \( C_1 \) does not create a triangle because \( G \) has no separating cycle of length 4 or 5. If \( q \) decreases, then some facial cycle distinct from \( C \) becomes equal to \( C \) after the vertex identification. But then \( G \) consists of \( C \) and a vertex of degree 2. But this is not possible because \( q > 10000 \). If one the \( q \) face boundaries is edge-disjoint from \( C \), then we may assume that this is also the case in the graph obtained by a vertex identification because that identification can be done in two distinct ways. \( \square \)

**Theorem 5.7.** Let \( G \) be a connected plane triangle-free graph with outer cycle \( C \) of length 4 or 5. Assume that \( G \) contains a collection of \( q \) cycles of length 5 such that each of them is edge-disjoint from \( C \), and any two of them are also edge-disjoint, and furthermore, if \( C_1, C_2 \) are two of the 5-cycles, then all edges of one of them are in the exterior of the other.

Let \( c \) be a 3-coloring of \( C \). Then there are at least \( 2^{q/10000} \) distinct 3-colorings of \( G \) extending \( c \).

**Proof.** The proof is by induction on \( q \). If \( q < 10000 \), we apply Theorem 5.2, so we proceed to the induction step. If none of the \( q \) cycles has another one in its interior, then we delete the interior of each of the \( q \) cycles and apply Theorem 5.6. So assume that at least one of the \( q \) cycles has another one in its interior. Now we delete the interior of each of those 5-cycles (in our collection) which is not contained in the interior of another one. We apply induction to the resulting graph. After that we apply induction to each of the 5-cycles whose interior was deleted. \( \square \)
**Theorem 5.8.** Let $G$ be a plane triangle-free graph with $n$ vertices. Then $G$ has at least $\frac{2n^{1/12}}{20000}$ distinct 3-colorings.

**Proof.** The proof is by induction on $n$. By adding edges, if necessary, we may assume that $G$ has an outer cycle of length 4 or 5. Euler’s formula implies that $G$ has more than $\frac{2n}{3}$ vertices of degree at most 10. Let $v$ be one of them. We delete $v$ and identify all its neighbors into one vertex. If the resulting graph is triangle-free we apply the induction hypothesis to it. Then there are two choices for the color of $v$ and the proof is complete. So we may assume that each vertex of degree at most 10 is contained in a 5-cycle. So $G$ has at least $\frac{2n}{15}$ distinct 5-cycles. Let $q$ be the largest natural number such that $10q^{12} \leq \frac{2n}{15}$. The outer cycle can be colored in at least 18 different way. By Lemmas 5.4, 5.5 and Theorem 5.7, $G$ has at least 18 times $\frac{2(q-5)}{10000}$ distinct 3-colorings because at most 5 cycles have an edge in common with $C$. □

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**References**